Title	Cases where the linear canonical transform of a signal has compact support or is band-limited
Authors(s)	Healy, John J., Sheridan, John T.
Publication date	2008-02-01
Publication information	Healy, John J., and John T. Sheridan. "Cases Where the Linear Canonical Transform of a Signal Has Compact Support or Is Band-Limited." Optical Society of America, February 1, 2008. https://doi.org/10.1364/OL.33.000228.
Publisher	Optical Society of America
Item record/more information	http://hdl.handle.net/10197/3366
Publisher's statement	This paper was published in Optics Letters and is made available as an electronic reprint with the permission of OSA. The paper can be found at the following URL on the OSA website: http://www.opticsinfobase.org/abstract.cfm?URI=ol-33-3-228. Systematic or multiple reproduction or distribution to multiple locations via electronic or other means is prohibited and is subject to penalties under law.
Publisher's version (DOI)	10.1364/OL.33.000228

Cases where the linear canonical transform of a signal has compact support or is band-limited

John J. Healy and John T. Sheridan*

School of Electrical, Electronic and Mechanical Engineering, University College Dublin, Republic of Ireland
**Corresponding author: john.sheridan@ee.ucd.ie*

Received August 14, 2007; revised November 2, 2007; accepted December 1, 2007;
posted December 17, 2007 (Doc. ID 86497); published January 25, 2008

A signal may have compact support, be band-limited (i.e., its Fourier transform has compact support), or neither ("unbounded"). We determine conditions for the linear canonical transform of a signal having these properties. We examine the significance of these conditions for special cases of the linear canonical transform and consider the physical significance of our results. © 2008 Optical Society of America

OCIS codes: 070.2465, 070.2590, 110.1220, 200.1130.

A theorem widely discussed in standard texts in engineering, optics, and mathematics, which we shall refer to as the "FT compact support theorem," states that a signal and its Fourier transform (FT) cannot both have compact support [1]. Strictly speaking, this is a corollary of the Paley–Wiener theorem [2,3].

The FT compact support theorem has physical significance in optics because of limits on image capture systems, i.e., finite camera apertures and pixel sizes, because of other apertures in the system, and for choosing ranges for numerical simulation [4].

The FT is a special case of the linear canonical transform (LCT) [5], a three-parameter class of linear transform that describes the effect of quadratic phase systems on a wavefield, generalizing many optical transforms. The LCT finds use in phase reconstruction [6], filter design [7], and graded-index media analysis [8]. In this paper, we consider how the FT compact support theorem generalizes for the LCT, which is of interest for signal separation [9].

Clarification is necessary because it has been assumed that a signal cannot have compact support in two LCT domains [10]. As we prove, this is not necessarily so. We will demonstrate that, e.g., the Fresnel transform (FST), a special case of the LCT, preserves band-limitedness, and chirp multiplication (CMT) preserves compact support.

Consider a signal, $f(x)$. The LCT of this signal is given by

$$L_T\{f(x)\}(x') = \begin{cases} \sqrt{\dfrac{1}{b}}e^{-j\pi/4}\displaystyle\int_{-\infty}^{\infty} f(x)e^{j\pi/b(ax^2-2xx'+dx'^2)}\mathrm{d}x & b\neq 0 \\ \sqrt{d}e^{j/2cdx'^2}f(dx') & b=0 \end{cases}. \tag{1}$$

The transform parameters are a, b, and d, which combine to form the ray transfer matrix, $T=[a\ b;\ c\ d]$. This must have unit determinant [5]. The effect of an LCT on a signal is an affine operation on the time–frequency representation as follows:

$$\begin{pmatrix} x' \\ k' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ k \end{pmatrix}, \tag{2}$$

where k and k' are the domains reached by taking the FT of a signal in x and x', respectively.

We propose that Eq. (2) indicates that given a compactly supported input signal, $f(x)$, to any quadratic phase system with parameter $d=0$, the output will be band-limited. This is because the operation shown in Eq. (3) will create frequency components only from spatial components ($k'=cx+0k$), which have compact support. Similarly, we propose a number of other properties, summarized in Table 1.

