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Stability and robustness of singular systems of fractional nabla difference equations

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Abstract. In this article we study the stability and robustness of a class of singular linear systems of fractional nabla difference equations whose coefficients are constant matrices. Firstly, by assuming that the singular fractional system has a unique solution for given initial conditions, we study the asymptotic stability of the equilibria of the homogeneous system. We also prove conditions on the input vector under which the solution of the non-homogeneous system converges. Next, since it is known that existence and uniqueness of solutions depend on the invariants of the pencil of the system, by taking into consideration the fact that small perturbations can change the invariants, we perturb the singular fractional system and obtain bounds on the perturbation effect of the invariants of the pencil. In addition, by using this result, we study the robustness of solutions of the system. Finally, we give numerical examples based on a real singular fractional nabla dynamical system to illustrate our theory.

Keywords: singular systems, fractional nabla operator, discrete time system, robustness, stability.

1 Introduction

Difference equations of fractional order have recently proven to be valuable tools in the modeling of many phenomena in various fields of science and engineering. Indeed, we can find numerous applications in viscoelasticity, electrochemistry, physics, control, porous media, electromagnetism and so forth, see [4], [5], [18], [19], [22], [24], [31], [35], [36], [39], [44]. Atici et al. [4], proved that fractional difference equations are used to provide models for tumor growth within fractional discrete Gompertz equation. Lee investigated in [31] the possibility that time can be regarded as a discrete dynamical variable through all phases of mechanics. At this point it is strongly believed that the fractional discrete operators can have important contribution in generalizing this idea to classical mechanics, non-relativistic quantum mechanics and relativistic quantum field theories.

The theory of discrete fractional equations is also a promising tool for several biological and physical applications where the memory effect appears. The dynamics of the complex systems are better described within this new powerful tool. The nanotechnology and its applications in biology for example as well as the discrete gravity are fields where the fractional discrete models will play an important role in the future, see [5].

There has been a significant development in the study of fractional differential/difference equations and inclusions in recent years; see the monographs of Baleanu et al. [5], Dzielinski et al. [19], Malinowska et al. [36] and the survey by Podlubny [39]. The stability of linear problems using fractional operators has been studied in [8], [9], [16], [29], [33], [37], [41],[45], [46]. For some recent contributions on fractional differential/difference equations, see [1], [2], [3], [7], [11], [12], [13], [14], [15], [17], [18], [20], [23], [25], [26], [27], ,[30], [32], [34], [38], [40], [42] and the references therein.

If we define $\mathbb{N}_\alpha$ by $\mathbb{N}_\alpha = \{\alpha, \alpha+1, \alpha+2, ...\}$, α integer, and n fractional then the nabla fractional operator of n-th order for any $Y_k : \mathbb{N}_a \to \mathbb{C}^m$ is defined by

$$\nabla_\alpha^{-n} Y_k = \frac{1}{\Gamma(n)} \sum_{j=\alpha}^{k} (k-j+1)^{\overline{n-1}} Y_j,$$

where $k^{\bar{n}} = \frac{\Gamma(k+n)}{\Gamma(k)}$. We consider the singular fractional discrete time system of the form

$$F\nabla_0^n Y_k = GY_k + V_k, \quad k = 1, 2, ... \ , \tag{1}$$

with known initial conditions

$$lim_{k\mapsto 0}\nabla_0^{-(1-n)}Y_k = Y_0. \tag{2}$$

Where $F, G \in \mathbb{C}^{r\times m}$ and $V_k, Y_k \in \mathbb{C}^m$. The matrices F, G can be non-square ($r \neq m$) or square ($r = m$) with F singular (det$F$=0).

A family of matrices $sF - G$, parametrized by a complex number s, is called matrix pencil, [10], [21], [28]. Given $F, G \in \mathbb{C}^{r\times m}$ and an arbitrary $s \in \mathbb{C}$, the matrix pencil $sF - G$ is called regular when $r = m$ and $\det(sF - G) \neq 0$ and singular when $r \neq m$ or $r = m$ and $\det(sF - G) \equiv 0$.

In this article we consider the system (1) with a *regular pencil*. Then the class of $sF-G$ is characterized by a uniquely defined element, known as the complex Weierstrass canonical form, see [10], [21], [28], specified by the complete set of invariants of $sF - G$. This is the set of elementary divisors of type $(s-a_j)^{p_j}$, called *finite elementary divisors*, where a_j is a finite eigenvalue of algebraic multiplicity p_j ($1 \leq j \leq \nu$), and the set of elementary divisors of type $\hat{s}^q = \frac{1}{s^q}$, called *infinite elementary divisors*, where q is the algebraic multiplicity of the infinite eigenvalue. Where $\sum_{j=1}^{\nu} p_j = p$ and $p+q = m$. The direct sum denoted by $B_{n_1} \oplus B_{n_2} \oplus \cdots \oplus B_{n_r}$ is the block diagonal matrix *blockdiag* $\begin{bmatrix} B_{n_1} & B_{n_1} & \dots & B_{n_r} \end{bmatrix}$. Where $B_{n_1} \in \mathbb{C}^{n_1\times n_1}$,$B_{n_2} \in \mathbb{C}^{n_2\times n_2}$, $\dots$, $B_{n_r} \in \mathbb{C}^{n_r\times n_r}$. From the regularity of $sF - G$, there exist non-singular matrices P, $Q \in \mathbb{C}^{m\times m}$ such that

$$\begin{aligned} PFQ &= I_p \oplus H_q, \\ PGQ &= J_p \oplus I_q. \end{aligned} \tag{3}$$

