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| Title | Harmonic functions which vanish on a cylindrical surface |
| Authors(s) | Gardiner, Stephen J., Render, Hermann |
| Publication date | 2016-01-15 |
| Publication information | Gardiner, Stephen J., and Hermann Render. "Harmonic Functions Which Vanish on a Cylindrical Surface." Elsevier, January 15, 2016. https://doi.org/10.1016/j.jmaa.2015.08.077 . |
| Publisher | Elsevier |
| Item record/more information | http://hdl.handle.net/10197/7132 |
| Publisher's statement | This is the author's version of a work that was accepted for publication in Journal of Mathematical Analysis and Applications. Changes resulting from the publishing process, such as peer review, editing, corrections, structural formatting, and other quality control mechanisms may not be reflected in this document. Changes may have been made to this work since it was submitted for publication. A definitive version was subsequently published in Journal of Mathematical Analysis and Applications (VOL 433, ISSUE 2, (2015)) DOI: 10.1016/j.jmaa.2015.08.077. |
| Publisher's version (DOI) | 10.1016/j.jmaa.2015.08.077 |

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Harmonic functions which vanish on a cylindrical surface

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Abstract

Suppose that a harmonic function h on a finite cylinder vanishes on the curved part of the boundary. This paper answers a question of Khavinson by showing that h then has a harmonic continuation to the infinite strip bounded by the hyperplanes containing the flat parts of the boundary. The existence of this extension is established by an analysis of the convergence properties of a double series expansion of the Green function of an infinite cylinder beyond the domain itself.

1 Introduction

The Schwarz reflection principle gives a formula for extending a harmonic function h on a domain $\Omega \subset \mathbb{R}^N$ through a relatively open subset E of the boundary $\partial\Omega$ on which h vanishes, provided E lies in a hyperplane (and is a relatively open subset thereof). By the Kelvin transformation there is a corresponding result where E lies in a sphere. When $N = 2$, such a reflection principle holds also when E is contained in an analytic arc (see Chapter 9 of [7]). However, when $N \geq 3$ and N is odd, Ebenfelt and Khavinson [4] (see also [6] and Chapter 10 of [7]) have shown that a reflection law can only hold when the containing real analytic surface is either a hyperplane or a sphere.

Now let $N \geq 3$, let Ω_a be the finite cylinder $B' \times (-a, a)$, where B' is the open unit ball in \mathbb{R}^{N-1} and $a > 0$, and let $\Omega = B' \times \mathbb{R}$. Dima Khavinson raised the following question with the authors:

Question. Given a harmonic function h on Ω which vanishes on $\partial\Omega$, does it follow that h must have a harmonic extension to \mathbb{R}^N ?

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⁰2010 *Mathematics Subject Classification* 31B05.

Keywords: harmonic continuation, Green function, cylindrical harmonics

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Although the above results show that there can be no pointwise reflection formula for such an extension, this paper will establish that such an extension does indeed exist.

We will use the notation $x = (x', x_N)$ to denote a typical point of $\mathbb{R}^N = \mathbb{R}^{N-1} \times \mathbb{R}$.

Theorem 1 *Let h be a harmonic function on Ω_a which continuously vanishes on $\partial B' \times (-a, a)$. Then h has a harmonic extension \tilde{h} to $\mathbb{R}^{N-1} \times (-a, a)$. Further, for any $b \in (0, a)$, there is a constant c , depending on a, b, N and h , such that*

$$|\tilde{h}(x)| \leq c \|x'\|^{1-N/2} \quad (x' \in \mathbb{R}^{N-1} \setminus B', |x_N| < b). \quad (1)$$

It is a classical fact that the Green function for a three-dimensional infinite cylinder can be represented as a double series involving Bessel functions and Chebychev polynomials: see, for example, p.62 of Dougall [3] or p.78 of Carslaw [2]. Our approach to proving Theorem 1 involves establishing such a double series representation in N dimensions and analysing its convergence properties outside the cylinder.

2 Preparatory material

Let J_ν and Y_ν denote the usual Bessel functions of order $\nu \geq 0$ of the first and second kinds (see Watson [12]). Thus these functions both satisfy the differential equation

$$z^2 \frac{d^2 y}{dz^2} + z \frac{dy}{dz} + (z^2 - \nu^2)y = 0. \quad (2)$$

Further, let $(j_{\nu, m})_{m \geq 1}$ denote the sequence of positive zeros of J_ν , in increasing order. We collect below some facts that we will need.

- Lemma 2** (i) $\frac{d}{dz} z^\nu J_\nu(z) = z^\nu J_{\nu-1}(z)$ and $\frac{d}{dz} \frac{J_\nu(z)}{z^\nu} = -\frac{J_{\nu+1}(z)}{z^\nu}$.
(ii) $J_{\nu-1}(z) + J_{\nu+1}(z) = \frac{2\nu J_\nu(z)}{z}$ and $J_{\nu-1}(z) - J_{\nu+1}(z) = 2J'_\nu(z)$.
(iii) $J_\nu(t)Y'_\nu(t) - Y_\nu(t)J'_\nu(t) = \frac{2}{\pi t}$ ($t > 0$).
(iv) $\{J_\nu(t)\}^2 + \{Y_\nu(t)\}^2 < \frac{2}{\pi} (t^2 - \nu^2)^{-1/2}$ ($t > \nu \geq \frac{1}{2}$).
(v) $|J_\nu(t)| \leq \left(\frac{t}{2}\right)^\nu \frac{1}{\Gamma(\nu+1)}$ ($t \geq 0$).
(vi) $j_{0, m} \geq (m + 3/4)\pi$.
(vii) $j_{\nu, m} \geq j_{0, m} + \nu$.
(viii) $|J_\nu(t)| < \nu^{-1/3}$ ($\nu > 0, t \geq 0$).

$$(ix) |J_\nu(t)| \leq \min\{1, t^{-1/3}\} \quad (t > 0).$$

$$(x) \{J_\nu(t)\}^2 + \{Y_\nu(t)\}^2 < \frac{2}{\pi t} \quad (0 \leq \nu \leq \frac{1}{2}, t > 0).$$

Proof. (i) and (ii). See p.45 of Watson [12].

