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# LEVELS AND SUBLEVELS OF COMPOSITION ALGEBRAS

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ABSTRACT. Lewis' and Leep's bounds on the level and sublevel of quaternion algebras are extended to the class of composition algebras. Some simple constructions of composition algebras of known level values are given. In addition, octonion algebras of sublevel 3 are presented.

## 1. INTRODUCTION

In [Lew2], Lewis constructed quaternion algebras of level  $2^k$  and  $2^k + 1$  for all integers  $k \geq 0$ . Laghribi and Mammone recovered these values using function field techniques (see [LM]). Pumplün employed their methodology in [Pu] to construct octonion algebras of level  $2^k$  and  $2^k + 1$ . These remain the only known values of the level of a composition algebra. Indeed, all of these constructions are of sublevel  $2^k$  for some  $k$ . Krüskemper and Wadsworth produced what was hitherto the only example of a composition algebra whose sublevel was not of this form, by constructing a quaternion algebra of sublevel 3 (see [KW]).<sup>1</sup>

To a large extent, the above constructions rely upon results regarding the equivalence between bounds on the level and sublevel and the isotropy of an associated quadratic form, the trace form. This strategy of considering the isotropy of the trace form to obtain bounds on the level and sublevel of quaternion algebras was first employed by Lewis in [Lew1]. Leep continued the process in [L].

In this paper, we extend their results to composition algebras and suggest simplifications of existing proofs. We also offer a simple method for recovering some known level values and construct octonion algebras of sublevel 3.

## 2. PRELIMINARIES

Let  $F$  denote a field of characteristic  $\neq 2$ ,  $F_0$  a formally real field and  $C$  a unital, not necessarily associative,  $F$ -algebra.

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<sup>1</sup>In the meantime, Detlev Hoffmann has kindly communicated a method for constructing infinitely many examples of quaternion algebras of level  $\notin \{2^k, 2^k + 1\}$ . This method also works for octonion algebras, allowing us to prove the existence of octonion algebras of level 6 and 7. Details will appear in a forthcoming publication.

A map  $*$  is called an *involution* on  $C$  if it is an anti-automorphism of period 2. We have  $C = \text{Sym}(C, *) \oplus \text{Skew}(C, *)$ , with  $\text{Sym}(C, *) = \{x \in C \mid x^* = x\}$  and  $\text{Skew}(C, *) = \{x \in C \mid x^* = -x\}$ . An involution  $*$  is called *scalar* if  $x^*x \in F$  and  $x^* + x \in F$  for all  $x \in C$ . For an algebra  $C$  with scalar involution  $*$ , we call  $t_C(x) = x + x^*$  the *trace* of  $C$  and the quadratic form  $T_C : C \rightarrow F$ ,  $T_C(x) = t_C(x^2)$  the *trace form*.

For  $a, b \in F^\times$ , the *quaternion algebra*  $\left(\frac{a, b}{F}\right)$  over  $F$  is a 4-dimensional  $F$ -vector space with basis  $\{1, i, j, k\}$ , satisfying  $i^2 = a, j^2 = b$  and  $ij = -ji = k$ .

The *Cayley-Dickson doubling process* is an algorithm for constructing new algebras with scalar involution from old ones. Applying the process to an algebra  $C$  with scalar involution  $*$ , together with a chosen scalar  $\mu \in F^\times$ , we will produce a new algebra,  $\text{Cay}(C, \mu)$ , the *Cayley-Dickson double* of  $C$ , whose scalar involution we will also denote by  $*$ .  $\text{Cay}(C, \mu)$  is the  $F$ -module  $C \times C$ , with multiplication defined by  $(u, v)(u', v') = (uu' + \mu v'^*v, v'u + vu'^*)$  and involution given by  $(u, v)^* = (u^*, -v)$ , for  $u, u', v, v' \in C$ .

An algebra  $C$  is a *composition algebra* if there exists a nondegenerate quadratic form  $q$  on  $C$  which allows composition, that is  $q(xy) = q(x)q(y)$  for all  $x, y \in C$ . Composition algebras are of rank 1, 2, 4 or 8. The composition algebras of rank 2 are the quadratic étale  $F$ -algebras; the composition algebras of rank 4 are the (non-commutative) quaternion algebras and those of rank 8 are the (non-commutative and non-associative) octonion algebras. For  $a, b, c \in F^\times$ , the *octonion algebra*  $\left(\frac{a, b, c}{F}\right)$  over  $F$  is defined as  $\left(\frac{a, b, c}{F}\right) := \text{Cay}\left(\left(\frac{a, b}{F}\right), c\right)$ . We note that  $\text{Cay}\left(\left(\frac{a, b}{F}\right), c\right) = \left(\frac{a, b}{F}\right) \oplus \left(\frac{a, b}{F}\right)e$  is an 8-dimensional  $F$ -vector space with basis  $\{1, i, j, k, e, ie, je, ke\}$ , satisfying  $i^2 = a, j^2 = b$  and  $e^2 = c$ . Since two composition algebras are isomorphic if and only if their norm forms are isometric, changing the order of the slots  $a, b, c$  in  $\left(\frac{a, b, c}{F}\right)$  yields an isomorphic octonion algebra. If we apply the Cayley-Dickson doubling process to a composition algebra  $C$  over  $F$ , we will obtain another composition algebra (if the dimension of the new algebra is at most 8), or alternatively what is known as a *generalised Cayley-Dickson algebra*.

