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## Homotopes of JB\*-triples and a Russo-Dye theorem

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In this note, we look at homotopes of Jordan triple structures and show that, following a renorming, an isotope of a JB\*-triple is also a JB\*-triple. We also provide a proof of the Russo–Due theorem for JBW\*-triples.

### 1. Introduction and Definitions

Generally, a homotope of an algebraic structure refers to the same underlying vector space but where the algebraic structure (generally a binary product) has been deformed in some way. Homotopes have proved quite useful in several ways: for example, some property of the original algebra may be established by checking it for all homotopes. In Jordan theory it is common to introduce the notion of quasi-invertible pair by looking at inverses (or generalised inverses) in a homotope of the original algebra (see [6, 2]). Homotopes of Jordan algebras are formed by fixing one element of the Jordan *triple* product. The question we address here is that of how one should define a homotope of a Jordan triple.

#### Definition 1.1.

- (1) A JORDAN ALGEBRA  $V$  is a vector space with a (non-associative) bilinear product  $\circ : V^2 \rightarrow V$  satisfying the Jordan identity

$$x^2 \circ (x \circ y) = x \circ (x^2 \circ y)$$

and which is commutative:  $x \circ y = y \circ x$ .

- (2) A JORDAN TRIPLE  $V$  is a vector space over a field  $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$  with a triple product  $\{\cdot, \cdot, \cdot\} : V^3 \rightarrow V$  which obeys the Jordan triple identity

$$\{a, b, \{x, y, z\}\} = \{\{a, b, x\}, y, z\} - \{x, \{b, a, y\}, z\} + \{x, y, \{a, b, z\}\}$$

and is symmetric and  $\mathbb{K}$ -linear in the outer variables, and  $\overline{\mathbb{K}}$ -linear in the inner variable.

- (3) A JORDAN BANACH TRIPLE is a Jordan triple which is also a Banach space in which the triple product is jointly continuous.
- (4) A JB\*-TRIPLE is a Jordan Banach triple over  $\mathbb{C}$  in which  $x \square x$  satisfies
- (a)  $\sigma(x \square x) \geq 0$

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(b)  $\exp(ix \square x)$  is an algebraic automorphism and a surjective linear isometry

(c)  $\|x \square x(x)\| = \|x\|^3$ .

(Throughout,  $x \square y$  denotes the linear map  $z \mapsto \{x, y, z\}$ .)

Every Jordan algebra is a Jordan triple via

$$\{x, y, z\} = (x \circ y) \circ z - (x \circ z) \circ y + (y \circ z) \circ x$$

and a complex Jordan algebra with involution is a complex Jordan triple via

$$\{x, y, z\} = (x \circ y^*) \circ z - (x \circ z) \circ y^* + (y^* \circ z) \circ x. \quad (1.1)$$

An element  $e$  is called a tripotent if  $\{e, e, e\} = e$ . Further, it is a UNITARY TRIPOTENT if  $\{e, e, x\} = x$  for all  $x$ . A unitary tripotent is the closest one comes in a triple setting to the concept of an identity.

## 2. Homotopes

Suppose  $A$  is a Jordan algebra over an arbitrary field and  $u \in A$ . Define a new structure  $A^u$  to be the same linear space with the product

$$x \circ_u y = \{x, u, y\}$$

where  $\{a, b, c\} = (a \circ b) \circ c + (b \circ c) \circ a - (a \circ c) \circ b$  is the associated Jordan triple product.  $A^u$  is called the  $u$ -homotope of  $A$  and is also a Jordan algebra. The Jordan algebra homotope  $A^u$  is called an ISOTOPE if  $u$  is invertible in the Jordan algebraic sense. Such isotopes of Jordan algebras satisfy properties of reflexivity, symmetry and transitivity (see for example [7]):

$$A^1 = A, \quad (2.1)$$

$$A = (A^u)^{u^{-2}}, \quad (2.2)$$

$$(A^u)^v = A^{Q_u v}. \quad (2.3)$$

The algebra homotope product  $\circ_u$  can be used to define a new triple product on  $A$ , as per (1.1), via

$$\begin{aligned} \{x, y, x\}_{(u)} &= 2(x \circ_u y) \circ_u x - (x \circ_u x) \circ_u y \\ &= 2\{\{x, u, y\}, u, x\} - \{\{x, u, x\}, u, y\} \\ &= \{x, \{u, y, u\}, x\}. \end{aligned}$$

The final equation here used the standard algebraic identities JP2 and JP9 [6].