We note that in Table 1, one may obtain results for some cases where a signal is known to be *not* band-limited or *not* compactly supported, e.g., if $a=0$ and the signal is not band-limited, then the input must be either compactly supported or unbounded. Since both result in an output that does not have compact support, we have a result for non-band-limited signals. However, for a signal without compact support the two possibilities give conflicting results, and we can draw no conclusion.

We now present mathematical proof of the claims summarized in Table 1:

Theorem 1. Given an input signal, $f(x)$, which is band-limited, $L_T\{f(x)\}(x')$ will have compact support if $a=0$.

Proof. We neglect the complex constant in Eq. (1) for convenience. This does not alter the result. Substituting $a=0$ into Eq. (1) for a, and moving the independent term outside the integral,

$$L_T\{f(x)\}(x') = e^{j\pi(dx'^2)/b}\int_{-\infty}^{\infty} f(x)e^{-j2\pi xx'/b}\mathrm{d}x. \tag{3}$$

The integral is now a scaled FT, giving

$$L_T\{f(x)\}(x') = e^{j\pi(dx'^2)/b}F(2\pi x'/b), \tag{4}$$

where $F(x')$ is the FT of $f(x)$. As $f(x)$ is band-limited, $F(2\pi x'/b)$ has compact support, and hence L_T has compact support.

Corollary 1. Given an input signal, $f(x)$, which is

Table 1. Conditions on the Input Signal (Columns) and the Parameters (Rows)[a]

	$a=0$	$b=0$	$c=0$	$d=0$	$a=d=0$	$b=c=0$	$a,b,c,d\neq 0$
CS	$\overline{CS}$	CS	$\overline{BL}$	BL	BL	CS	•
	C1	T2	C3	T4	T4	T2	T5
BL	CS	$\overline{CS}$	BL	$\overline{BL}$	CS	BL	•
	T1	C2	T3	C4	T1	T3	T5
•	$\overline{CS}$	$\overline{CS}$	$\overline{BL}$	$\overline{BL}$	•	•	—
	C1	C2	C3	C4	C1, C4	C2, C3	T5

[a]CS and BL mean compactly supported and band-limited respectively. The convention $\overline{CS}$ means "not CS," and • means neither CS nor BL. A hyphen means "cannot be determined". The abbreviations T1, C2, etc., refer to "Theorem 1," "Corollary 2," etc., where these prove the property.

not band-limited, $L_T\{f(x)\}(x')$ will not have compact support if $a=0$.

Proof. As $a=0$, Eq. (4) holds. As $f(x)$ is not band-limited, $F(2\pi x'/b)$ does not have compact support, and hence L_T does not have compact support.

Theorem 2. Given an input signal, $f(x)$, which has compact support, $L_T\{f(x)\}(x')$ will have compact support if $b=0$.

Proof. From Eq. (1)

$$L_T\{f(x)\}(x') = \sqrt{d}e^{jcdx'^2/2}f(dx'). \qquad (5)$$

The proof is trivial, as the output is multiplied by a scaled copy of the input, which has compact support.

Corollary 2. Given an input signal, $f(x)$, that does not have compact support, $L_T\{f(x)\}(x')$ will not have compact support if $b=0$.

Proof. We note that Eq. (5) applies, but $f(x)$ is assumed *not* to have compact support.

Theorem 3. Given an input signal, $f(x)$, which is band-limited, $L_T\{f(x)\}(x')$ will be band-limited if $c=0$.

Proof. We note that $f(x)$ and the output of the optical system are related by an LCT with parameters $T=[a\ b;0\ a^{-1}]$. The additive property of the LCT gives the result that the FT of the input signal, $\Im\{f(x)\}(k)$, and of the output, $\Im\{L_T(x')\}(k')$, are related as follows:

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} a^{-1} & 0 \\ -b & a \end{pmatrix}. \qquad (6)$$

We note that this matrix has zero "b" parameter. By Theorem 2, compact support is preserved in this operation, i.e., if the $\Im\{f(x)\}(k)$ has compact support, $\Im\{L_T(x')\}(k')$ will have compact support. Equivalently, if $f(x)$ is band-limited, $L_T\{f(x)\}(x')$ will be band-limited. Similar proofs show the equivalence of Theorems 1–4.