The *level* of  $F$ , denoted  $s(F)$ , is the least integer  $n$  such that  $-1$  is a sum of  $n$  squares in  $F$ . If no such integer exists, we say that  $s(F) = \infty$ . In [Pfl], Pfister showed that the level of a field, if finite, is a power of 2, and moreover that any prescribed power of 2 may be realised as the level of a field.

Another classical field invariant is the Pythagoras number. The *Pythagoras number* of  $F$ , denoted  $p(F)$ , is the least integer  $m \geq 1$  such that each nonzero sum of

squares in  $F$  can be written as a sum of  $\leq m$  squares. If no such integer exists, we say that  $p(F) = \infty$ . As a consequence of Pfister's results on the level of fields,  $p(F)$  is always of the form  $2^k$  or  $2^k + 1$  for  $F$  a nonformally real field, and all integers of this form are realisable as the Pythagoras numbers of a nonformally real field. In [H2], Hoffmann showed that every positive integer is realisable as the Pythagoras number of a formally real field.

The concept of level has different generalisations in a non-commutative setting. For  $D$  a division algebra, we define the *level* of  $D$ , denoted  $s(D)$ , in the same manner as for fields. The *sublevel* of  $D$ , denoted  $\underline{s}(D)$ , is the least integer  $n$  for which 0 is a sum of  $n + 1$  squares of elements in  $D$ . If 0 is not expressible in this manner, we say that the sublevel of  $D$  is infinite. Note that  $\underline{s}(D) \leq s(D)$ .

A composition algebra is *split* if it contains a composition subalgebra which is isomorphic to  $F \oplus F$ , which is the case if and only if  $C$  contains zero divisors.

Let us consider the case where  $C$  is a quadratic étale  $F$ -algebra. If  $C$  is split, then  $C = F \oplus F$ . We remark that  $s(F \oplus F) = s(F)$ . Hence, Pfister's classification of  $s(F)$  extends to split quadratic étale  $F$ -algebras. Alternatively, if  $C$  is non-split then  $C = F(\sqrt{d})$ , for  $d \notin F^{\times 2}$ . Since  $F(\sqrt{d})$  is itself a field, we conclude that Pfister's classification applies to all quadratic étale  $F$ -algebras.

Hence, to complete the classification of the level of composition algebras, one needs only consider quaternion and octonion algebras. Moreover, since split quaternion and octonion algebras have level 1 (a consequence of a split quaternion algebra being isomorphic to  $M_2F$ ), one may further restrict one's attention to quaternion and octonion algebras which are division.

For  $C = \left(\frac{a,b}{F}\right)$ , we have  $T_C \simeq 2\langle 1, a, b, -ab \rangle$ . For  $C = \left(\frac{a,b,c}{F}\right)$ , we have  $T_C \simeq 2\langle 1, a, b, -ab, c, -ac, -bc, abc \rangle$ . For our purposes, we may disregard this scalar factor of 2. We define the *pure trace form* of a composition algebra  $C$ , denoted  $T_P$ , via the following relation:  $T_C = \langle 1 \rangle \perp T_P$ .

A quadratic form  $q$  over  $F$  is said to be *isotropic* if there exists a non-zero vector  $x$  such that  $q(x) = 0$ . The *value set* of  $q$ , denoted  $D_F(q)$ , is the set of elements of  $F^\times$  which are represented by  $q$ . We note that  $D_F(T_P) = \{x^2 | x \in \text{Skew}(C, *)\}$  for  $*$  given by conjugation.

For further definitions and notation regarding quadratic forms, we refer the reader to [S].

## 3. BOUNDS AND TRACE FORMS

Let  $C$  denote a quaternion or octonion division algebra over  $F$  and  $n$  a positive integer.