(iii) See p.76, (1) of [12].

(iv) See p.447, (1) of [12].

(v) See p.49, (1) of [12].

(vi) See p.489 of [12].

(vii) See Laforgia and Muldoon [8], (2.4).

(viii) See Landau [9].

(ix) We know from p.406, (10) of [12] that $|J_\nu| \leq 1$, and from [9] that $|J_\nu(t)| \leq t^{-1/3}$.

(x) By Section 13.74 of [12] the function $t \mapsto t \left(\{J_\nu(t)\}^2 + \{Y_\nu(t)\}^2 \right)$ is non-decreasing when $0 \leq \nu \leq \frac{1}{2}$, and has limit $2/\pi$ at ∞ . ■

Some consequences of Lemma 2 are noted below.

Lemma 3 (i) $j_{\nu,m}^2 \{J_{\nu+1}(j_{\nu,m})\}^2 \geq \frac{2}{\pi} \sqrt{j_{\nu,m}^2 - \nu^2} \quad (m \geq 1).$

(ii) $j_{\nu,m} \geq (m + 3/4)\pi + \nu.$

(iii) $|J_\nu(j_{\nu,m}s)| \leq j_{\nu,m}(1-s) \quad (0 \leq s \leq 1).$

(iv) $|J_\nu(j_{\nu,m}s)| \leq \frac{1}{\pi} \sqrt{\frac{2}{ms}} \quad (s \geq 1).$

Proof. (i) By the second equation of Lemma 2(i), $J'_\nu(j_{\nu,m}) = -J_{\nu+1}(j_{\nu,m})$. If $\nu \geq \frac{1}{2}$, then, by (iii), (iv) and (vii) of that result,

$$j_{\nu,m}^2 \{J_{\nu+1}(j_{\nu,m})\}^2 = j_{\nu,m}^2 \{J'_\nu(j_{\nu,m})\}^2 = \frac{4}{\pi^2 \{Y_\nu(j_{\nu,m})\}^2} > \frac{2}{\pi} \sqrt{j_{\nu,m}^2 - \nu^2}.$$

If $0 \leq \nu < \frac{1}{2}$, we use part (x) of Lemma 2 in place of part (iv).

(ii) This follows from Lemma 2(vi),(vii).

(iii) By the mean value theorem,

$$|J_\nu(j_{\nu,m}s)| \leq j_{\nu,m}(1-s) \sup_{1/2 \leq t \leq 1} |J'_\nu(j_{\nu,m}t)| \quad (1/2 \leq s \leq 1).$$

We know from Lemma 2(i),(ii) that $J'_\nu = (J_{\nu-1} - J_{\nu+1})/2$ when $\nu \geq 1$ and $J'_\nu(z) = (\nu/z)J_\nu(z) - J_{\nu+1}(z)$ otherwise. In either case we see from Lemma 2(ix) and part (ii) of this lemma that $|J'_\nu(j_{\nu,m}t)| \leq 1$ when $1/2 \leq t \leq 1$, so the desired inequality is established when $1/2 \leq s \leq 1$. It is clearly also valid when $0 \leq s < 1/2$, since $|J_\nu| \leq 1$, and $j_{\nu,m} \geq 2$ by part (ii).

(iv) By (ii) we have $(j_{\nu,m}s)^2 \geq (m\pi s)^2 + \nu^2$ when $s \geq 1$. If $\nu \geq 1/2$, then we see from Lemma 2(iv) that

$$\{J_\nu(j_{\nu,m}s)\}^2 < \frac{2}{\pi^2 ms} \quad (s \geq 1),$$

as desired. If $0 \leq \nu < 1/2$, we can instead appeal to Lemma 2(x). ■

The following result is taken from Sections 18.24–18.26 of [12].

Proposition 4 *Let $f : (0, 1] \rightarrow \mathbb{R}$ be a continuous function of locally bounded variation such that $f(1) = 0$ and $\int_0^1 t^{1/2} |f(t)| dt < \infty$, and let*

$$a_m = \frac{2}{\{J_{\nu+1}(j_{\nu,m})\}^2} \int_0^1 t f(t) J_{\nu}(j_{\nu,m}t) dt.$$

Then the series $\sum_{m=1}^{\infty} a_m J_{\nu}(j_{\nu,m}t)$ converges to $f(t)$ locally uniformly on $(0, 1]$.

Formula (3) below is stated without proof by Carslaw [2].

Lemma 5 *Let $0 < s < 1$.*

(a) If $\nu > 0$, then

$$4\nu \sum_{m=1}^{\infty} \frac{J_{\nu}(j_{\nu,m}s) J_{\nu}(j_{\nu,m}t)}{j_{\nu,m}^2 \{J_{\nu+1}(j_{\nu,m})\}^2} = \begin{cases} t^{\nu}(s^{-\nu} - s^{\nu}) & (0 \leq t \leq s) \\ s^{\nu}(t^{-\nu} - t^{\nu}) & (s < t \leq 1) \end{cases}, \quad (3)$$

and the series converges uniformly for $t \in [0, 1]$.

(b) In the case where $\nu = 0$ we have

$$2 \sum_{m=1}^{\infty} \frac{J_0(j_{0,m}s) J_0(j_{0,m}t)}{j_{0,m}^2 \{J_1(j_{0,m})\}^2} = \begin{cases} -\log s & (0 \leq t \leq s) \\ -\log t & (s < t \leq 1) \end{cases}, \quad (4)$$

and the series converges uniformly for $t \in [0, 1]$.