Where J_p, H_q appropriate matrices with H_q a nilpotent matrix with index q_*, i.e. $H_q^{q_*} = 0_{q,q}$, J_p a Jordan matrix and $p + q = m$. The matrices P, Q can be written as

$$P = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix}, \quad Q = \begin{bmatrix} Q_p & Q_q \end{bmatrix}. \tag{4}$$

Where with $P_1 \in \mathbb{C}^{p\times m}$, $P_2 \in \mathbb{C}^{q\times m}$, $Q_p \in \mathbb{C}^{m\times p}$ and $Q_q \in \mathbb{C}^{m\times q}$.

With $F_{n,n}(J_p(k+n)^{\bar{n}})$ we will denote the discrete Mittag-Leffler function with two parameters defined by, see [3], [**?**]

$$F_{n,n}(J_p(k+n)^{\bar{n}}) = \sum_{i=0}^{\infty} \frac{(k+n)^{\overline{in}}}{\Gamma((i+1)n)} J_p^i. \tag{5}$$

The following Theorem has been proved, see [11], [12], [13], [14].

Theorem 1.1. For system (1) with a regular pencil there exists at least one solution if and only if all finite eigenvalues of the pencil are distinct and lie within the open disk $S = \{s \in \mathbb{C} : |s| < 1\}$. Then, the solutions of system (1) for $k \geq 0$ are given by the formula

$$Y_k = (k+1)^{\overline{n-1}} Q_p F_{n,n}(J_p(k+n)^{\bar{n}})(I_p - J_p)C + QD_k.$$

In addition, if there exists a solution, the initial value problem (1), (2) has a unique solution if and only if

$$Y_0 \in colspan Q_p - Q_q \sum_{i=0}^{q_*-1} \nabla_0^{in} H_q^i P_2 V_0 \tag{6}$$

and it is given by

$$Y_k = (k+1)^{\overline{n-1}} Q_p F_{n,n}(J_p(k+n)^{\bar{n}})(I_p - J_p) Z_0^p + Q D_k. \tag{7}$$

Where Z_0^p is the unique solution of the linear system

$$Y_0 = Q_p Z_0^p + D_0.$$

Where $D_k = \begin{bmatrix} \sum_{i=1}^{k}(k-i+1)^{\overline{n-1}} F_{n,n}(J_p(k+n-i)^{\bar{n}}) P_1 V_i \\ -\sum_{i=0}^{q_*-1} \nabla_0^{in} H_q^i P_2 V_k \end{bmatrix}$, $C \in \mathbb{C}^p$ is a constant vector. The matrices J_p, H_q are defined in (3) and the matrices P_1, P_2, Q_p, Q_q are given by (4). The discrete Mittag-Leffler function with two parameters is defined by (5).

Throughout the paper with $\|\cdot\|$ we will denote an induced norm and with 0_{ij} the zero matrix with i rows, j columns. The paper is organized as follows: in Section 2 we study the asymptotic stability of the equilibrium for the homogeneous system of (1) and prove conditions for the input vector under which the solution of the non-homogeneous system converges. In Section 3 we perturb system (1) and provide a result regarding the robustness of its solution. In Section 4 we provide a numerical example to justify our theory.

2 Stability

We begin this Section with the following definition

Definition 2.1. For any system in the form of (1), Y_* is an equilibrium state if it does not change under the initial condition, i.e.: Y_* is an equilibrium state if and only if $Y_0 = Y_*$ implies that $Y_k = Y_*$, $\forall k \geq 0$.

If Y_* is an equilibrium of system (1), then $Y_* \in colspanQ_p - Q_q \sum_{i=0}^{q_*-1} \nabla_0^{in} H_q^i P_2 V_0$. This is a direct result from Theorem 1.1 and (6). If there exists a solution for system (1) then it is unique if and only if (6) holds, i.e. $Y_* \in colspanQ_p - Q_q \sum_{i=0}^{q_*-1} \nabla_0^{in} H_q^i P_2 V_0$. We can now state the following Theorem

Theorem 2.1. We consider the initial value problem (1), (2) and assume that it has a unique solution. Then

(a) For $V_k = 0_{m,1}$ the equilibrium state of the homogenous system is asymptotic stable.

(b) For the unique solution Y_k of the initial value problem (1), (2) if

  (i) $Q_p P_1 V_k = 0_{m,1}$, for every $k \in \mathbb{N}$;

  (ii) The series $\sum_{j=0}^{+\infty} V_j$ converges;

  (iii) $lim_{k\to+\infty}[Q_q \sum_{i=0}^{q_*-1} \nabla_0^{in} H_q^i P_2 V_k] = V$.