**Proposition 3.1.** *If  $(n - 1) \times T_C$  is anisotropic but  $n \times T_C$  is isotropic, then  $n - 1 \leq \underline{s}(C) \leq 2n - 1$ .*

*Proof.* If  $\underline{s}(C) \leq n - 2$  then  $0 = \sum_{i=1}^{n-1} x_i^2$  for  $x_i \in C$ , so that  $(n - 1) \times T_C$  is isotropic. Thus,  $(n - 1) \times T_C$  anisotropic implies that  $\underline{s}(C) \geq n - 1$ . Now if  $n \times T_C$  is isotropic, that is  $n \times \langle 1 \rangle \perp n \times T_P$  is isotropic, 0 is a sum of  $2n$  squares in  $C$ . Hence  $\underline{s}(C) \leq 2n - 1$ .  $\blacksquare$

In [Lew1], Lewis proved that for  $D$  a finite-dimensional, associative division algebra of degree  $d$  over  $F$ , if  $(n - 1) \times T_D$  is anisotropic but  $n \times T_D$  is isotropic, then  $n - 1 \leq \underline{s}(D) \leq dn - 1$ .

Since  $T_C$  isotropic implies  $\underline{s}(C) = 1$ , the upper bound in Proposition 3.1 is optimal in the case where  $n = 1$ . Note the following:  $2^k \times T_C$  isotropic implies that  $\underline{s}(C) \leq 2^k$ , which in turn implies that  $\underline{s}(C) \leq 2^k$ . This arises as an immediate consequence of the next proposition. In particular, the above offers a reduction on the upper bound in Proposition 3.1 when  $n \neq 1$ , as for  $n \neq 1$ ,  $2^k < 2n - 1$  where  $2^{k-1} < n \leq 2^k$ .

**Proposition 3.2.**  *$2^k \times T_C$  is isotropic if and only if  $\langle 1 \rangle \perp 2^k \times T_P$  is isotropic.*

*Proof.* That the isotropy of  $\langle 1 \rangle \perp 2^k \times T_P$  implies that of  $2^k \times T_C$  clearly follows from the fact that  $\langle 1 \rangle \perp 2^k \times T_P$  is a subform of  $2^k \times T_C$ .

Conversely, suppose  $2^k \times T_C$  is isotropic. Hence, for  $C$  a quaternion algebra, we have

$$\sum_{i=1}^{2^k} p_i^2 + a \sum_{i=1}^{2^k} q_i^2 + b \sum_{i=1}^{2^k} r_i^2 - ab \sum_{i=1}^{2^k} s_i^2 = 0,$$

for  $p_i, q_i, r_i, s_i \in F$ . If  $p_i = 0$  for all  $i$ , this statement reads  $2^k \times T_P$  is isotropic. Alternatively, if  $p_i \neq 0$  for some  $i$ , multiplication by  $\sum_{i=1}^{2^k} p_i^2$  yields

$$\left( \sum_{i=1}^{2^k} p_i^2 \right)^2 + a \sum_{i=1}^{2^k} q_i'^2 + b \sum_{i=1}^{2^k} r_i'^2 - ab \sum_{i=1}^{2^k} s_i'^2 = 0,$$

for certain  $q_i', r_i', s_i' \in F$  by [S, p.151]. Hence,  $\langle 1 \rangle \perp 2^k \times T_P$  is isotropic in this case also. A similar proof works in the case where  $C$  is an octonion algebra.  $\blacksquare$

**Proposition 3.3.** *Let  $n+1$  be a power of two. Then  $\underline{s}(C) \leq n$  if and only if either  $\langle 1 \rangle \perp n \times T_P$  is isotropic or  $(n+1) \times T_P$  is isotropic.*

*Proof.* For the proof where  $C$  is a quaternion algebra, see [Lew1, Proposition 2].

Assume that  $C = \left( \frac{a,b,c}{F} \right)$ .

Clearly the right to left implication is valid for all  $n$ .

Working left to right,  $\underline{s}(C) \leq n$  implies that

$$0 = \sum_{i=1}^{n+1} p_i^2 + a \sum_{i=1}^{n+1} q_i^2 + b \sum_{i=1}^{n+1} r_i^2 - ab \sum_{i=1}^{n+1} s_i^2 + c \sum_{i=1}^{n+1} t_i^2 - ac \sum_{i=1}^{n+1} u_i^2 - bc \sum_{i=1}^{n+1} v_i^2 + abc \sum_{i=1}^{n+1} w_i^2$$

and

$$\sum_{i=1}^{n+1} p_i q_i = \sum_{i=1}^{n+1} p_i r_i = \sum_{i=1}^{n+1} p_i s_i = \sum_{i=1}^{n+1} p_i t_i = \sum_{i=1}^{n+1} p_i u_i = \sum_{i=1}^{n+1} p_i v_i = \sum_{i=1}^{n+1} p_i w_i = 0.$$

If all  $p_i = 0$ , then  $(n+1) \times T_P$  is isotropic. If not all  $p_i = 0$ , then multiplication by  $\sum_{i=1}^{n+1} p_i^2$  yields that  $n \times T_P$  represents  $-1$ , by [S, p.151], so  $\langle 1 \rangle \perp n \times T_P$  is isotropic.  $\blacksquare$

We may further clarify the above result (see Theorem 3.5).