Proof. (a) Let $f_s(t)$ denote the expression on the right hand side of (3). By Proposition 4 it is sufficient to show that

$$\int_0^1 t f_s(t) J_{\nu}(j_{\nu,m}t) dt = \frac{2\nu}{j_{\nu,m}^2} J_{\nu}(j_{\nu,m}s).$$

In fact, using parts (i) and (ii) of Lemma 2, we see that

$$\begin{aligned}
\int_0^1 t f_s(t) J_\nu(j_{\nu,m} t) dt &= (s^{-\nu} - s^\nu) \int_0^s t^{\nu+1} J_\nu(j_{\nu,m} t) dt \\
&\quad + s^\nu \int_s^1 (t^{1-\nu} - t^{\nu+1}) J_\nu(j_{\nu,m} t) dt \\
&= (s^{-\nu} - s^\nu) \int_0^{j_{\nu,m} s} \frac{u^{\nu+1} J_\nu(u)}{j_{\nu,m}^{\nu+2}} du \\
&\quad + s^\nu \int_{j_{\nu,m} s}^{j_{\nu,m}} \left(\frac{j_{\nu,m}^{\nu-2}}{u^{\nu-1}} - \frac{u^{\nu+1}}{j_{\nu,m}^{\nu+2}} \right) J_\nu(u) du \\
&= (s^{-\nu} - s^\nu) \left[\frac{u^{\nu+1} J_{\nu+1}(u)}{j_{\nu,m}^{\nu+2}} \right]_0^{j_{\nu,m} s} \\
&\quad + s^\nu \left[-\frac{j_{\nu,m}^{\nu-2} J_{\nu-1}(u)}{u^{\nu-1}} - \frac{u^{\nu+1} J_{\nu+1}(u)}{j_{\nu,m}^{\nu+2}} \right]_{j_{\nu,m} s}^{j_{\nu,m}} \\
&= \frac{s}{j_{\nu,m}} \{J_{\nu+1}(j_{\nu,m} s) + J_{\nu-1}(j_{\nu,m} s)\} \\
&\quad - \frac{s^\nu}{j_{\nu,m}} \{J_{\nu-1}(j_{\nu,m}) + J_{\nu+1}(j_{\nu,m})\} \\
&= \frac{2\nu}{j_{\nu,m}^2} J_\nu(j_{\nu,m} s).
\end{aligned}$$

To see the uniformity of convergence of the series on all of $[0, 1]$, we note from parts (viii) and (ix) of Lemma 2 that

$$|J_\nu(j_{\nu,m} s) J_\nu(j_{\nu,m} t)| \leq (j_{\nu,m} s \nu)^{-1/3},$$

and thus from Lemma 3(i),(ii) that

$$\left| \frac{J_\nu(j_{\nu,m} s) J_\nu(j_{\nu,m} t)}{j_{\nu,m}^2 \{J_{\nu+1}(j_{\nu,m})\}^2} \right| \leq \frac{\pi (j_{\nu,m} s \nu)^{-1/3}}{2 \sqrt{j_{\nu,m}^2 - \nu^2}} \leq \frac{1}{2(\pi s \nu)^{1/3}} \frac{1}{m^{4/3}}.$$

(b) Now let $f_s(t)$ denote the expression on the right hand side of (4). Then

$$\begin{aligned}
\int_0^1 t f_s(t) J_0(j_{0,m} t) dt &= (-\log s) \int_0^s t J_0(j_{0,m} t) dt - \int_s^1 (\log t) t J_0(j_{0,m} t) dt \\
&= (-\log s) \int_0^{j_{0,m} s} \frac{u J_0(u)}{j_{0,m}^2} du - \int_{j_{0,m} s}^{j_{0,m}} \frac{u J_0(u)}{j_{0,m}^2} \log \frac{u}{j_{0,m}} du \\
&= (-\log s) \left[\frac{u J_1(u)}{j_{0,m}^2} \right]_0^{j_{0,m} s} + \log j_{0,m} \left[\frac{u J_1(u)}{j_{0,m}^2} \right]_{j_{0,m} s}^{j_{0,m}} \\
&\quad - \left[\frac{\log u}{j_{0,m}^2} u J_1(u) \right]_{j_{0,m} s}^{j_{0,m}} + \int_{j_{0,m} s}^{j_{0,m}} \frac{J_1(u)}{j_{0,m}^2} du \\
&= \frac{J_0(j_{0,m} s)}{j_{0,m}^2},
\end{aligned}$$

and (4) follows as before from Proposition 4. The proof of uniform convergence is also similar. ■

We recall (see Section 4.7 of Szegő [11], or Chapter IV of Stein and Weiss [10]) that, when $\lambda > 0$,

$$(1 - 2tu + u^2)^{-\lambda} = \sum_{n=0}^{\infty} P_n^{(\lambda)}(t) u^n \quad (|t| \leq 1, |u| < 1),$$

where $P_n^{(\lambda)}$ is the usual ultraspherical (Gegenbauer) polynomial. Also,

$$-\log(1 - 2tu + u^2) = \sum_{n=1}^{\infty} \frac{2}{n} T_n(t) u^n \quad (|t| \leq 1, |u| < 1), \quad (5)$$

where $T_n(t)$ is the Chebychev polynomial given by $\cos(n \cos^{-1} t)$ when $|t| \leq 1$. In each case, for a given t , the series is locally uniformly convergent in $u \in (-1, 1)$. These polynomials are related to each other by the equation

$$T_n(t) = (n/2) \lim_{\lambda \rightarrow 0^+} \lambda^{-1} P_n^{(\lambda)}(t) \quad (n \geq 1).$$

The next result summarizes properties that we will use.

Lemma 6 (i) $|P_n^{(\lambda)}(t)| \leq P_n^{(\lambda)}(1) = \binom{n+2\lambda-1}{n} \quad (|t| \leq 1).$

(ii) $P_n^{(\lambda)}$ satisfies the differential equation

$$(1-t^2) \frac{d^2 y}{dt^2} - (2\lambda+1)t \frac{dy}{dt} + n(n+2\lambda)y = 0.$$

(iii) T_n satisfies the differential equation

$$(1 - t^2) \frac{d^2 y}{dt^2} - t \frac{dy}{dt} + n^2 y = 0.$$

Proof. (i) See Theorem 2.7.1 and Corollary 2.3.8 of [1] for the inequality, and p.80 of [11] for the equality.