Following this, we can take any Jordan triple  $E$  and let  $E^{(u)}$  be  $E$  with the product

$$\{x, y, x\}_{(u)} = \{x, \{u, y, u\}, x\}. \quad (2.4)$$

which also satisfies the Jordan triple identity. To reiterate, if  $E$  is a Jordan triple and  $u \in E$  then  $E^{(u)} := (E, \{\cdot, \cdot, \cdot\}_{(u)})$  is also a Jordan triple which we call the  $u$ -homotope of  $E$ .

### 2.1. Transitivity property for $E^{(u)}$

Denote the product in  $(E^{(u)})^{(v)}$  by  $\{\cdot, \cdot, \cdot\}_{(u,v)}$ . We see that  $\{x, y, x\}_{(u,v)} = Q_x^{(u)} Q_v^{(u)} y$  where  $Q_x^{(u)}$  denotes the operator  $y \mapsto \{x, y, x\}_{(u)}$ . Thus  $Q_x^{(u)} = Q_x Q_u$  and we have

$$\begin{aligned} \{x, y, x\}_{(u,v)} &= Q_x^{(u)} Q_v^{(u)} y \\ &= Q_x Q_u Q_v Q_u y \\ &= Q_x Q_{Q_u v} y \\ &= \{x, y, x\}_{(Q_u v)} \end{aligned}$$

and so

$$E^{(u,v)} := (E^{(u)})^{(v)} = E^{(Q_u v)} \quad (2.5)$$

which is consistent with the situation for algebras (2.3).

In dealing with complex Jordan triples (we are thinking in particular of JB\*-triples) it is preferable to have a homotope product which is  $\mathbb{C}$ -linear in the inner variable. Consequently,  $\{x, y, x\}_u = \{x, \{u, y, u\}, x\}$  is inconsistent and we will instead use the homotope  $E^{[u]}$  with product

$$\{x, y, x\}_{[u]} := Q_x Q_u^2 y.$$

The reader may verify that the Jordan triple identity also holds for this triple product. Notice that  $E^{[u]} = E$  if  $Q_u^2 = I$  which corresponds to  $u$  being a unitary tripotent, thus providing an analogue of (2.1). We would similarly like an analogue of the transitivity property (2.3) and (2.5).

**Lemma 2.1.**  $(E^{[u]})^{[v]} = E^{[Q_u^2 v]}$  if  $u$  is a tripotent in the JB\*-triple  $E$ .

**Proof.**

$$\begin{aligned} \{x, y, x\}_{[u],[v]} &= Q_x^{[u]} (Q_v^{[u]})^2 y \\ &= Q_x Q_u^2 (Q_v Q_u^2)^2 y \\ &= Q_x Q_u^2 Q_v Q_u^2 Q_v Q_u^2 y \end{aligned}$$

while

$$\begin{aligned} \{x, y, x\}_{[Q_u^2 v]} &= Q_x Q_{Q_u^2 v}^2 y \\ &= Q_x Q_{Q_u^2 v} Q_{Q_u^2 v} y \\ &= Q_x Q_u Q_u Q_v Q_u Q_u Q_v Q_u Q_u^2 y \\ &= Q_x Q_u^2 Q_v Q_u^4 Q_v Q_u^2 y \end{aligned}$$

These are equal if  $Q_u^2 = Q_u^4$  which occurs if and only if  $u$  is a tripotent.  $\square$

### 3. Structure Maps

Throughout  $V$  is a JB\*-triple, though much of what we say will hold for more general Jordan Banach triples.  $\mathcal{L}(E)$  denotes the bounded linear operators on a complex Banach space while  $\overline{\mathcal{L}}(E)$  consists of the bounded conjugate-linear maps.

**Definition 3.1.** Let  $(V, \{\cdot, \cdot, \cdot\})$  be a JB\*-triple. We call  $(S, T) \in \mathcal{L}(V)^2 \cup \overline{\mathcal{L}}(V)^2$

- (a) a LEFT STRUCTURE MAP if  $\{Sx, y, Sx\} = S\{x, Ty, x\}$
- (b) a RIGHT STRUCTURE MAP if  $\{Tx, y, Tx\} = T\{x, Sy, x\}$
- (c) a STRUCTURE MAP if it is a left- and right structure map.

and we denote the set of structure maps by  $\mathcal{S}(V)$ .