Firstly, however, we need to extend a result of Leep regarding quaternion algebras ([L, Lemma 2.3]) to octonion algebras:

**Lemma 3.4.** *Let  $C = \left( \frac{a,b,c}{F} \right)$  be an octonion division algebra. Suppose  $2^k \times T_P$  is isotropic, where  $k \geq 0$ . Then  $(1 + [\frac{2}{3} \cdot 2^k]) \times T_P$  is isotropic, where  $[ \ ]$  denotes the greatest integer function.*

*Proof.* If  $2^k \times \langle a, b, -ab, c, -ac, -bc, abc \rangle$  is isotropic, then  $2^k \times \langle -a, -b, ab, -c, ac, bc, -abc \rangle$  is isotropic and thus  $2^k \times \langle \langle -a, -b, -c \rangle \rangle$  is hyperbolic. After multiplying by  $-1$  we note that any subform of  $2^k \times \langle -1, a, b, -ab, c, -ac, -bc, abc \rangle$  of dimension greater than  $4 \cdot 2^k$  is isotropic. The conclusion follows since  $7(1 + [\frac{2}{3} \cdot 2^k]) > 7(\frac{2}{3} \cdot 2^k) > 4 \cdot 2^k$ .  $\blacksquare$

We may now present our refinement of Proposition 3.3.

**Theorem 3.5.**  *$\underline{s}(C) = 1$  if and only if either  $T_C$  or  $2 \times T_P$  is isotropic.*

*For  $k > 1$ ,  $\underline{s}(C) \leq 2^k - 1$  if and only if  $\langle 1 \rangle \perp (2^k - 1) \times T_P$  is isotropic.*

*Proof.* For  $\underline{s}(C) = 1$ , we invoke Proposition 3.3.

Let us now consider the case where  $k > 1$ . Given Proposition 3.3, to prove that  $\underline{s}(C) \leq 2^k - 1$  if and only if  $\langle 1 \rangle \perp (2^k - 1) \times T_P$  is isotropic, it suffices to show that  $2^k \times T_P$  isotropic implies that  $\langle 1 \rangle \perp (2^k - 1) \times T_P$  is isotropic in this case.

Lemma 3.4 states that  $2^k \times T_P$  isotropic implies that  $(1 + [\frac{2}{3} \cdot 2^k]) \times T_P$  is isotropic. For  $k \geq 2$ ,  $1 + [\frac{2}{3} \cdot 2^k] \leq 2^k - 1$ , implying that  $(1 + [\frac{2}{3} \cdot 2^k]) \times T_P$  is a subform of  $\langle 1 \rangle \perp (2^k - 1) \times T_P$ . Hence,  $2^k \times T_P$  isotropic implies that  $\langle 1 \rangle \perp (2^k - 1) \times T_P$  is isotropic for  $k > 1$ .  $\blacksquare$

That  $1 + [\frac{2}{3} \cdot 2] \not\leq 1$  justifies why we cannot extend this proof to the case where  $k = 1$ . Indeed it is necessary to consider the isotropy of both  $T_C$  and  $2 \times T_P$  in this case, as the following examples attest:

**Example 3.6.** Let  $C = (\frac{-1, -1, -1}{\mathbb{R}})$ . Then  $s(C) = \underline{s}(C) = 1$ ,  $2 \times T_P$  is anisotropic and  $T_C$  is isotropic.

**Example 3.7.** Let  $C = (\frac{2, 5, t}{\mathbb{Q}_5(t)})$ . Then  $s(C) = 1$ , since  $-1 \in \mathbb{Q}_5^{\times 2}$ ,  $\underline{s}(C) = 1$ ,  $T_C$  is anisotropic and  $2 \times T_P$  is isotropic.

We remark that “ $\underline{s}(C) \leq n$ ” and “ $\langle 1 \rangle \perp n \times T_P$  is isotropic” are not equivalent for all  $n \neq 1$ . In [K], Koprowski produced an example of a quaternion division algebra of level  $2^k$  such that  $\langle 1 \rangle \perp 2^k \times T_P$  is anisotropic. Since for  $k \geq 2$ ,  $s(C) = 2^k$  implies  $\underline{s}(C) = 2^k$  (Theorem 3.11 below), Koprowski's example is such that  $\underline{s}(C) = 2^k$  and  $\langle 1 \rangle \perp 2^k \times T_P$  is anisotropic. Indeed, we may now present a simplified version of Koprowski's proof of the existence of such a quaternion division algebra:

**Theorem 3.8.** (Koprowski) For any positive integer  $k$  there exists a quaternion division algebra  $D$  such that  $-1$  is a sum of  $2^k$  squares of quaternions in  $D$  but  $-1$  is not a sum of  $2^k$  squares of pure quaternions. Moreover,  $s(D) = 2^k$ .