Then

$$lim_{k\to+\infty} Y_k = Y_* + V.$$

Where Y_* equilibrium of the homogeneous system of (1).

Proof. For the proof of (a), if we set in (1) $V_k = 0_{m,1}$ the homogeneous systems that appears has the form

$$F\nabla_0^n Y_k = GY_k.$$

Then from Theorem 1.1 there exists a solution if and only if all finite eigenvalues of the pencil of the system are distinct and lie within the open disk $S = \{s \in \mathbb{C} : |s| < 1\}$, i.e.

$\|J_p\| < 1$. In addition from (6) the solution is unique if and only if $Y_0 \in colspan Q_p$. Then the solution is then given by (7) and has the form

$$Y_k = (k+1)^{\overline{n-1}} Q_p F_{n,n}(J_p (k+n)^{\bar{n}})(I_p - J_p) Z_0^p.$$

Where Z_0^p is the unique solution of the linear system $Y_0 = Q_p Z_0^p$. This implies that if Y_* is an equilibrium state of the homogeneous system, for $Y_0 = Y_*$ we have $Y_* = Q_p Z_0^p$. By taking the limit of the above written solution we have

$$lim_{k\to+\infty} Y_k = lim_{k\to+\infty}(k+1)^{\overline{n-1}} Q_p F_{n,n}(J_p (k+n)^{\bar{n}})(I_p - J_p) Z_0^p =$$

$$Q_p[lim_{k\to+\infty}(k+1)^{\overline{n-1}} F_{n,n}(J_p (k+n)^{\bar{n}})](I_p - J_p) Z_0^p =$$

$$Q_p[lim_{k\to+\infty}\frac{\Gamma(k+n)}{\Gamma(k+1)}\sum_{i=0}^{\infty} J_p^i \frac{(k+n)^{\overline{in}}}{\Gamma((i+1)n)}](I_p - J_p) Z_0^p =$$

$$Q_p[lim_{k\to+\infty}\sum_{i=0}^{\infty} J_p^i \frac{\Gamma(k+n+in)}{\Gamma(k+1)\Gamma((i+1)n)}](I_p - J_p) Z_0^p =$$

$$Q_p[lim_{k\to+\infty}\sum_{i=0}^{\infty} J_p^i \frac{\Gamma(k+n+in)}{\Gamma(k+1)\Gamma((i+1)n)}](I_p - J_p) Z_0^p =$$

$$Q_p[lim_{k\to+\infty}\sum_{i=0}^{\infty} J_p^i \frac{\Gamma((i+1)n)[(i+1)n+k-1][(i+1)n+k-2]...[(i+1)n]}{\Gamma(k+1)\Gamma((i+1)n)}](I_p - J_p) Z_0^p =$$

$$Q_p[lim_{k\to+\infty}\sum_{i=0}^{\infty} J_p^i \frac{[(i+1)n+k-1][(i+1)n+k-2]...[(i+1)n]}{k(k-1)...1}](I_p - J_p) Z_0^p =$$

$$Q_p[lim_{k\to+\infty}\sum_{i=0}^{\infty} J_p^i \frac{[k-1+(i+1)n]}{k}\frac{[k-2+(i+1)n]}{k-1}...\frac{[(i+1)n]}{1}](I_p - J_p) Z_0^p =$$

$$Q_p(I_p - J_p)^{-1}(I_p - J_p) Z_0^p = Q_p Z_0^p = Y_*.$$

Thus

$$lim_{k\to+\infty} Y_k = Y_*.$$

For the proof of (b) we consider system (1) with known initial conditions (2) and a unique solution. Then from (7)

$$lim_{k\to+\infty}Y_k = lim_{k\to+\infty}[Y_k^{(h)} + QD_k].$$

Where $Y_k^{(h)} = (k+1)^{\overline{n-1}}Q_pF_{n,n}(J_p(k+n)^{\bar{n}})(I_p-J_p)Z_0^p$. In (a) we proved that $lim_{k\to+\infty}Y_k^{(h)}$ converges. For the other limit

$$lim_{k\to+\infty}[Q_p\sum_{i=1}^{k}(k-i+1)^{\overline{n-1}}F_{n,n}(J_p(k+n-i)^{\bar{n}})P_1V_i - Q_q\sum_{i=0}^{q_*-1}\nabla_0^{in}H_q^iP_2V_k],$$

we have to compute two limits separately. The first is

$$lim_{k\to+\infty}[Q_p\sum_{i=1}^{k}(k-i+1)^{\overline{n-1}}F_{n,n}(J_p(k+n-i)^{\bar{n}})P_1V_i],$$

which converges $\forall k\in\mathbb{N}$ if and only if $Q_pP_1V_k = 0_{m,1}$. The other limit is

$$lim_{k\to+\infty}[Q_q\sum_{i=0}^{q_*-1}\nabla_0^{in}H_q^iP_2V_k] =$$

$$lim_{k\to+\infty}[Q_q\sum_{i=0}^{q_*-1}\frac{1}{\Gamma(-in)}\sum_{j=0}^{k}(k-j+1)^{\overline{-in-1}}H_q^iP_2V_j] =$$

$$Q_q\sum_{i=0}^{q_*-1}\frac{1}{\Gamma(-in)}H_q^iP_2(lim_{k\to+\infty}[\sum_{j=0}^{k}(k-j+1)^{\overline{-in-1}}V_j]).$$