Clearly  $(S, T)$  is a left structure map if and only if  $(T, S)$  is a right structure map and so  $(S, T) \in \mathcal{S}(V)$  if and only if  $(T, S) \in \mathcal{S}(V)$ . For  $\alpha = (S, T) \in \mathcal{S}(V)$ , we let  $\bar{\alpha} = (T, S)$  and call this the ADJOINT structure map of  $(S, T)$ . Self-adjoint structure maps are thus those of the type  $(T, T)$ . Notice that, in the definition of structure map, conditions (a) and (b) are not equivalent, for example  $(0, T)$  satisfies (a) for any  $T \in \mathcal{L}(V)$  but not (b). Also, given  $S$ , the fact that  $(S, T) \in \mathcal{S}(V)$  may not uniquely determine  $T$ . If however, either  $S$  or  $T$  is invertible then uniqueness of the paired map does hold.

**Lemma 3.1.** *Let  $V$  be a JB\*-triple. If  $(S, T_1)$  and  $(S, T_2)$  are elements in  $\mathcal{S}(V)$  and  $S$  is 1-1 then  $T_1 = T_2$ .*

**Proof.**  $S\{x, T_1y, x\} = S\{x, T_2y, x\}$  implies  $\{x, T_1y, x\} = \{x, T_2y, x\}$  and so  $\{x, (T_1 - T_2)y, x\} = 0$  for all  $x, y \in V$ . Using  $\|\{a, a, a\}\| = \|a\|^3$  then implies that  $(T_1 - T_2)y = 0$  for all  $y$  and so  $T_1 = T_2$ .  $\square$

It is easy to check that  $\mathcal{S}(V)$  has a monoid structure: if  $(S_1, T_1) \in \mathcal{S}(V)$  and  $(S_2, T_2) \in \mathcal{S}(V)$  then  $(S_1S_2, T_2T_1) \in \mathcal{S}(V)$  with  $(I, I)$  being the identity element.

**Definition 3.2.** If  $(S, T) \in \mathcal{S}(V)$  and both  $S$  and  $T$  are invertible then we call the pair  $(S, T)$  an INVERTIBLE STRUCTURE MAP. The set of invertible structure maps will be denoted  $\text{SG}(V)$ .

It is easy to check that if  $(S, T) \in \text{SG}(V)$  then so too is  $(S^{-1}, T^{-1})$  and it follows that in fact  $\text{SG}(V)$  is a group (which we call the STRUCTURE GROUP [5]) with identity  $(I, I)$  and  $(S, T)^{-1} = (S^{-1}, T^{-1})$ .

#### 3.1. Examples of Structure Maps

- (i) Suppose  $T \in \mathcal{L}(V)$  is a triple homomorphism, that is  $T\{x, x, x\} = \{Tx, Tx, Tx\}$  for all  $x$ . If  $T$  has a right inverse ( $TT^r = I$ ) then  $(T, T^r)$  is a left structure map (and hence  $(T^r, T)$  is a right structure map). If  $T$  is invertible then  $(T, T^{-1})$  is an invertible structure map.

(ii) Let  $u \in V$ . Then  $(S, T) = (Q_u, Q_u) \in \mathcal{S}(V)$  although  $Q_u$  is not linear. Indeed

$$\begin{aligned} \{Sx, y, Sx\} &= \{Q_u x, y, Q_u x\} \\ &= Q_{Q_u x} y \\ &= Q_u Q_x Q_u y \\ &= T\{x, Sy, x\} \end{aligned}$$

so (a) holds. Then (b) holds on swapping the pair of operators. For linearity, we take  $(Q_u^2, Q_u^2)$  to get a structure map. This is invertible if and only if  $u$  is a unitary tripotent.

(iii) By (JP26), the Bergmann operator pair  $(B(u, v), B(v, u))$  forms a structure map. It is an invertible structure map if  $(u, v)$  is a quasi-invertible pair.

(iv) Let  $P$  be a structural projection, that is, a real linear projection  $R$  on  $V$  such that  $\{Rx, y, Rx\} = R\{x, Ry, x\}$  for all  $x$  and  $y$ . Then  $(R, R)$  satisfies the properties of a structure map except possibly for linearity.