(ii) See p.80 of [11].

(iii) See p.60 of [11]. ■

3 Expansions in a ball

We denote by $G_{B'}(\cdot, \cdot)$ the Green function of the unit ball B' of \mathbb{R}^{N-1} ($N \geq 3$), and define

$$\nu_n = n + \frac{N-3}{2} \quad (n \geq 0).$$

Proposition 7 Suppose that $y' \in B' \setminus \{0'\}$.

(i) Let $N \geq 4$. If $0 < \|x'\| < \|y'\|$, then

$$G_{B'}(x', y') = \sum_{n=0}^{\infty} (\|x'\| \|y'\|)^{\frac{3-N}{2}} P_n^{\left(\frac{N-3}{2}\right)} \left(\frac{\langle x', y' \rangle}{\|x'\| \|y'\|} \right) \|x'\|^{\nu_n} \left\{ \|y'\|^{-\nu_n} - \|y'\|^{\nu_n} \right\}; \quad (6)$$

and, if $\|y'\| < \|x'\| < 1$, then

$$G_{B'}(x', y') = \sum_{n=0}^{\infty} (\|x'\| \|y'\|)^{\frac{3-N}{2}} P_n^{\left(\frac{N-3}{2}\right)} \left(\frac{\langle x', y' \rangle}{\|x'\| \|y'\|} \right) \|y'\|^{\nu_n} \left\{ \|x'\|^{-\nu_n} - \|x'\|^{\nu_n} \right\}. \quad (7)$$

Further, for any $\varepsilon > 0$, these series converge uniformly on $\{x' : 0 < \|x'\| \leq \|y'\| - \varepsilon\}$ and $\{x' : \|y'\| + \varepsilon \leq \|x'\| < 1\}$, respectively.

(ii) Let $N = 3$. If $0 < \|x'\| < \|y'\|$, then

$$G_{B'}(x', y') = -\log \|y'\| + \sum_{n=1}^{\infty} \frac{1}{n} T_n \left(\frac{\langle x', y' \rangle}{\|x'\| \|y'\|} \right) \|x'\|^n \left\{ \|y'\|^{-n} - \|y'\|^n \right\}; \quad (8)$$

and, if $\|y'\| < \|x'\| < 1$, then

$$G_{B'}(x', y') = -\log \|x'\| + \sum_{n=1}^{\infty} \frac{1}{n} T_n \left(\frac{\langle x', y' \rangle}{\|x'\| \|y'\|} \right) \|y'\|^n \left\{ \|x'\|^{-n} - \|x'\|^n \right\}. \quad (9)$$

Again, for any $\varepsilon > 0$, these series converge uniformly on $\{x' : 0 < \|x'\| \leq \|y'\| - \varepsilon\}$ and $\{x' : \|y'\| + \varepsilon \leq \|x'\| < 1\}$, respectively.

Proof. (i) Let $N \geq 4$. Then (see Chapter 4 of [1])

$$\begin{aligned}
G_{B'}(x', y') &= \|x' - y'\|^{3-N} - \|y'\|^{3-N} \left\| x' - \frac{y'}{\|y'\|^2} \right\|^{3-N} \\
&= \|y'\|^{3-N} \left\| \frac{x'}{\|y'\|} - \frac{y'}{\|y'\|} \right\|^{3-N} - \left\| \|y'\| x' - \frac{y'}{\|y'\|} \right\|^{3-N} \\
&= \|x'\|^{3-N} \left\| \frac{x'}{\|x'\|} - \frac{y'}{\|x'\|} \right\|^{3-N} - \left\| \|y'\| x' - \frac{y'}{\|y'\|} \right\|^{3-N}.
\end{aligned}$$

We know that

$$\begin{aligned}
\left\| \|y'\| x' - \frac{y'}{\|y'\|} \right\|^{3-N} &= \left(1 - 2 \frac{\langle x', y' \rangle}{\|x'\| \|y'\|} \|x'\| \|y'\| + \|x'\|^2 \|y'\|^2 \right)^{\frac{3-N}{2}} \\
&= \sum_{n=0}^{\infty} P_n^{\left(\frac{N-3}{2}\right)} \left(\frac{\langle x', y' \rangle}{\|x'\| \|y'\|} \right) (\|x'\| \|y'\|)^n,
\end{aligned}$$

and the series converges uniformly for $x' \in B' \setminus \{0\}$. If $\|x'\| < \|y'\|$, then

$$\begin{aligned}
\left\| \frac{x'}{\|y'\|} - \frac{y'}{\|y'\|} \right\|^{3-N} &= \left(1 - 2 \frac{\langle x', y' \rangle}{\|x'\| \|y'\|} \frac{\|x'\|}{\|y'\|} + \frac{\|x'\|^2}{\|y'\|^2} \right)^{\frac{3-N}{2}} \\
&= \sum_{n=0}^{\infty} P_n^{\left(\frac{N-3}{2}\right)} \left(\frac{\langle x', y' \rangle}{\|x'\| \|y'\|} \right) \left(\frac{\|x'\|}{\|y'\|} \right)^n,
\end{aligned}$$

where the series converges uniformly for $0 < \|x'\| \leq \|y'\| - \varepsilon$, and so

$$G_{B'}(x', y') = \sum_{n=0}^{\infty} P_n^{\left(\frac{N-3}{2}\right)} \left(\frac{\langle x', y' \rangle}{\|x'\| \|y'\|} \right) \|x'\|^n \left\{ \|y'\|^{3-N-n} - \|y'\|^n \right\},$$

which yields the desired formula. If $\|y'\| < \|x'\| < 1$, then

$$\begin{aligned}
\left\| \frac{x'}{\|x'\|} - \frac{y'}{\|x'\|} \right\|^{3-N} &= \left(1 - 2 \frac{\langle x', y' \rangle}{\|x'\| \|y'\|} \frac{\|y'\|}{\|x'\|} + \frac{\|y'\|^2}{\|x'\|^2} \right)^{(3-N)/2} \\
&= \sum_{n=0}^{\infty} P_n^{\left(\frac{N-3}{2}\right)} \left(\frac{\langle x', y' \rangle}{\|x'\| \|y'\|} \right) \left(\frac{\|y'\|}{\|x'\|} \right)^n,
\end{aligned}$$

where the series converges uniformly for $\|y'\| + \varepsilon \leq \|x'\| < 1$, and so

$$G_{B'}(x', y') = \sum_{n=0}^{\infty} P_n^{\left(\frac{N-3}{2}\right)} \left(\frac{\langle x', y' \rangle}{\|x'\| \|y'\|} \right) \|y'\|^n \left\{ \|x'\|^{3-N-n} - \|x'\|^n \right\},$$

which again yields the desired formula.