We are interested in the convergence of $lim_{k\to+\infty}[\sum_{j=0}^{k}(k-j+1)^{\overline{-in-1}}V_j]$. For every $k\in\mathbb{N}$ and $0\le j\le k$ we have

$$\left|(k-j+1)^{\overline{-in-1}}\right| = \left|\frac{\Gamma(k-j-in)}{\Gamma(k-j+1)}\right| \le \left|(k-j+1)^{-in-1}\right| \le 1.$$

Thus

$$\left|(k-j+1)^{\overline{-in-1}}V_j\right| \leq |V_j|\,.$$

This means that if the series $lim_{k\to+\infty}\sum_{j=0}^{k}V_j = \sum_{j=0}^{+\infty}V_j$ converges, then the limit $lim_{k\to+\infty}[\sum_{j=0}^{k}(k-j+1)^{\overline{-in-1}}V_j]$ also converges, i.e. $lim_{k\to+\infty}[Q_q\sum_{i=0}^{q_*-1}\nabla_0^{in}H_q^iP_2V_k]$ converges. From (7)

$$lim_{k\to+\infty}Y_k = lim_{k\to+\infty}[Y_k^{(h)} + QD_k].$$

Where $Y_k^{(h)}$ is the solution of the homogeneous system of (1) and D_k is given by Theorem 1.1 and

$$QD_k = Q_p\sum_{i=1}^{k}(k-i+1)^{\overline{n-1}}F_{n,n}(J_p(k+n-i)^{\bar{n}})P_1V_i + Q_q\sum_{i=0}^{q_*-1}\nabla_0^{in}H_q^iP_2V_k.$$

From (a)

$$lim_{k\to+\infty}Y_k^{(h)} = Y_*,$$

where Y_* equilibrium of the homogeneous system of (1). In (b)(i) it was assumed that $Q_pP_1V_k = 0_{m,1}$. Hence

$$lim_{k\to+\infty}[Q_p\sum_{i=1}^{k}(k-i+1)^{\overline{n-1}}F_{n,n}(J_p(k+n-i)^{\bar{n}})P_1V_i = 0_{m,1}.$$

If

$$lim_{k\to+\infty}[Q_q\sum_{i=0}^{q_*-1}\nabla_0^{in}H_q^iP_2V_k] = V,$$

then

$$lim_{k\to+\infty}[Y_k^{(h)} + QD_k] = Y_* + 0_{m,1} + V,$$

or, equivalently,

$$lim_{k\to+\infty}Y_k = Y_* + V.$$

The proof is completed.

3 Perturbation and robustness

Singular systems of fractional nabla difference equations in the form of (1) with given initial conditions (2) aren't always guaranteed to have solution. From Theorem 1.1 existence of solutions depends on the finite eigenvalues of the pencil of the system. Let $s_j \in \mathbb{C}$, $j = 1, 2, ..., \nu$ s_j, $j = 1, 2, ..., \nu$, be a finite eigenvalue of $sF - G$ with algebraic multiplicity p_j, $\sum_{j=1}^{\nu} p_j = p$. We define the arbitrary matrix $E \in \mathbb{C}^{m\times m}$ such that

$$F\nabla_0^n Y_k = (G + E)Y_k + V_k. \tag{8}$$

In addition, let the inverse of the non-singular matrix Q, as defined in (3), be written in the form

$$Q^{-1} = \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix}, \tag{9}$$

where $Q_1 \in \mathbb{C}^{p\times m}$, $Q_2 \in \mathbb{C}^{q\times m}$.

Theorem 3.1. Consider the system (1) with a regular pencil and finite eigenvalues of type s_j, $j = 1, 2, ..., \nu$ and of algebraic multiplicity p_j. Then

(a) For a random finite eigenvalue $\tilde{s}$ of the pencil of the perturbed system (8), we have

$$min_{1\leq j\leq \nu} |\tilde{s} - s_j| \leq max\left\{p_* K,\ \sqrt[p_*]{p_*}\ \sqrt[p_*]{K}\right\}. \tag{10}$$

Where $K = \dfrac{\left\|\begin{bmatrix} P_1 E \\ 0_{q,m} \end{bmatrix}\right\|_1}{\left\|\begin{bmatrix} Q_1 \\ 0_{q,m} \end{bmatrix}\right\|_1}$, $p_* = max_{1\leq j\leq \nu} p_j$ and P_1, Q_1 are defined in (4), (9), respectively.