### 3.2. Homotopes again

Suppose  $\alpha = (S, T)$  is a structure map. Consider the ‘‘triple product’’ on  $V$  given by

$$\{x, y, z\}^\alpha := \{x, Ty, z\}. \quad (3.1)$$

Clearly this is linear in  $x$  and  $z$  and, if  $T$  is linear, anti-linear in  $y$ . Does it satisfy the Jordan triple identity? Well, consider the box operator  $x \square^\alpha x$  with respect to the  $\alpha$ -product.

$$\begin{aligned} z \square^\alpha z \{x, y, x\}^\alpha &= \{z, Tz, \{x, Ty, x\}\} \\ &= \{\{z, Tz, x\}, Ty, x\} - \{x, \{Tz, z, Ty\}, x\} + \{x, Ty, \{z, Tz, x\}\} \\ &= \{\{z, z, x\}^\alpha, y, x\}^\alpha - \{x, \{z, Sz, y\}, x\}^\alpha + \{x, y, \{z, z, x\}^\alpha\}^\alpha \end{aligned}$$

In particular, if  $S = T$  then this becomes

$$= \{\{z, z, x\}^\alpha, y, x\}^\alpha - \{x, \{z, z, y\}^\alpha, x\}^\alpha + \{x, y, \{z, z, x\}^\alpha\}^\alpha$$

and thus  $iz \square^\alpha z$  is a derivation on  $\{\cdot, \cdot, \cdot\}^\alpha$  and consequently the Jordan triple identity holds. But, in general, if  $(S, T)$  is a structure map then (3.1) does not define a Jordan triple product. This might then provide sufficient motivation to restrict attention to linear<sup>a</sup> self-adjoint structure maps (see [1]). However, there are advantages in not doing so and instead making the following definition.

#### Definition 3.3.

(i) Let  $\alpha = (S, T)$  be a structure map. We define a new triple product  $\{\cdot, \cdot, \cdot\}_\alpha$  by

$$\{x, y, x\}_\alpha = \{x, STy, x\}.$$

<sup>a</sup>That is, both  $S$  and  $T$  are linear.

We note that since  $(S, T)$  (and consequently  $(T, S)$ ) are structure maps, the monoid structure guarantees that  $(ST, ST)$  is a linear self-adjoint structure map. The above discussion then shows that  $\{\cdot, \cdot, \cdot\}_\alpha$  satisfies the Jordan triple identity.

- (ii) If  $\alpha = (S, T)$  is a structure map then  $V$  endowed with the triple product given in (i) is called the  $\alpha$ -HOMOTOPE of  $V$ , denoted  $V^\alpha$ .
- (iii) If  $\alpha$  is an invertible structure map then we say that the homotope  $V^\alpha$  is an ISOTOPE of  $V$ .

**Lemma 3.2.** *If  $V^\alpha$  is a homotope of  $V$  then there is a triple product preserving map<sup>b</sup> from  $(V^\alpha, \{\cdot, \cdot, \cdot\}^\alpha)$  to  $(V, \{\cdot, \cdot, \cdot\})$ .*

**Proof.** Since  $\alpha = (S, T) \in \mathcal{S}(V)$  we have that

$$\begin{aligned} T\{x, y, x\}_\alpha &= T\{x, STy, x\} \\ &= \{Tx, Ty, Tx\} \end{aligned}$$

as required. □

If  $V^\alpha$  is an isotope of  $V$  then so is  $V^{\bar{\alpha}}$  and also  $V$  is an isotope of  $V^\alpha$ . Thus  $V^\alpha$  and  $V^{\bar{\alpha}}$  are Jordan triple isomorphic. However, it is not immediately clear that  $V^\alpha$  is an isotope of  $V^{\bar{\alpha}}$ . This raises the question of symmetry and transitivity of homotopes, to which we will return.

**Note.**

- (1) For  $\alpha = (Q_u, Q_u)$ , the homotope  $E^\alpha$  agrees with  $E^{[u]}$  given in Section 2.
- (2) Denoting the quadratic and box operators in the  $\alpha$ -homotope by  $Q_x^\alpha$  and  $x \square^\alpha y$  respectively, it is clear that

$$Q_x^\alpha = Q_x \circ ST, \quad x \square^\alpha y = x \square STy.$$

From this, one sees the  $\alpha$ -Bergmann operator  $B^\alpha(x, y) = I - 2x \square STy + Q_x STQ_y ST$ . The structure properties guarantee that  $STQ_y ST = Q_{STy}$  and so  $B^\alpha(x, y) = B(x, STy)$ .