*Proof.* Let  $F$  be a formally real field of Pythagoras number  $2^k + 1$ . Let  $c_0 \in F$  be a sum of  $2^k + 1$  squares in  $F$  but no fewer and consider  $D := (\frac{-c_0, x}{F(x)})$ .

Since  $(2^k + 1) \times \langle 1 \rangle \perp (2^k - 1) \times T_P$  is isotropic over  $F(x)$ ,  $s(D) \leq 2^k$  ([L, Theorem 2.2]).

Now  $\langle 1 \rangle \perp 2^k \times \langle -c_0, x, c_0x \rangle$  is anisotropic over  $F(x)$  if and only if  $\langle 1 \rangle \perp 2^k \times \langle -c_0 \rangle$  and  $2^k \times \langle 1, c_0 \rangle$  are anisotropic over  $F$ , by Springer's Theorem (see [S, p.209]).  $\langle 1 \rangle \perp 2^k \times \langle -c_0 \rangle$  isotropic over  $F$  implies that  $c_0$  is a sum of  $2^k$  squares, which is a contradiction.  $2^k \times \langle 1, c_0 \rangle$  is also anisotropic over  $F$ , since it is a positive definite

form over a formally real field. Thus  $\langle 1 \rangle \perp 2^k \times \langle -c_0, x, c_0x \rangle$  is anisotropic. Hence,  $s(D) \not\leq 2^k - 1$  by Lemma 3.9 below. Thus  $s(D) = 2^k$ .

Finally,  $D$  is division since  $s(D) \neq 1$ . ■

**Lemma 3.9.** *Let  $n + 1$  be a power of 2. Then  $s(C) \leq n \iff \langle 1 \rangle \perp n \times T_P$  is isotropic.*

*Proof.* For  $C$  a quaternion algebra, see [Lew2, Comment]. Let us assume that  $C = \left( \frac{a, b, c}{F} \right)$ .

Clearly  $\langle 1 \rangle \perp n \times T_P$  isotropic implies that  $-1$  is a sum of  $n$  squares in  $C$ .

Conversely, suppose  $s(C) \leq n$ . Thus

$$-1 = \sum_{i=1}^{2^k-1} p_i^2 + a \sum_{i=1}^{2^k-1} q_i^2 + b \sum_{i=1}^{2^k-1} r_i^2 - ab \sum_{i=1}^{2^k-1} s_i^2 + c \sum_{i=1}^{2^k-1} t_i^2 - ac \sum_{i=1}^{2^k-1} u_i^2 - bc \sum_{i=1}^{2^k-1} v_i^2 + abc \sum_{i=1}^{2^k-1} w_i^2$$

and

$$\sum_{i=1}^{2^k-1} p_i q_i = \sum_{i=1}^{2^k-1} p_i r_i = \sum_{i=1}^{2^k-1} p_i s_i = \sum_{i=1}^{2^k-1} p_i t_i = \sum_{i=1}^{2^k-1} p_i u_i = \sum_{i=1}^{2^k-1} p_i v_i = \sum_{i=1}^{2^k-1} p_i w_i = 0.$$

Letting  $p_{2^k} = 1$  and  $q_{2^k} = r_{2^k} = s_{2^k} = t_{2^k} = u_{2^k} = v_{2^k} = w_{2^k} = 0$ , we obtain

$$0 = \sum_{i=1}^{2^k} p_i^2 + a \sum_{i=1}^{2^k} q_i^2 + b \sum_{i=1}^{2^k} r_i^2 - ab \sum_{i=1}^{2^k} s_i^2 + c \sum_{i=1}^{2^k} t_i^2 - ac \sum_{i=1}^{2^k} u_i^2 - bc \sum_{i=1}^{2^k} v_i^2 + abc \sum_{i=1}^{2^k} w_i^2$$

and

$$\sum_{i=1}^{2^k} p_i q_i = \sum_{i=1}^{2^k} p_i r_i = \sum_{i=1}^{2^k} p_i s_i = \sum_{i=1}^{2^k} p_i t_i = \sum_{i=1}^{2^k} p_i u_i = \sum_{i=1}^{2^k} p_i v_i = \sum_{i=1}^{2^k} p_i w_i = 0.$$