(b) If there exist solutions for the system (1), after a perturbation accordingly to (8), there will still exist solutions for the system if

$$s_* + max\left\{p_* K,\ \sqrt[p_*]{p_*}\ \sqrt[p_*]{K}\right\} < 1. \tag{11}$$

Where $s_* = max_{1\leq j\leq \nu} |s_j|$.

Proof. For the proof of (a) let $\tilde{s}$ be a random finite eigenvalue of the pencil $sF - (G + E)$ with $\tilde{s} \neq s_j$. Let $U \in \mathbb{C}^m$ be an eigenvector of $\tilde{s}$, i.e. $\tilde{s}FU = (G + E)U$, or, equivalently,

$$(\tilde{s}F - G)U = EU.$$

We consider the matrices P, Q as defined in (4). By substituting the transformation $U = QW$ into the above expression, $W \in \mathbb{C}^m$, we obtain

$$(\tilde{s}F - G)QW = EQW,$$

whereby, multiplying by P, we get

$$(\tilde{s}F_w - G_w)W = PEQW.$$

While $\tilde{s} \neq s_j$ we have $\det(\tilde{s}F_w - G_w) \neq 0$ and

$$W = (\tilde{s}F_w - G_w)^{-1}PEQW.$$

Where F_w, G_w are defined in (3) and thus

$$(\tilde{s}F_w - G_w)^{-1} = \begin{bmatrix} (\tilde{s}I_p - J_p)^{-1} & 0_{p,q} \\ 0_{q,p} & (\tilde{s}H_q - I_q)^{-1} \end{bmatrix}.$$

By using (4), the matrix PEQ can be written as

$$PEQ = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} E \begin{bmatrix} Q_p & Q_q \end{bmatrix} = \begin{bmatrix} P_1EQ_p & P_1EQ_q \\ P_2EQ_p & P_2EQ_q \end{bmatrix}.$$

Moreover, let $W = \begin{bmatrix} W_p \\ W_q \end{bmatrix}$, $W_p \in \mathbb{C}^{p\times m}$, $W_q \in \mathbb{C}^{q\times m}$. Then

$$\begin{bmatrix} W_p \\ W_q \end{bmatrix} = \begin{bmatrix} (\tilde{s}I_p - J_p)^{-1} & 0_{p,q} \\ 0_{q,p} & (\tilde{s}H_q - I_q)^{-1} \end{bmatrix} \begin{bmatrix} P_1EQ_p & P_1EQ_q \\ P_2EQ_p & P_2EQ_q \end{bmatrix} \begin{bmatrix} W_p \\ W_q \end{bmatrix}.$$

From the above equation we get

$$W_p = (\tilde{s}I_p - J_p)^{-1} \begin{bmatrix} P_1EQ_p & P_1EQ_q \end{bmatrix} \begin{bmatrix} W_p \\ W_q \end{bmatrix},$$

which can be written as

$$\begin{bmatrix} W_p \\ 0_{q,1} \end{bmatrix} = \begin{bmatrix} (\tilde{s}I_p - J_p)^{-1} & 0_{p,q} \\ 0_{q,p} & 0_{q,q} \end{bmatrix} \begin{bmatrix} P_1EQ_p & P_1EQ_q \\ 0_{q,p} & 0_{q,q} \end{bmatrix} \begin{bmatrix} W_p \\ W_q \end{bmatrix}.$$

From the transformation $U = QW$ and by using (9), we get $Q^{-1}U = W$, or, equivalently, $\begin{bmatrix} Q_1U \\ Q_2U \end{bmatrix} = \begin{bmatrix} W_p \\ W_q \end{bmatrix}$ and thus $Q_1U = W_p$. Hence by using this observation into the above expression we get

$$\begin{bmatrix} Q_1 \\ 0_{q,m} \end{bmatrix} U = \begin{bmatrix} (\tilde{s}I_p - J_p)^{-1} & 0_{p,q} \\ 0_{q,p} & 0_{q,q} \end{bmatrix} \begin{bmatrix} P_1EQ_p & P_1EQ_q \\ 0_{q,p} & 0_{q,q} \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} U.$$

Furthermore

$$\left\| \begin{bmatrix} Q_1 \\ 0_{m,q} \end{bmatrix} U \right\| = \left\| \begin{bmatrix} (\tilde{s}I_p - J_p)^{-1} & 0_{p,q} \\ 0_{q,p} & 0_{q,q} \end{bmatrix} \begin{bmatrix} P_1EQ_p & P_1EQ_q \\ 0_{q,p} & 0_{q,q} \end{bmatrix} Q^{-1}U \right\|$$

and

$$sup \left\{ \frac{\left\| \begin{bmatrix} Q_1 \\ 0_{m,q} \end{bmatrix} U \right\|}{\|U\|} \right\} = sup \left\{ \frac{\left\| \begin{bmatrix} (\tilde{s}I_p - J_p)^{-1} & 0_{p,q} \\ 0_{q,p} & 0_{q,q} \end{bmatrix} \begin{bmatrix} P_1EQ_p & P_1EQ_q \\ 0_{q,p} & 0_{q,q} \end{bmatrix} Q^{-1}U \right\|}{\|U\|} \right\},$$

since $U \neq 0_{m,1}$ is an eigenvector of $\tilde{s}$. Hence

$$\left\| \begin{bmatrix} Q_1 \\ 0_{q,m} \end{bmatrix} \right\| = \left\| \begin{bmatrix} (bI_p - J_p(a))^{-1} & 0_{p,q} \\ 0_{q,p} & 0_{q,q} \end{bmatrix} \begin{bmatrix} P_1EQ_p & P_1EQ_q \\ 0_{q,p} & 0_{q,q} \end{bmatrix} Q^{-1} \right\|.$$