(ii) Let $N = 3$. Then

$$\begin{aligned}
G_{B'}(x', y') &= -\log \|x' - y'\| + \log \left(\|y'\| \left\| x' - \frac{y'}{\|y'\|^2} \right\| \right) \\
&= -\log \|y'\| - \log \left\| \frac{x'}{\|y'\|} - \frac{y'}{\|y'\|} \right\| + \log \left\| \|y'\| x' - \frac{y'}{\|y'\|} \right\| \\
&= -\log \|x'\| - \log \left\| \frac{x'}{\|x'\|} - \frac{y'}{\|x'\|} \right\| + \log \left\| \|y'\| x' - \frac{y'}{\|y'\|} \right\|.
\end{aligned}$$

We know from (5), multiplied by the factor $1/2$, that

$$-\log \left\| \|y'\| x' - \frac{y'}{\|y'\|} \right\| = \sum_{n=1}^{\infty} \frac{1}{n} T_n \left(\frac{\langle x', y' \rangle}{\|x'\| \|y'\|} \right) (\|x'\| \|y'\|)^n.$$

Also,

$$-\log \left\| \frac{x'}{\|y'\|} - \frac{y'}{\|y'\|} \right\| = \sum_{n=1}^{\infty} \frac{1}{n} T_n \left(\frac{\langle x', y' \rangle}{\|x'\| \|y'\|} \right) \left(\frac{\|x'\|}{\|y'\|} \right)^n \quad \text{if } \|x'\| < \|y'\|,$$

and

$$-\log \left\| \frac{x'}{\|x'\|} - \frac{y'}{\|x'\|} \right\| = \sum_{n=1}^{\infty} \frac{1}{n} T_n \left(\frac{\langle x', y' \rangle}{\|x'\| \|y'\|} \right) \left(\frac{\|y'\|}{\|x'\|} \right)^n \quad \text{if } \|y'\| < \|x'\| < 1.$$

Thus we obtained the desired formulae as before. ■

The following result is sufficient for our purposes, but is known to hold under much weaker smoothness assumptions (see [5] and the references therein).

Proposition 8 *Let $f \in C^\infty(\partial B')$ and let $c_{i,j}$ be the Fourier coefficients of f with respect to an orthonormal basis $\{H_{i,j} : j = 1, \dots, M(i)\}$ of the spherical harmonics of degree i in \mathbb{R}^{N-1} . Then the series $\sum_{i=0}^{\infty} \sum_{j=1}^{M(i)} c_{i,j} H_{i,j}$ converges uniformly on $\partial B'$ to f , and so the series*

$$\sum_{i=0}^{\infty} \|x'\|^i \sum_{j=1}^{M(i)} c_{i,j} H_{i,j} \left(\frac{x'}{\|x'\|} \right)$$

converges uniformly on $B' \setminus \{0'\}$ to the Poisson integral of f in B' .

Let $y' \in B' \setminus \{0'\}$ and $\delta \in (0, 1)$, and let $S_{y'}$ be the sphere in \mathbb{R}^{N-1} centred at $0'$ that contains y' . We define $\mu_{y',\delta}$ to be the probability measure on $S_{y'}$ which has density with respect to surface measure proportional to

$$\exp \left(- \left(1 - \frac{\|z' - y'\|^2}{\delta^2 \|y'\|^2} \right)^{-1} \right) \quad \text{when } \|z' - y'\| < \delta \|y'\|, \quad \text{and } 0 \text{ otherwise.}$$

We further define the Green potential

$$G_{B'}\mu_{y',\delta}(x') = \int G_{B'}(x', z') d\mu_{y',\delta}(z') \quad (x' \in B'),$$

and the function

$$P_n^{\left(\frac{N-3}{2}\right)}\mu_{y',\delta}(x') = \int P_n^{\left(\frac{N-3}{2}\right)}\left(\frac{\langle x', z' \rangle}{\|x'\| \|z'\|}\right) d\mu_{y',\delta}(z') \quad (x' \in \mathbb{R}^{N-1} \setminus \{0'\}).$$

When $N = 3$ the function $T_n\mu_{y',\delta}$ is defined from T_n analogously.

Remark 9 *By Proposition 7 we obtain formulae for $G_{B'}\mu_{y',\delta}(x')$ if we replace $P_n^{\left(\frac{N-3}{2}\right)}\left(\frac{\langle x', y' \rangle}{\|x'\| \|y'\|}\right)$ by $P_n^{\left(\frac{N-3}{2}\right)}\mu_{y',\delta}(x')$ in (6) and (7), and $T_n\left(\frac{\langle x', y' \rangle}{\|x'\| \|y'\|}\right)$ by $T_n\mu_{y',\delta}(x')$ in (8) and (9). Further, the series in (6) and (8) would now converge uniformly on $\{x' : 0 < \|x'\| \leq \|y'\|\}$, by Proposition 8, because the restriction of $G_{B'}\mu_{y',\delta}$ to $S_{y'}$ is C^∞ (cf. Theorem 3.3.3 of [1]). Also, the series in (7) and (9) would converge uniformly on $\{x' : \|y'\| \leq \|x'\| < 1\}$. To see this in the case of (9) we write the series as the difference of*

$$\sum_{n=1}^{\infty} \frac{1}{n} T_n\mu_{y',\delta}(x') \|y'\|^n \|x'\|^{-n} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{n} T_n\mu_{y',\delta}(x') \|y'\|^n \|x'\|^n.$$

The second of these series clearly converges uniformly on B' , and we can use inversion together with Proposition 8 to see that the first series converges uniformly on $\{x' : \|x'\|^{-1} \leq \|y'\|^{-1}\}$. A similar argument, based on the Kelvin transformation, applies to the series in (7).