Corollary 3. Given an input signal, $f(x)$, which is band-limited, $L_T\{f(x)\}(x')$ will not be band-limited if $c=0$.

Proof. We note that Eq. (6) holds and invoke Corollary 2.

Theorem 4. Given an input signal, $f(x)$, that has compact support, if $d=0$, $L_T\{f(x)\}(x')$ will be band-limited.

Proof. We neglect the complex constant from Eq. (1) for convenience. This does not alter the result. Given that $d=0$ and that $f(x)$ has compact support over the range $-\Gamma$ to Γ, the LCT of this signal is

$$L_T\{f(x)\}(x') = \int_{-\Gamma}^{\Gamma} f(x)e^{j\pi/b(ax^2-2xx')}\mathrm{d}x. \qquad (7)$$

Taking the FT of both sides and changing the order of integration gives

$$\Im[L_T\{f(x)\}(x')](k') = \int_{-\Gamma}^{\Gamma} f(x)e^{j\pi/b(ax^2)}\int_{-\infty}^{\infty} e^{j\pi/b(-2xx')}e^{jk'x'}\mathrm{d}x'\mathrm{d}x. \qquad (8)$$

By the sifting property of the Dirac delta function,

$$\Im[L_T\{f(x)\}(x')](k') = \int_{-\Gamma}^{\Gamma} f(x)e^{j\pi/b(ax^2)}\delta\left(k' - 2\pi\frac{x}{b}\right)\mathrm{d}x = f\left(\frac{bk'}{2\pi}\right)e^{jabk'^2/4\pi}. \qquad (9)$$

Here, as $f(x)$ has compact support, $\Im[L_T\{f(x)\}(x')](k')$ has compact support. Therefore $L_T\{f(x)\}(x')$ is band-limited.

Corollary 4. Given an input, $f(x)$, that does not have compact support, and given that $d=0$, $L_T\{f(x)\}(x')$ will not be band-limited.

Proof. We note that Eq. (9) applies. As $f(x)$ does not have compact support, $\Im[L_T\{f(x)\}(x')]$ does not have compact support. Therefore $L_T\{f(x)\}(x')$ is not band-limited.

Theorem 5. Given an input, $f(x)$, which either has compact support or is band-limited, and given that no transform parameter is zero, $L_T\{f(x)\}(x')$ is neither band-limited nor has compact support.

Proof. The LCT may be decomposed in many ways, including a chirp, a scaled FT, and another chirp [5], as follows:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ d/b & 1 \end{pmatrix}\begin{pmatrix} 0 & b \\ -1/b & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ a/b & 1 \end{pmatrix}. \qquad (10)$$

As every matrix in this decomposition has at least one zero element, we can track the properties of a signal as it propagates through each LCT. First, consider an input, $f(x)$, that has compact support. It is transformed by a chirp, which preserves the compact support according to Theorem 2. Next, the scaled FT transforms the compact support into band-limitedness according to Theorem 3. Finally, the band-limited signal is transformed by the other chirp. As the band-limited signal does not have compact support (from the FT compact support theorem), Corollary 2 shows that the output does not have compact support.

However, $\Im\{f(x)\}(k)$ is related to $L_T\{f(x)\}(x')$ by the transform with parameters

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} b & -a \\ d & -c \end{pmatrix}, \tag{11}$$

which is also a general LCT. As $f(x)$ has compact support, $\Im\{f(x)\}(k)$ is band-limited. Therefore, a band-limited signal passing through a general LCT with no zero elements in its ray transfer matrix cannot have compact support. Similarly, we can relate $f(x)$ and $\Im\{L_T(x')\}(k)$, which is not band-limited, by a general LCT with all nonzero parameters, and also $\Im\{f(x)\}(k)$, which is band-limited, with $\Im\{L_T(x')\}(k)$, which is not. Taken together, these results indicate that given any $f(x)$ which is band-limited or has compact support, $L_T\{f(x)\}(x')$ cannot be band-limited or have compact support.