Where

$$\begin{bmatrix} P_1EQ_p & P_1EQ_q \\ 0_{q,p} & 0_{q,q} \end{bmatrix} = \begin{bmatrix} P_1E \\ 0_{q,m} \end{bmatrix} \begin{bmatrix} Q_p & Q_q \end{bmatrix} = \begin{bmatrix} P_1E \\ 0_{q,m} \end{bmatrix} Q.$$

Thus

$$\left\| \begin{bmatrix} Q_1 \\ 0_{q,m} \end{bmatrix} \right\| = \left\| \begin{bmatrix} (bI_p - J_p(a))^{-1} & 0_{p,q} \\ 0_{q,p} & 0_{q,q} \end{bmatrix} \begin{bmatrix} P_1E \\ 0_{q,m} \end{bmatrix} \right\|,$$

or, equivalently,

$$\left\| \begin{bmatrix} Q_1 \\ 0_{q,m} \end{bmatrix} \right\| = \left\| \begin{bmatrix} (\tilde{s}I_p - J_p)^{-1}P_1E \\ 0_{q,m} \end{bmatrix} \right\|.$$

or, equivalently,

$$\left\| \begin{bmatrix} Q_1 \\ 0_{q,m} \end{bmatrix} \right\| \leq \left\| (\tilde{s}I_p - J_p)^{-1} \right\| \left\| \begin{bmatrix} P_1E \\ 0_{q,m} \end{bmatrix} \right\|,$$

or, equivalently,

$$\frac{1}{\|(\tilde{s}I_p - J_p)^{-1}\|} \leq \frac{\left\| \begin{bmatrix} P_1 E \\ 0_{q,m} \end{bmatrix} \right\|}{\left\| \begin{bmatrix} Q_1 \\ 0_{q,m} \end{bmatrix} \right\|}.$$

Where $(\tilde{s}I_p - J_p)^{-1} := [\tilde{s}I_{p_1} - J_{p_1}(s_1)]^{-1} \oplus \tilde{s}I_{p_2} - J_{p_2}(s_2)]^{-1} \oplus \dots \oplus [\tilde{s}I_{p_\nu} - J_{p_\nu}(s_\nu)]^{-1}$ and

$$[\tilde{s}I_{p_j} - J_{p_j}(s_j)]^{-1} = \begin{bmatrix} \frac{1}{\tilde{s}-s_j} & \frac{1}{(\tilde{s}-s_j)^2} & \dots & \frac{1}{(\tilde{s}-s_j)^{p_j}} \\ 0 & \frac{1}{\tilde{s}-s_j} & \dots & \frac{1}{(\tilde{s}-s_j)^{p_j-1}} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \frac{1}{\tilde{s}-s_j} \end{bmatrix}, \quad \forall j = 1, 2, ..., \nu.$$

Let $M_j = max\left\{\frac{1}{|\tilde{s}-s_j|}, \frac{1}{|\tilde{s}-s_j|^{p_j}}\right\}$. By taking the norm $\|\cdot\|_1$ of $(\tilde{s}I_p - J_p)^{-1}$ we have

$$\left\|[\tilde{s}I_p - J_p]^{-1}\right\|_1 = max_{1\leq j\leq \nu} \sum_{l=1}^{p_j} \frac{1}{|\tilde{s}-s_j|^l} \leq max_j M_j p_j \leq (max_{1\leq j\leq \nu} M_j)(max_{1\leq j\leq \nu} p_j),$$

or, equivalently,

$$\frac{1}{(max_{1\leq j\leq \nu} M_j)(max_{1\leq j\leq \nu} p_j)} \leq \frac{1}{\|[\tilde{s}I_p - J_p]^{-1}\|_1},$$

or, equivalently,

$$\frac{min_{1\leq j\leq \nu}(min\left\{|\tilde{s}-s_j|, |\tilde{s}-s_j|^{p_j}\right\})}{max_{1\leq j\leq \nu} p_j} \leq \frac{1}{\|[\tilde{s}I_p - J_p]^{-1}\|_1},$$

or, equivalently,

$$\frac{min_{1\leq j\leq \nu}(min\left\{\left|\tilde{s}-s_j\right|,\left|\tilde{s}-s_j\right|^{p_j}\right\})}{max_{1\leq j\leq \nu}p_j}\leq\frac{\left\|\begin{bmatrix} P_1E \\ 0_{q,m} \end{bmatrix}\right\|_1}{\left\|\begin{bmatrix} Q_1 \\ 0_{q,m} \end{bmatrix}\right\|_1}.$$