- (3) We will say the structure map  $\alpha = (S, T)$  is *commuting* if  $ST = TS$ . Clearly if  $\alpha$  is commuting then  $V^\alpha = V^{\bar{\alpha}}$ . Conversely, if  $V^\alpha = V^{\bar{\alpha}}$  then  $\{x, STy, x\} = \{x, TSy, x\}$  for all  $x$  and  $y$  leads to  $((TS - ST)y)^{(3)} = 0$  for all  $y$  and consequently  $TS = ST$ . That is,  $V^\alpha = V^{\bar{\alpha}}$  if and only if  $\alpha$  is a commuting structure map.

<sup>b</sup>It is only the possible lack of linearity that prevents us using the term ‘‘Jordan triple homomorphism’’

- (4) Suppose  $\alpha = (S, T)$  is invertible. Then of course  $\alpha^{-1}$  is a structure map on  $V$ , but also we have

$$\begin{aligned} S^{-1}\{x, T^{-1}y, x\}_\alpha &= S^{-1}\{x, Sy, x\} \\ &= S^{-1}\{x, T^{-1}TSy, x\} \\ &= \{S^{-1}x, TSy, S^{-1}x\} \\ &= \{S^{-1}x, y, S^{-1}x\}_{\bar{\alpha}} \end{aligned}$$

and similarly with  $S$  and  $T$  reversed. In particular, if  $\alpha$  is a commuting structure map then it follows that  $\alpha^{-1}$  is a structure map on  $V^\alpha$  (as well as on  $V$ ) and so we may talk about the structure  $(V^\alpha)^{\alpha^{-1}}$ . There isn't much to say about this structure other than  $(V^\alpha)^{\alpha^{-1}} = V$ .

This last remark leads us into the area of isotope reflexivity, transitivity and symmetry. We emphasize one point suggested above by stating the following, which is easily checked from the definitions.

**Lemma 3.3.** *Suppose  $\alpha \in \text{SG}(V)$  and  $\beta \in \text{SG}(V)$ . Then  $\beta \in \text{SG}(V^\alpha)$  if  $S_\alpha T_\alpha$  commutes with  $T_\beta$  and  $S_\beta$ .*

**Corollary 3.1.** *If  $\alpha = (S, T) \in \text{SG}(V)$  then  $\alpha^{-1} \in \text{SG}(V^\alpha)$  if  $ST = TS$ .*

#### 4. Renorming

It is well known that, analogous to the situation for  $C^*$ -algebras, the norm and the triple product on a  $\text{JB}^*$ -triple uniquely determine each other. Thus, if the triple product is changed, for example by passing to a homotope, then one cannot expect the resulting structure to be a  $\text{JB}^*$ -triple without also altering the norm.

**Definition 4.1.** For  $\alpha = (S, T) \in \mathcal{S}(V)$  we define  $\|x\|_\alpha := \|Tx\|$ .

Of course,  $\|\cdot\|_\alpha$  is a semi-norm on  $V$  and a norm if  $\alpha \in \text{SG}(V)$  or indeed if  $T$  is injective.

**Lemma 4.1.** *Suppose  $\alpha = (S, T) \in \mathcal{S}(V)$ . Then  $\|\{x, x, x\}_\alpha\|_\alpha = \|x\|_\alpha^3$ .*

**Proof.**

$$\begin{aligned} \|\{x, x, x\}_\alpha\|_\alpha &= \|\{x, STx, x\}\|_\alpha \\ &= \|T\{x, STx, x\}\| \\ &= \|\{Tx, Tx, Tx\}\| \\ &= \|Tx\|^3 \\ &= \|x\|_\alpha^3 \quad \square \end{aligned}$$

This would suggest that an isotope  $V^\alpha$  may itself form a  $\text{JB}^*$ -triple under this norm. To verify, we have to determine the spectrum of  $x \square_\alpha x$  and show that

$\exp(ix \square_{\alpha} x)$  is a surjective linear isometry. (The requirement that  $\exp(ix \square_{\alpha} x)$  is an algebraic automorphism follows automatically from the Jordan triple identity.)