Multiplying by  $\sum_{i=1}^{2^k} p_i^2$ , we have

$$0 = \left( \sum_{i=1}^{2^k} p_i^2 \right)^2 + a \sum_{i=1}^{2^k} q_i'^2 + b \sum_{i=1}^{2^k} r_i'^2 - ab \sum_{i=1}^{2^k} s_i'^2 + c \sum_{i=1}^{2^k} t_i'^2 - ac \sum_{i=1}^{2^k} u_i'^2 - bc \sum_{i=1}^{2^k} v_i'^2 + abc \sum_{i=1}^{2^k} w_i'^2,$$

for certain  $q_i', r_i', s_i', t_i', u_i', v_i', w_i' \in F$  by [S, p.151], where we may assume that  $q_1' = \sum_{i=1}^{2^k} p_i q_i = 0$ ;  $r_1' = \sum_{i=1}^{2^k} p_i r_i = 0$ ;  $s_1' = \sum_{i=1}^{2^k} p_i s_i = 0$ ;  $t_1' = \sum_{i=1}^{2^k} p_i t_i = 0$ ;  $u_1' = \sum_{i=1}^{2^k} p_i u_i = 0$ ;  $v_1' = \sum_{i=1}^{2^k} p_i v_i = 0$  and  $w_1' = \sum_{i=1}^{2^k} p_i w_i = 0$ . Thus,  $\langle 1 \rangle \perp (2^k - 1) \times T_P$  is isotropic. ■

**Corollary 3.10.** (i) For  $k \geq 2$ ,  $\underline{s}(C) \leq 2^k - 1 \iff s(C) \leq 2^k - 1$ .

(ii) Let  $\underline{s}(C) = n$  and let  $k \geq 2$  be such that  $2^{k-1} \leq n < 2^k$ . Then  $s(C) \leq 2^k - 1$ .

If  $\underline{s}(C) = 1$ , we have that  $s(C) \leq 2$ .



*Proof.* Part (i) follows immediately from combining Theorem 3.5 and Lemma 3.9. For  $2 \leq \underline{s}(C) < \infty$ , part (ii) merely represents the left to right implication in (i). If  $\underline{s}(C) = 1$ , then either  $T_C$  or  $2 \times T_P$  is isotropic, by Proposition 3.3. If  $2 \times T_P$  is isotropic then it is universal, and hence represents  $-1$ , implying that  $s(C) \leq 2$ . Alternatively,  $T_C$  isotropic implies that  $-1 \in D_F(T_P)$ , and hence that  $s(C) = 1$ . ■

As a consequence of Corollary 3.10, we may now extend the following results of Leep ([L, Theorem 2.5]) to composition algebras:

**Theorem 3.11.** (i) For  $k \geq 2$ ,  $\underline{s}(C) = 2^k - 1 \implies s(C) = 2^k - 1$ .  
(ii) For  $k \geq 2$ ,  $s(C) = 2^k \implies \underline{s}(C) = 2^k$ .  
(iii) For  $k \geq 1$ ,  $s(C) = 2^k + 1 \implies \underline{s}(C) = 2^k$  or  $2^k + 1$ .

*Proof.* For  $k \geq 2$ , these results follow from Corollary 3.10. For  $k = 1$  in (iii),  $s(C) = 3$  implies  $\underline{s}(C) \leq 3$ . Since  $\underline{s}(C) = 1$  implies  $s(C) \leq 2$  (Corollary 3.10), we have that  $s(C) = 3$  implies  $\underline{s}(C) = 2$  or  $3$ . We note that we cannot extend (i) or (ii) to the case  $k = 1$ , as for  $C = \left(\frac{2.5}{\mathbb{Q}}\right)$ ,  $\underline{s}(C) = 1$  and  $s(C) = 2$ . ■

**Proposition 3.12.** If  $-1 \notin F^{\times 2}$ , then  $s(C) = 1$  if and only if  $T_C$  is isotropic.

*Proof.* Suppose  $s(C) = 1$ . Thus,  $-1 = y^2$ , for some  $y = p + qi + rj + sk + te + uie + vje + wke \in C$ . Hence, applying the trace we have  $t_C(-1) = t_C(y^2)$ , which in turn implies that  $-1 = T_C(y)$ . So

$$-1 = p^2 + aq^2 + br^2 - abs^2 + ct^2 - acu^2 - bcv^2 + abcw^2$$

and

$$-1 = y^2 = p^2 + aq^2 + br^2 - abs^2 + ct^2 - acu^2 - bcv^2 + abcw^2 + \underbrace{2p(qi + \dots + wke)}_0.$$

Thus if  $q = \dots = w = 0$ , we obtain the contradiction that  $-1 \in F^{\times 2}$ . Hence  $p = 0$  and  $-1 = aq^2 + \dots + abcw^2$ , implying that  $0 = 1 + aq^2 + \dots + abcw^2$ , that is  $T_C$  is isotropic.