Let $max_{1\leq j\leq \nu}p_j=p_*$. If $\left|\tilde{s}-s_j\right|<\left|\tilde{s}-s_j\right|^{p_j}$, $(\left|\tilde{s}-s_j\right|>1)$ then

$$min_{1\leq j\leq \nu}\left|\tilde{s}-s_j\right|\leq p_*\frac{\left\|\begin{bmatrix} P_1E \\ 0_{q,(mn)} \end{bmatrix}\right\|_1}{\left\|\begin{bmatrix} Q_1 \\ 0_{q,(mn)} \end{bmatrix}\right\|_1}=p_*K.$$

If $\left|\tilde{s}-s_j\right|>\left|\tilde{s}-s_j\right|^{p_j}$, $(\left|\tilde{s}-s_j\right|<1)$ then

$$min_{1\leq j\leq \nu}\left|b-a_j\right|\leq\ \sqrt[p_*]{p_*}\ \sqrt[p_*]{\frac{\left\|\begin{bmatrix} P_1E \\ 0_{q,m} \end{bmatrix}\right\|_1}{\left\|\begin{bmatrix} Q_1 \\ 0_{q,m} \end{bmatrix}\right\|_1}}\ =\ \sqrt[p_*]{p_*}\ \sqrt[p_*]{K}.$$

By using the above two inequalities

$$min_{1\leq j\leq \nu}\left|b-a_j\right|\leq max\left\{p_*K,\ \sqrt[p_*]{p_*}\ \sqrt[p_*]{K}\right\}.$$

Note that the norm $\left\|\cdot\right\|_1$ can be replaced with $\left\|\cdot\right\|_\infty$.

For the proof of (b) let as in (a) $\tilde{s}$ be a random eigenvalue of the perturbed system (8) and $\tilde{s}_*$ an eigenvalue of the same system such that $\left|\tilde{s}_*\right|=max\left|\tilde{s}\right|$. Then from (a) we have

$$min_{1\leq j\leq \nu}\left|\tilde{s}_* - s_j\right| \leq max\left\{p_*K,\ \sqrt[p_*]{p_*}\ \sqrt[p_*]{K}\right\},$$

or, equivalently,

$$min_{1\leq j\leq \nu}\left||\tilde{s}_*| - |s_j|\right| \leq max\left\{p_*K,\ \sqrt[p_*]{p_*}\ \sqrt[p_*]{K}\right\},$$

or, equivalently,

$$\left||\tilde{s}_*| - s_*\right| \leq max\left\{p_*K,\ \sqrt[p_*]{p_*}\ \sqrt[p_*]{K}\right\},$$

$$s_* - max\left\{p_*K,\ \sqrt[p_*]{p_*}\ \sqrt[p_*]{K}\right\} \leq |\tilde{s}_*| \leq s_* + max\left\{p_*K,\ \sqrt[p_*]{p_*}\ \sqrt[p_*]{K}\right\}.$$

Thus there exist solutions for the perturbed system (8) if $|\tilde{s}_*| < 1$, i.e.

$$s_* + max\left\{p_*K,\ \sqrt[p_*]{p_*}\ \sqrt[p_*]{K}\right\} < 1.$$

The proof is completed.

4 Numerical Example

In [6], [43] we get a description of one of the first modifications of the original Leontief model. In this Subsection, by using the nabla fractional operator, we propose a modification of the typical Leontief model, incorporating delayed variables into the system of equations, see [11], [12]. With this modification, decisions (for consumption or investment levels) can be taken based on information from the past, i.e. upon the experience of the last k years, $k \geq 0$. It is constructed as follows. Suppose the economy is divided into m sectors. Each sector is producing every year an amount of a new product. Let Y_k have i^{th} component the output in the k^{th} time period of sector i, $i = 1, 2, ..., m$. Let b_{ij} be the amount of commodity i that sector j, $j = 1, 2, ..., m$, must have to produce one unit of commodity j and let a_{ij} be the proportion of commodity j that gets transferred to commodity i in the k_{th} time period. Where a_{ij}, b_{ij} are elements of the matrices A, B, respectively. If F is the matrix that describes the amount of commodity i given to j and the amount of commodity i needed from j for the past $k-1$ years, $k \geq 0$. In addition let V_k have i^{th} component the final demand (demand in the k^{th} time period excluding investment demand).

Then the amount of commodity i is equal to the amount of commodity i inputted by all sectors plus the amount needed for production at year k plus the remaining from previous years amount of product plus amount used to meet non-investment demand (consumption).