**Lemma 4.2.** *Suppose  $\alpha = (S, T) \in \mathcal{S}(V)$ . Then  $\exp(ix \square_{\alpha} x)$  is a surjective linear isometry on the homotope  $V^{\alpha}$ .*

**Proof.** First notice that  $x \square_{\alpha} x = x \square STx$  and so  $T(x \square_{\alpha} x)y = T\{x, STx, y\} = \{Tx, Tx, Ty\}$ . In other words,  $T \circ (x \square_{\alpha} x) = (Tx \square Tx) \circ T$ . Iterating this formula shows that  $T((ix \square_{\alpha} x)^k y) = (iTx \square Tx)^k(Ty)$  and following series expansion we obtain

$$\begin{aligned} \|\exp(ix \square_{\alpha} x)(y)\|_{\alpha} &= \|T \exp(ix \square_{\alpha} x)(y)\| \\ &= \|\exp(iTx \square Tx)(Ty)\| \\ &= \|Ty\| \end{aligned}$$

(since  $Tx \square Tx$  is Hermitian on  $V$ )

$$= \|y\|_{\alpha}$$

which proves the claim.  $\square$

**Lemma 4.3.** *Suppose  $\alpha = (S, T) \in \mathcal{S}\mathcal{G}(V)$ . Then  $x \square_{\alpha} x$  has positive spectrum.*

**Proof.** We already seen above that  $x \square_{\alpha} x = x \square STx = T^{-1}(Tx \square Tx)T$ . Thus  $\sigma(x \square_{\alpha} x) = \sigma(Tx \square Tx)$  which is positive since  $V$  is a  $\text{JB}^*$ -triple.  $\square$

**Corollary 4.1.** *Let  $V$  be a  $\text{JB}^*$ -triple and  $\alpha = (S, T) \in \mathcal{S}\mathcal{G}(V)$ . Then the isotope  $V^{\alpha}$  is also a  $\text{JB}^*$ -triple.*

#### 4.1. Transitivity of Homotopes

Recall the neat transitivity formula (2.3) for homotopes of Jordan algebras:  $(A^u)^v = A^{Q_u v}$ . Lemma 2.1, in particular its rather unnatural tripotent requirement, suggests that an analogous transitivity formula for triple homotopes defined in terms of structure maps may not be so easily expressed. Let us make a notational definition.

**Definition 4.2.** Let  $V$  be a  $\text{JB}^*$ -triple with  $\alpha = (S_1, T_1) \in \mathcal{S}\mathcal{G}(V)$  and  $\beta = (S_2, T_2) \in \mathcal{S}(V^{\alpha})$ . Then

$$\begin{aligned} \alpha \diamond \beta &= (S_1, T_1) \diamond (S_2, T_2) \\ &:= (S_1 T_1 S_2 T_1^{-1}, T_1 T_2). \end{aligned}$$

**Lemma 4.4.** *Suppose  $\alpha = (S_1, T_1) \in \mathcal{S}\mathcal{G}(V)$  and  $\beta = (S_2, T_2) \in \mathcal{S}(V^{\alpha})$ . Then  $\alpha \diamond \beta \in \mathcal{S}(V)$  and  $(V^{\alpha})^{\beta} = V^{\alpha \diamond \beta}$ . In particular,*

$$\{x, y, x\}_{\alpha, \beta} = \{x, y, x\}_{\alpha \diamond \beta} = \{x, S_1 T_1 S_2 T_2 y, x\}. \quad (4.1)$$

**Proof.** It is clear that  $\{x, y, x\}_{\alpha, \beta} = \{x, S_2 T_2 y, x\}_{\alpha} = \{x, S_1 T_1 S_2 T_2 y, x\}$  and so one must only check that  $\alpha \diamond \beta \in \mathcal{S}(V)$ . Firstly,  $\alpha \in \text{SG}(V)$  and  $\beta \in \text{SG}(V^\alpha)$  can be expressed thus:

$$\{S_1 x, y, S_1 x\} = S_1 \{x, T_1 y, x\} \quad (4.2)$$

$$\{T_1 x, y, T_1 x\} = T_1 \{x, S_1 y, x\} \quad (4.3)$$

$$\{S_2 x, S_1 T_1 y, S_2 x\} = S_2 \{x, S_1 T_1 T_2 y, x\} \quad (4.4)$$

$$\{T_2 x, S_1 T_1 y, T_2 x\} = T_2 \{x, S_1 T_1 S_2 y, x\}. \quad (4.5)$$

It is then easy to see that

$$\begin{aligned} \{S_1 T_1 S_2 T_1^{-1} x, y, S_1 T_1 S_2 T_1^{-1} x\} &\stackrel{(4.2)}{=} S_1 \{T_1 S_2 T_1^{-1} x, T_1 y, T_1 S_2 T_1^{-1} x\} \\ &\stackrel{(4.3)}{=} S_1 T_1 \{S_2 T_1^{-1} x, S_1 T_1 y, S_2 T_1^{-1} x\} \\ &\stackrel{(4.4)}{=} S_1 T_1 S_2 \{T_1^{-1} x, S_1 T_1 T_2 y, T_1^{-1} x\} \\ &\stackrel{(4.2)}{=} S_1 T_1 S_2 T_1^{-1} \{x, T_1 T_2 y, x\} \end{aligned}$$

so  $\alpha \diamond \beta$  is a left structure map. A similar verification shows that it is also a right structure map.  $\square$

Remark that for  $\alpha \in \mathcal{S}(V)$  and  $\beta \in \mathcal{S}(V^\alpha)$  the composition of homotopes  $(V^\alpha)^\beta$

$$\{x, y, x\}_{\alpha, \beta} = \{x, S_1 T_1 S_2 T_2 y, x\}$$

is indeed a Jordan triple, that is, the triple product satisfies the Jordan triple identity. The requirement above that  $\alpha$  be invertible serves only to guarantee that the resulting Jordan triple  $(V^\alpha)^\beta$  is a homotope of  $V$ .

## 5. The Russo-Dye theorem

This section is essentially independent of those previous. Recall the following classical theorem of functional analysis [8].

**Theorem. 5.1 (Russo–Dye).** In a unital  $C^*$ -algebra the unit ball is the closed convex hull of the unitary elements.

We begin with another look at polar decomposition in a  $C^*$ -algebra.

**Lemma 5.1.** *Let  $a$  be an invertible element in the von Neumann algebra  $A$ . Then  $a = vpv$  for some positive  $p$  and unitary  $v$ .*

**Proof.** It is well known that an invertible  $a$  in a  $C^*$ -algebra can be uniquely written as  $a = u_1 p_1$  and also as  $a = p_2 u_2$  where  $u_i$  is unitary and  $p_i$  is positive (polar decomposition). Indeed,  $p_1 = (a^* a)^{\frac{1}{2}}$ ,  $u_1 = a(a^* a)^{-\frac{1}{2}}$ ,  $p_2 = (a a^*)^{\frac{1}{2}}$  and  $u_2 = (a a^*)^{-\frac{1}{2}} a$ .

Remark that in a von Neumann algebra every unitary element is a square and, indeed, an exponential of a purely imaginary element [4] and so it follows that

every unitary has a square root which is a unitary. Thus we may consider the unitary  $v = u^{\frac{1}{2}}$ . It is clear that  $a = vpv$  where  $p = vp_1v^*$  is positive.  $\square$

**Lemma 5.2 (Polar Decomposition).** *Let  $a$  be invertible in a  $JBW^*$ -algebra. Then  $a = Q_u(p)$  where  $u$  is unitary and  $p$  is positive.*

**Proof.** Consider the (weak\*) closed Jordan \*-algebra generated by  $a$ ,  $a^*$  and 1 which is a von Neumann algebra. Then by Lemma 5.1,  $a$  has the desired form.  $\square$

**Theorem 5.1.** *In a  $JBW^*$ -algebra, the unit ball is the closed convex hull of the unitary elements.*

**Proof.** We follow the elegant proof of the  $C^*$ -algebra version as given in [3]. It suffices to show that any element  $x$  in the open unit ball of a  $JBW^*$ -algebra  $A$  lies in the closed convex hull of the unitary elements, i.e.  $x \in \overline{\text{co}}(U)$ . We claim first that if  $u$  is any unitary element then  $\frac{1}{2}(x + u) \in \overline{\text{co}}(U)$ .