Conversely,  $T_C$  isotropic implies that there exists  $y \in C^\times$  such that  $T_C(y) = p^2 + aq^2 + br^2 - abs^2 + ct^2 - acu^2 - bcv^2 + abcw^2 = 0$ . If  $T_P$  is isotropic, it is universal and hence  $-1 \in D_F(T_P)$ , implying that  $s(C) = 1$ . Alternatively,  $p \neq 0$  implies that  $1 + a\left(\frac{q}{p}\right)^2 + \dots + abc\left(\frac{w}{p}\right)^2 = 0$ , or that  $\left(\left(\frac{q}{p}\right)i + \dots + \left(\frac{w}{p}\right)ke\right)^2 = -1$ . ■

Recall that neither “ $s(C) \leq n$ ” nor “ $\underline{s}(C) \leq n$ ” is equivalent to “ $\langle 1 \rangle \perp n \times T_P$  is isotropic” for all values of  $n$ . However, for any value of  $n$  there exists a field  $F$  such that “ $s(C) \leq n$ ”, “ $\underline{s}(C) \leq n$ ” and “ $\langle 1 \rangle \perp n \times T_P$  is isotropic” are equivalent:

**Theorem 3.13.** *Suppose that the Pythagoras number  $p(F) = n$  of  $F$  is finite. Then*

$$s(C) < \infty \iff s(C) \leq n \iff \langle 1 \rangle \perp n \times T_P \text{ is isotropic.}$$

*Proof.* Clearly  $\langle 1 \rangle \perp n \times T_P$  isotropic implies that  $s(C) \leq n$ , which obviously implies that  $s(C) < \infty$ .

Conversely, suppose  $s(C) < \infty$ . Hence  $\langle 1 \rangle \perp m \times T_C$  is isotropic for some  $m \in \mathbb{N}$ , that is  $\sum_{i=1}^{m+1} p_i^2 + a \sum_{i=1}^m q_i^2 + b \sum_{i=1}^m r_i^2 - ab \sum_{i=1}^m s_i^2 = 0$ . We may assume that  $p_i \neq 0$  for some  $i$ , without loss of generality. Multiply across by  $\sum_{i=1}^{m+1} p_i^2$ . Since a product of sums of squares is itself a sum of squares,  $p(F) = n$  implies that  $\langle 1 \rangle \perp n \times T_P$  is isotropic.  $\blacksquare$

**Corollary 3.14.** *Suppose that  $p(F) = n$  is finite. Then  $\underline{s}(C) < \infty \iff \underline{s}(C) \leq n \iff \langle 1 \rangle \perp n \times T_P$  is isotropic.*

We remark that  $p(F) < \infty$  does not imply that  $s(D) < \infty$ , as  $p(\mathbb{Q}(x, y)) \leq 8$  ([Pf2]) whereas  $s\left(\left(\frac{x, y}{\mathbb{Q}(x, y)}\right)\right) = \infty$  for example.

The above fact can be viewed as a demonstration that  $\mathbb{Q}(x, y)$  is not a SAP field. A field  $F$  is said to satisfy the *Strong Approximation Property* (or is SAP, for short) if for all  $a, b \in F^\times$ , there exists  $n \in \mathbb{N}$  such that  $n \times \langle 1, a, b, -ab \rangle$  is isotropic. We note that this condition is equivalent to all quaternion algebras over  $F$  having finite level (and equivalently, finite sublevel).

Finally, we present the following bounds for the level and sublevel of quaternion and octonion algebras over some specified fields.

**Proposition 3.15.** *Let  $D$  be a quaternion division algebra and  $O$  an octonion division algebra.*

- (i) *If  $F = \mathbb{Q}_p$  for  $p$  a prime, then  $s(O) = \underline{s}(O) = 1$ .*
- (ii) *If  $F$  is an algebraic number field, then  $s(O) = \underline{s}(O) = 1$ .*
- (iii) *Let  $F = \mathbb{Q}_p(t)$ , for  $p \neq 2$  a prime. Then  $\underline{s}(D) \leq 2$  and  $s(D) \leq 2$ , whereas  $\underline{s}(O) = 1$  and  $s(O) \leq 2$ .*
- (iv) *Let  $F = \mathbb{Q}(t)$ . If  $s(D) < \infty$ ,  $\underline{s}(D) \leq 5$  and  $s(D) \leq 5$ . Also, for  $s(O) < \infty$ ,  $\underline{s}(O) \leq 5$  and  $s(O) \leq 5$ . Moreover, there exists a  $D$  such that  $s(D) = \underline{s}(D) = \infty$ .*

*Proof.* (i) If  $F = \mathbb{Q}_p$ , then any quadratic form of dimension  $\geq 5$  is isotropic over  $F$ . Hence,  $T_P$  is isotropic and so represents  $-1$ . Thus,  $-1$  is the square of a pure octonion, implying that  $s(O) = 1$ .