Let

$$f_{n,m}(s, t) = \frac{J_{\nu_n}(j_{\nu_n,ms})J_{\nu_n}(j_{\nu_n,mt})}{j_{\nu_n,m} \{J_{\nu_n+1}(j_{\nu_n,m})\}^2} \quad (s \geq 0, t \geq 0, n \geq 0, m \geq 1).$$

The above remark and Lemma 5 combine to yield the following.

Proposition 10 *Suppose that $x', y' \in B' \setminus \{0'\}$ and $\delta \in (0, 1)$.*

(i) *If $N \geq 4$, then*

$$G_{B'}\mu_{y',\delta}(x') = \sum_{n=0}^{\infty} (\|x'\| \|y'\|)^{\frac{3-N}{2}} P_n^{\left(\frac{N-3}{2}\right)}\mu_{y',\delta}(x') 4\nu_n \sum_{m=1}^{\infty} \frac{f_{n,m}(\|x'\|, \|y'\|)}{j_{\nu_n,m}}.$$

(ii) *If $N = 3$, then*

$$G_{B'}\mu_{y',\delta}(x') = 2 \sum_{m=1}^{\infty} \frac{f_{0,m}(\|x'\|, \|y'\|)}{j_{0,m}} + 4 \sum_{n=1}^{\infty} T_n\mu_{y',\delta}(x') \sum_{m=1}^{\infty} \frac{f_{n,m}(\|x'\|, \|y'\|)}{j_{\nu_n,m}}.$$

4 Proof of Theorem 1

We define $a_N = \sigma_N(N - 2)$ when $N \geq 3$, and $a_2 = \sigma_2$, where σ_N denotes the surface area of the unit sphere in \mathbb{R}^N .

Lemma 11 *For any $n \geq 0, m \geq 1$ and any $y \in (B' \setminus \{0'\}) \times \mathbb{R}$, let $u_{n,m,y}$ be the function defined by*

$$x \mapsto \|x'\|^{\frac{3-N}{2}} P_n^{\left(\frac{N-3}{2}\right)} \left(\frac{\langle x', y' \rangle}{\|x'\| \|y'\|} \right) f_{n,m}(\|x'\|, \|y'\|) e^{-j\nu_{n,m}|x_N - y_N|} \quad (N \geq 4),$$

$$x \mapsto T_n \left(\frac{\langle x', y' \rangle}{\|x'\| \|y'\|} \right) f_{n,m}(\|x'\|, \|y'\|) e^{-j\nu_{n,m}|x_N - y_N|} \quad (N = 3).$$

Then $u_{n,m,y}$

- (i) is harmonic on $(\mathbb{R}^{N-1} \setminus \{0'\}) \times (\mathbb{R} \setminus \{y_N\})$;
- (ii) has a harmonic continuation to $\mathbb{R}^{N-1} \times (\mathbb{R} \setminus \{y_N\})$;
- (iii) continuously vanishes on $\partial\Omega$;
- (iv) is, in Ω , the Green potential of the signed measure given by

$$2a_N^{-1} j\nu_{n,m} u_{n,m,y}(x', y_N) dx' \quad \text{on } B' \times \{y_N\}; \quad (10)$$

(v) satisfies

$$|u_{n,m,y}(x)| \leq \frac{1 - \|y'\|}{\|x'\|^{N/2-1}} \binom{n + N - 4}{n} \frac{e^{-j\nu_{n,m}\varepsilon}}{\{J_{\nu_{n+1}}(j\nu_{n,m})\}^2}$$

on $\mathbb{R}^{N-1} \times ((-\infty, y_N - \varepsilon) \cup (y_N + \varepsilon, \infty))$, where $\varepsilon > 0$ and the binomial coefficient is interpreted as 1 when $N = 3$.

Proof. (i) If $N \geq 4$, we know from Theorem 2.7.1 of [1] that the function

$$x' \mapsto \|x'\|^n P_n^{\left(\frac{N-3}{2}\right)} \left(\frac{\langle x', y' \rangle}{\|x'\| \|y'\|} \right),$$

interpreted as 0 at $0'$, is harmonic on \mathbb{R}^{N-1} , so the function

$$x' \mapsto P_n^{\left(\frac{N-3}{2}\right)} \left(\frac{\langle x', y' \rangle}{\|x'\| \|y'\|} \right)$$

is an eigenfunction of the Laplace-Beltrami operator on $\partial B'$ with eigenvalue $-n(n + N - 3)$. Next, the Laplacian of the radial function

$$x' \mapsto \|x'\|^{\frac{3-N}{2}} J_{\nu_n}(j\nu_{n,m} \|x'\|)$$

is, by direct computation,

$$r^{-(N+1)/2} \left\{ j_{\nu_{n,m}}^2 r^2 J_{\nu_n}''(j\nu_{n,m}r) + j_{\nu_{n,m}} r J_{\nu_n}'(j\nu_{n,m}r) - \left(\frac{N-3}{2} \right)^2 J_{\nu_n}(j\nu_{n,m}r) \right\},$$

where $r = \|x'\|$. Hence, by (2), the function

$$x' \mapsto \|x'\|^{\frac{3-N}{2}} P_n^{\left(\frac{N-3}{2}\right)} \left(\frac{\langle x', y' \rangle}{\|x'\| \|y'\|} \right) J_{\nu_n}(j_{\nu_n, m} \|x'\|)$$

is an eigenfunction of the Laplacian on $\mathbb{R}^{N-1} \setminus \{0'\}$, with eigenvalue $-j_{\nu_n, m}^2$. The harmonicity of $u_{n, m, y}$ in $(\mathbb{R}^{N-1} \setminus \{0'\}) \times (\mathbb{R} \setminus \{y_N\})$ is now clear.