In matrix form this model is characterized by the following system of fractional difference equations of inputs and outputs

$$Y_k = AY_k - BY_k + F\sum_{j=0}^{k-1} \frac{(k-j+1)^{\overline{-n-1}}}{\Gamma(-n)} Y_j + V_k,$$

or, equivalently,

$$Y_k = AY_k - BY_k - FY_k + F\nabla_0^n Y_k + V_k.$$

Where $A, B, F \in \mathbb{R}^{m\times m}$. A is called the Leontief input-output coefficient matrix or the matrix of flow coefficients, B is the capital coefficient matrix, Y_k is the vector of output levels in period k and V_k the vector of final demand, exogenously determined in k. The first term on the right-hand side of the above notation, AY_k, denotes intermediate demand for goods by industries; whereas the second term, BY_k, reflects the distribution of inputs to investment. The third term $-FY_k + F\nabla_0^n Y_k$ is related to the existing product in each sector from previous years. This system is actually a delayed model because of the term $\nabla_0^n Y_k = \frac{1}{\Gamma(-n)} \sum_{j=0}^{k} (k-j+1)^{\overline{-n-1}} Y_j$ and is quite interesting from the mathematical point of view. These terms produce oscillatory trajectories for the solutions and consequently for the output levels in period k. Delayed information is very important because decisions can be taken by using information not only from the current year but also from previous years. For $G = I_m - A + B + F$ we get system (1). We consider this system with initial conditions of type (2) and $n = \frac{3}{2}$. Let

$$F = \begin{bmatrix} 0 & 0 & -1 & 0 & 1 \\ 0 & -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, G = \frac{1}{5}\begin{bmatrix} -5 & 0 & 8 & 5 & -3 \\ -11 & -1 & 14 & 11 & -8 \\ -2 & -2 & 2 & 2 & 0 \\ 11 & 2 & -14 & -11 & 8 \\ -5 & 0 & 10 & 5 & -5 \end{bmatrix},$$

and

$$V_k = \begin{bmatrix} \frac{k^2}{k^4+1} \\ \frac{1-2^k}{k!} \\ 0 \\ -\frac{2^k-1}{k!} \\ \frac{k^2}{k^4+1} \end{bmatrix}, Y_0 = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Then $\det(sF - G){=}s(s - \frac{1}{5})(s - \frac{2}{5})$ and the pencil is regular. The pencil $sF - G$ has three finite eigenvalues ($p = 3$) 0, $\frac{1}{5}$, $\frac{2}{5}$ and two infinite eigenvalues ($q{=}2$). The Jordan matrix J_p and the nilpotent matrix H_q (with index $q^* = 2$) have the form

$$J_p = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{5} & 0 \\ 0 & 0 & \frac{2}{5} \end{bmatrix}, H_q = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

It is easy to observe that $\|J_p\| < 1$. By calculating the eigenvectors of the finite eigenvalues and the eigenvectors of the infinite eigenvalues we get the matrices

$$Q_p = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}, Q_q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$$

Moreover

$$P_1 = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 \end{bmatrix}, P_2 = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Since $V_0 = 0_{5,1}$, we get $\sum_{i=0}^{1} \nabla_0^{in} Q_q H_q^i P_2 V_0 = 0_{5,1}$ and hence it is easy to observe that

$$Y_0 \in colspan Q_p - \sum_{i=0}^{q_*-1} \nabla_0^{in} Q_q H_q^i P_2 V_0.$$

From Theorem 1.1 there exists a solution and it is unique. Since $\sum_{k=0}^{+\infty} \frac{2^k-1}{k!} = e^2 - e$ and $lim_{k\to+\infty} \left| \frac{\frac{k^2}{k^4+1}}{\frac{1}{k^2}} \right| = 1 \neq 0$ & $\sum_{k=1}^{+\infty} \frac{1}{k^2}$ converges, we have that $\sum_{k=1}^{+\infty} \frac{k^2}{k^4+1}$ converges, i.e. the series $\sum_0^{+\infty} V_k$ is converging. In addition $Q_p P_1 V_k = 0_{5,1}$ and hence from Theorem 2.1, the solution converges.

We assume now the perturbed system (8) with

$$E = \begin{bmatrix} 0 & 0 & 0 & 0.2 & 0.3 \\ 0 & 0 & 0 & -0.3 & 0.2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -0.1 & 0 & 0 & 0 \\ 0.1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Since

$$Q^{-1} = \begin{bmatrix} 0 & 0 & 1 & 1 & -1 \\ 1 & 1 & -1 & -1 & 0 \\ 0 & 0 & -1 & 0 & 1 \\ 1 & 0 & -1 & -1 & 1 \\ -1 & 0 & 2 & 1 & -1 \end{bmatrix},$$

we have $K = \frac{1}{6}$, $p_* = 1$, $s_* = \frac{2}{5}$ and from Theorem 3.1

$$s_* + max\left\{p_*K,\ \sqrt[p_*]{p_*}\,\sqrt[p_*]{K}\right\} = \frac{1}{6} + \frac{2}{5} = \frac{23}{30} < 1,$$

which means that there exist solutions for the perturbed system.

Conclusions

In this article we studied the stability and the robustness of a class of singular systems of fractional nabla difference equations whose coefficients are constant matrices. We proved that the equilibrium of the homogeneous system of (1) is asymptotic stable and provided conditions for the input vector under which the solution of the non-homogeneous system (1) converges. In addition, we applied perturbation to the system and obtained a result for robustness of solutions.

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