We now briefly consider the implications of the above results for a number of transforms used in optics that are special cases of the LCT. The optical FT has parameters $T=[0\ \lambda f;-1/\lambda f\ 0]$ and becomes the FT if $\lambda f=1$. As $a=0$, the FT of a band-limited signal will have compact support. $d=0$ implies the FT of a signal with compact support will itself be band-limited. These are consistent with well-known properties of the FT. With both $a=d=0$, Theorems 1 and 4 may be combined to form a new proof of the FT compact support theorem.

The fractional Fourier transform (FRT) of order p has parameters $T=[\cos(p\pi/2)\ \sin(p\pi/2);-\sin(p\pi/2)\ \cos(p\pi/2)]$. Only integer values of p result in zero-valued transform parameters. If $p\equiv 0 \bmod 4$, we have a noninverting, unit magnification imaging system. In this case, $a=d=1$ and $b=c=0$. An input with compact support will result in an output with compact support, while a band-limited input will result in a band-limited output. These are consistent with the properties of imaging systems. If $p\equiv 2 \bmod 4$, we have an inverting imaging system, with the same zero-valued parameters and properties discussed for the noninverting imaging system. If $p\equiv 1 \bmod 4$, the FRT becomes the FT, which we have already discussed. Finally, if $p\equiv 3 \bmod 4$, the FRT becomes the inverse FT, with zero-valued parameters and properties as discussed for the FT.

The parameters of the FST for a distance z and wavelength λ are $T=[1\ \lambda z;0\ 1]$. As $c=0$, the FST of a band-limited signal will remain band-limited, independent of z. $b=0$ is either the trivial case of propagating by a distance $z=0$, or alternatively letting the wavelength $\lambda=0$ (geometrical optics limit), which ignores all diffractive effects. This is equivalent to the noninverting imaging system discussed for the FRT. For $b\neq 0$, $c=0$ implies that the Fresnel transform of any compactly supported signal will be unbounded in space and frequency.

Magnification systems have parameters $T=[M\ 0;0\ 1/M]$. With $b=c=0$, they behave in a similar manner to the imaging system previously discussed.

CMT (models a thin lens) has parameters $T=[1\ 0;-1/\lambda f\ 1]$. $b=0$, meaning compact support is preserved. $c=0$ when either wavelength or focal length is infinite. A band-limited signal undergoing a CMT will be unbounded in space and frequency.

In this paper, we have provided a mathematical basis for and an optical interpretation of the conditions under which the LCT of a signal has compact support or is band-limited. These properties of the LCT can be applied like demagnification to reduce an input to pass through a limiting aperture or to match a signal to the sampling rate or aperture size of the available camera, thus increasing the information content of the captured signal. The additional parameters grant system designers flexibility, and it may be possible to leverage this to preserve more information than a simple magnification system allows. We are currently examining experimental and numerical implications.

The authors thank Guohai Situ for his helpful comments and Brian Mulkeen for locating a reference. We acknowledge the support of Enterprise Ireland and Science Foundation Ireland. We also acknowledge the support of the Irish Research Council for Science, Engineering and Technology. J. J. Healy acknowledges the SPIE's support through an Educational Scholarship in Optical Science and Engineering.

References

1. H. Baher, *Analog & Digital Signal Processing* (Wiley, 1990), p. 121.
2. A. Papoulis, *The Fourier Integral and its Applications* (McGraw-Hill, 1962), p. 215.
3. R. E. A. C. Paley and N. Wiener, *Fourier Transforms in the Complex Domain* (American Mathematical Society, 1934), p. 12.
4. D. P. Kelly, B. M. Hennelly, W. T. Rhodes, and J. T. Sheridan, Opt. Eng. **45**, 088201 (2006).
5. B. M. Hennelly and J. T. Sheridan, J. Opt. Soc. Am. A **22**, 917 (2005).
6. M. J. Bastiaans and K. B. Wolf, in *Seventh International Symposium on Signal Processing and Its Applications* (IEEE, 2003), Vol. 1, pp. 589–592.
7. B. Barshan, M. Alper Kutay, and H. M. Ozaktas, Opt. Commun. **135**, 32 (1997).
8. S.-C. Pei and J.-J. Ding, J. Opt. Soc. Am. A **17**, 2355 (2000).
9. K. K. Sharma and S. D. Joshi, Opt. Commun. **265**, 454 (2006).
10. A. Stern, Signal Processing **86**, 1421 (2006).