Indeed, since  $\|x\| < 1$ ,  $u + x$  is invertible and  $y = (u + x)/2$  is also in the unit ball. By polar decomposition, we can then write  $y = Q_v(p)$  where  $v$  is unitary and  $p$  is positive of norm  $\leq 1$ . But  $p = \frac{1}{2}(w + w^*)$  where  $w = p + i(1 - p^2)^{\frac{1}{2}}$  is unitary. Consequently,  $y$  is also in the convex hull of the unitary elements (since  $a, b$  unitary imply  $Q_a(b)$  is unitary).

Now, given  $x$  and choosing any  $x_0 \in U$ , it is clear that  $x_1 = (x + x_0)/2$  is contained in  $\overline{\text{co}}(U)$  by the above claim. Indeed the sequence of elements generated by  $x_{n+1} = (x_n + x)/2$  is (a) contained in  $\overline{\text{co}}(U)$ , and (b) convergent to  $x$  as required.  $\square$

**Corollary 5.1.** *In a  $JBW^*$ -triple with a unitary tripotent, the closed unit ball is the closed convex hull of the unitary tripotents.*

**Proof.** Let  $e$  be a unitary tripotent in the  $JBW^*$ -triple  $E$ . Then  $E^e := (E, \circ_e)$  is a unital  $JBW^*$ -algebra with respect to  $x \circ_e y := \{x, e, y\}$ ,  $x^* = \{e, x, e\}$ , and so the previous result implies that the closed unit ball of  $E_e$  is the closed convex hull of the unitary elements of  $E_e$ . Thus it merely suffices to show that a unitary element  $f$  of  $E^e$  is in fact a unitary tripotent of  $E$ .

Suppose then that  $f$  is a unitary in  $E^e$ , that is  $f$  is invertible with  $f^{-1} = f^*$ . Invertibility of  $f$  in the  $JB^*$ -algebra  $E_e$  means  $Q_f^{(e)}$  is an invertible operator on  $E^e$  and we have  $(Q_f^{(e)})^{-1}f = f^* = Q_e f$ . Here  $Q_f^{(e)}$  denotes the quadratic operator generated by  $f$  in the  $JB^*$ -algebra  $E^e$ . That is,  $Q_f^{(e)} = 2(L_f^{(e)})^2 - L_{f^2}^{(e)}$  where  $L_f^{(e)}(y) = f \circ_e y = \{f, e, y\}$ .

Notice that  $Q_f^{(e)} = Q_f Q_e$  (see Section 2) and so  $f = Q_f^{(e)} f^* = Q_f Q_e(Q_e f) = Q_f f$  implying that  $f$  is a tripotent. Also  $Q_f^{(e)} = Q_f Q_e$  invertible implies that  $Q_f$  and the Peirce 1-projection  $P_f^1 = Q_f^2$  is invertible. Thus the Peirce 0- and  $\frac{1}{2}$ -spaces must be zero. Hence  $P_f^1$  is the identity and  $f$  is a unitary tripotent as asserted.  $\square$

## References

1. BERTRAM, W. Homotopes and conformal deformations of symmetric spaces. *J. Lie Theory* 18, 2 (2008), 301–333.
2. DINEEN, S., MACKEY, M., AND MELLON, P. The density property for JB\*-triples. *Studia Math.* 137, 4 (1999), 143–160.
3. GARDNER, L. T. An elementary proof of the Russo-Dye theorem. *Proceedings of the American Mathematical Society* 90, 1 (January 1984), 171.
4. KADISON, R. V., AND RINGROSE, J. R. *Fundamentals of the theory of operator algebras. Vol. I*, vol. 100 of *Pure and Applied Mathematics*. Academic Press Inc. [Harcourt Brace Jovanovich Publishers], New York, 1983. Elementary theory.
5. KAUP, W. A Riemann mapping theorem for bounded symmetric domains in complex Banach spaces. *Math. Z.* 138 (1983), 503–529.
6. LOOS, O. *Bounded symmetric domains and Jordan pairs*. University of California at Irvine, Lecture Notes, 1977.
7. MCCRIMMON, K. *A Taste of Jordan Algebras*. UTX. Springer, 2004.
8. RUSSO, B., AND DYE, H. A. A note on unitary operators in  $C^*$ -algebras. *Duke Math. J.* 33 (1966), 413–416.