The case where $N = 3$ is similar (and simpler).

(ii) This follows from Lemma 2(v) and the fact that any line segment, being polar, is a removable singularity for bounded harmonic functions in \mathbb{R}^N ($N \geq 3$). (See Corollary 5.2.3 of [1].)

(iii) This is obvious.

(iv) Let Ψ be a C^∞ function on Ω with compact support. This support is contained in Ω_a for some $a > |y_N|$. By applying Green's theorem to the cylinders $B' \times (-a, y_N - \varepsilon)$ and $B' \times (y_N + \varepsilon, a)$, we obtain (in the sense of distributions)

$$\begin{aligned} (\Delta u_{n, m, y})(\Psi) &= \int_{\Omega} u_{n, m, y} \Delta \Psi \, dx \\ &= \lim_{\varepsilon \rightarrow 0^+} \left\{ \begin{aligned} &\int_{B' \times \{y_N + \varepsilon\}} \left(\Psi \frac{\partial u_{n, m, y}}{\partial x_N} - u_{n, m, y} \frac{\partial \Psi}{\partial x_N} \right) dx' \\ &- \int_{B' \times \{y_N - \varepsilon\}} \left(\Psi \frac{\partial u_{n, m, y}}{\partial x_N} - u_{n, m, y} \frac{\partial \Psi}{\partial x_N} \right) dx' \end{aligned} \right\} \\ &= - \int_{B' \times \{y_N\}} 2j_{\nu_n, m} u_{n, m, y} \Psi dx'. \end{aligned}$$

Thus, if v denotes the Green potential in Ω of the signed measure given by (10), we see from Theorems 4.3.8(i) and 4.3.5 of [1] that $u_{n, m, y} - v$ is harmonic on Ω . Since this difference vanishes on $\partial\Omega$ and at infinity, we conclude that $u_{n, m, y} = v$ on Ω .

(v) Lemma 3(iv) shows that $|J_{\nu_n}(j_{\nu_n, m} \|x'\|)| \leq \|x'\|^{-1/2}$ when $\|x'\| \geq 1$, and this inequality remains valid for $\|x'\| < 1$ by Lemma 2(ix). Also, Lemma 3(iii) shows that $|J_{\nu_n}(j_{\nu_n, m} \|y'\|)| \leq j_{\nu_n, m}(1 - \|y'\|)$. On $\mathbb{R}^{N-1} \times ((-\infty, y_N - \varepsilon) \cup (y_N + \varepsilon, \infty))$ we thus have the stated estimate, in view of Lemma 6(i). ■

Remark 12 Now let $\delta \in (0, 1)$ and let $u_{n, m, y}^\delta$ have the same definition as $u_{n, m, y}$, except that we use $P_n^{\left(\frac{N-3}{2}\right)} \mu_{y', \delta}(x')$ (respectively, $T_n \mu_{y', \delta}(x')$) in place of $P_n^{\left(\frac{N-3}{2}\right)} \left(\frac{\langle x', y' \rangle}{\|x'\| \|y'\|} \right)$ (respectively, $T_n \left(\frac{\langle x', y' \rangle}{\|x'\| \|y'\|} \right)$). Then Lemma 11 clearly remains true if we replace $u_{n, m, y}$ by $u_{n, m, y}^\delta$ throughout.

Theorem 13 Let $y \in (B' \setminus \{0'\}) \times \mathbb{R}$. Then $G_\Omega(\cdot, y)$ has a harmonic extension $\tilde{G}_\Omega(\cdot, y)$ to $\mathbb{R}^{N-1} \times (\mathbb{R} \setminus \{y_N\})$. On this set,

$$\tilde{G}_\Omega(x, y) = \frac{a_N}{a_{N-1}} \|y'\|^{\frac{3-N}{2}} \sum_{n=0}^{\infty} 2\nu_n \sum_{m=1}^{\infty} u_{n,m,y}(x) \quad (N \geq 4), \quad (11)$$

$$\tilde{G}_\Omega(x, y) = 2 \sum_{m=1}^{\infty} u_{0,m,y}(x) + 4 \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} u_{n,m,y}(x) \quad (N = 3). \quad (12)$$

Further, if $\varepsilon > 0$, then there is a positive constant c , depending on ε and N , such that

$$\left| \tilde{G}_\Omega(x, y) \right| \leq c \frac{1 - \|y'\|}{\|x'\|^{N/2-1}} \quad \text{on } \{(x', x_N) : |x_N - y_N| \geq \varepsilon\}. \quad (13)$$

Proof. Let $\delta \in (0, 1)$. If $|x_N - y_N| \geq \varepsilon > 0$, then Lemma 11(v), Remark 12 and Lemma 3(i) together show that

$$\left| u_{n,m,y}^\delta(x) \right| \leq \frac{\pi}{2} \frac{1 - \|y'\|}{\|x'\|^{N/2-1}} \binom{n+N-4}{n} \frac{j_{\nu_n,m}^2 e^{-j\nu_n,m\varepsilon}}{\sqrt{j_{\nu_n,m}^2 - \nu_n^2}}. \quad (14)$$

Since

$$\frac{1}{\sqrt{j_{\nu_n,m}^2 - \nu_n^2}} \leq \frac{1}{j_{0,m}} \leq \frac{1}{\pi}$$

by Lemma 2(vii), and $j_{\nu_n,m} \geq (m + 3/4)\pi + n$ by Lemma 3(ii), we see that

$$\left| \frac{u_{n,m,y}^\delta(x)}{j_{\nu_n,m}^2} \right| \leq \frac{1 - \|y'\|}{\|x'\|^{N/2-1}} \binom{n+N-4}{n} e^{-(\pi m+n)\varepsilon} \quad (n \geq 0, m \geq 1).$$

It follows that the series

$$\begin{aligned} & \frac{a_N}{a_{N-1}} \|y'\|^{\frac{3-N}{2}} \sum_{n=0}^{\infty} 2\nu_n \sum_{m=1}^{\infty} \frac{u_{n,m,y}^\delta(x)}{j_{\nu_n,m}^2} \quad (N \geq 4), \\ & 2 \sum_{m=1}^{\infty} \frac{u_{0,m,y}^\delta(x)}{j_{0,m}^2} + 4 \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{u_{n,m,y}^\delta(x)}{j_{\nu_n,m}^2} \quad (N = 3) \end{aligned}$$

converge absolutely and locally uniformly on $(\mathbb{R}^{N-1} \setminus \{0'\}) \times (\mathbb{R} \setminus \{y_N\})$, and indeed (by the volume mean value property of harmonic functions) on $\mathbb{R}^{N-1} \times (\mathbb{R} \setminus \{y_N\})$, to a harmonic function g satisfying the bound in (13). We know from Remark 12 and Lemma 11(iv) that $(a_N/a_{N-1})j_{\nu_n,m}^{-2}u_{n,m,y}^\delta$ is, in Ω , the potential of the signed measure given by

$$2a_{N-1}^{-1}j_{\nu_n,m}^{-1}u_{n,m,y}^\delta(x', y_N)dx' \quad \text{on } B' \times \{y_N\}.$$

(When $N = 2$, we have $a_N/a_{N-1} = 2$ and $\nu_n = n$.) Thus, by Proposition 10, and the uniformity of convergence in Lemma 5 and Remark 9, the function g is, in Ω , the potential $G_\Omega \mu_1^\delta$ of the measure given by

$$d\mu_1^\delta = a_{N-1}^{-1} G_{B' \mu_{y', \delta}}(x') dx' \quad \text{on } B' \times \{y_N\}.$$

By the invariance of $G_\Omega(\cdot, \cdot)$ under translation in the x_N -direction, and harmonicity,

$$\frac{\partial^2 G_\Omega \mu_1^\delta}{\partial x_N^2} = \int_{B'} G_\Omega(\cdot, (z', y_N)) d\mu_2^\delta(z') \quad \text{on } \Omega,$$

where

$$d\mu_2^\delta = -a_{N-1}^{-1} \Delta G_{B' \mu_{y', \delta}} \quad \text{on } B' \times \{y_N\}$$

in the sense of distributions. Hence

$$\frac{\partial^2 G_\Omega \mu_1^\delta}{\partial x_N^2} = \int_{B'} G_\Omega(\cdot, (z', y_N)) d\mu_{y', \delta}(z') \quad \text{on } \Omega. \quad (15)$$

The left hand side of (15) equals

$$\begin{aligned} & \frac{a_N}{a_{N-1}} \|y'\|^{\frac{3-N}{2}} \sum_{n=0}^{\infty} 2\nu_n \sum_{m=1}^{\infty} u_{n,m,y}^\delta(x) \quad (N \geq 4), \\ & 2 \sum_{m=1}^{\infty} u_{0,m,y}^\delta(x) + 4 \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} u_{n,m,y}^\delta(x) \quad (N = 3) \end{aligned}$$

on $\Omega \setminus (\mathbb{R}^{N-1} \times \{y_N\})$ by the local uniform convergence of the above series there. Further, as $\delta \rightarrow 0+$, the functions defined by these series converge locally uniformly on the same set to the expressions on the right hand side of (11) and (12). Since the right hand side of (15) converges to $G_\Omega(\cdot, y)$ as $\delta \rightarrow 0+$, we have established (11) and (12). Further, (13) follows easily from (14). ■

Remark 14 *The proof of the above results can be simplified when $N = 3$: the use of the smoothing measure $\mu_{y', \delta}$ can be avoided, since the partial sums of the series in Proposition 7(ii) are then dominated by a multiple of the integrable function $x' \mapsto -\log \left(1 - \frac{\min\{\|x'\|, \|y'\|\}}{\max\{\|x'\|, \|y'\|\}} \right)$ on B' .*

Proof of Theorem 1. Let h be a harmonic function on Ω_a which vanishes on $\partial B' \times (-a, a)$. There is no loss of generality in assuming that h is continuous on $\bar{\Omega}_a$. Further, since h (being the integral of $h|_{\partial\Omega_a}$ against harmonic measure) can be written as the difference of two such functions which are positive, there is no loss of generality in assuming that $h > 0$.

Next, let $0 < b < a$, and let h^* be defined as h on $B' \times [-b, b]$, as 0 on $B' \times (-\infty, -a]$ and $B' \times [a, \infty)$ and $\partial\Omega$, and extended to Ω by solving the Dirichlet problem in $B' \times (-a, -b)$ and in $B' \times (b, a)$. Then h^* is subharmonic on $B' \times ((-\infty, -b) \cup (b, \infty))$ and superharmonic on $B' \times (-a, a)$, and continuously vanishes on $\partial\Omega$. It can be written as $G_\Omega\mu$, where μ is a signed measure on $\tilde{B}' \times \{\pm a, \pm b\}$ satisfying

$$\int (1 - |y'|) d|\mu|(y) < \infty.$$

It now follows from Theorem 13 that h can be extended to a harmonic function \tilde{h} on $\mathbb{R}^{N-1} \times (-b, b)$. The estimate (1) is a consequence of (13). Since b can be arbitrarily close to a , the result follows. ■

Remark 15 *The rate of decay in (1) is sharp. This follows from the observations that each of the functions $u_{n,m,y}$ in Lemma 11 satisfies the hypotheses of Theorem 1 (with $a = |y_N|$), and that (for a fixed ν)*

$$J_\nu(t) = \sqrt{\frac{2}{\pi t}} \cos\left(t - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) + O(t^{-3/2}) \quad (t \rightarrow \infty)$$

(see [11], equation (1.71.7)).

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