(ii) If  $F$  is an algebraic number field, then  $T_O = \langle 1 \rangle \perp T_P$  is isotropic since it is indefinite and of dimension  $> 4$  (see [S, p.224]), hence  $T_P$  represents  $-1$  and  $s(O) = 1$ .

(iii) For  $F = \mathbb{Q}_p(t)$ ,  $s(D) \leq 2$  since  $s(\mathbb{Q}_p(t)) \leq 2$ . Indeed this bound for  $s(D)$  and  $\underline{s}(D)$  is the best possible, as for  $p \equiv 3 \pmod{4}$ ,  $\underline{s}\left(\left(\frac{-p,t}{\mathbb{Q}_p(t)}\right)\right) = s\left(\left(\frac{-p,t}{\mathbb{Q}_p(t)}\right)\right) = 2$ . Parimala and Suresh proved that all forms over  $\mathbb{Q}_p(t)$  of dimension greater than 10 are isotropic ([PS]). Hence  $2 \times T_P$  is isotropic, which implies that  $\underline{s}(O) = 1$ .

(iv) Finally, for  $F = \mathbb{Q}(t)$ , we know that  $p(\mathbb{Q}(t)) = 5$  (see [Pf2, p.100]). That 5 is an upper bound for  $s(C)$ , when finite, follows immediately from Theorem 3.13. Since  $\mathbb{Q}(t)$  is not SAP, there exists a quaternion division algebra of infinite level and sublevel over  $\mathbb{Q}(t)$ .  $\blacksquare$

For  $F = \mathbb{Q}_p(t)$ , we have no example of an octonion algebra of level 2. Finding such an  $O$  over  $\mathbb{Q}_p(t)$ , where  $p \equiv 3 \pmod{4}$ , is equivalent to finding an anisotropic  $T_O$ .

#### 4. CONSTRUCTING QUATERNION AND OCTONION ALGEBRAS OF PRESCRIBED LEVEL AND SUBLEVEL

Our constructions will employ transcendental field extensions. Firstly, we note that moving to a transcendental field extension alone will not reduce the sublevel of a quaternion algebra:

**Proposition 4.1.** *Let  $x$  be transcendental over  $F$ . Then  $\underline{s}\left(\left(\frac{a,b}{F}\right)\right) = \underline{s}\left(\left(\frac{a,b}{F(x)}\right)\right)$ .*

*Proof.* Clearly  $\underline{s}\left(\left(\frac{a,b}{F(x)}\right)\right) \leq \underline{s}\left(\left(\frac{a,b}{F}\right)\right)$ .

Suppose  $\underline{s}\left(\left(\frac{a,b}{F(x)}\right)\right) = n$ . Then

$$\sum_{i=1}^{n+1} p_i(x)^2 + a \sum_{i=1}^{n+1} q_i(x)^2 + b \sum_{i=1}^{n+1} r_i(x)^2 - ab \sum_{i=1}^{n+1} s_i(x)^2 = 0$$

and

$$\sum_{i=1}^{n+1} p_i(x)q_i(x) = \sum_{i=1}^{n+1} p_i(x)r_i(x) = \sum_{i=1}^{n+1} p_i(x)s_i(x) = 0.$$

By clearing out the denominators, we may assume that  $p_i(x), q_i(x), r_i(x)$  and  $s_i(x) \in F[x]$ . Moreover, dividing across by the lowest power of  $x$  in  $\sum_{i=1}^{n+1} p_i(x)^2, \sum_{i=1}^{n+1} q_i(x)^2,$

$\sum_{i=1}^{n+1} r_i(x)^2$  and  $\sum_{i=1}^{n+1} s_i(x)^2$ , we obtain

$$\sum_{i=1}^{n+1} \alpha_i^2 + a \sum_{i=1}^{n+1} \beta_i^2 + b \sum_{i=1}^{n+1} \gamma_i^2 - ab \sum_{i=1}^{n+1} \delta_i^2 = 0$$

and

$$\sum_{i=1}^{n+1} \alpha_i \beta_i = \sum_{i=1}^{n+1} \alpha_i \gamma_i = \sum_{i=1}^{n+1} \alpha_i \delta_i = 0,$$

for  $\alpha_i, \beta_i, \gamma_i, \delta_i \in F$ , not all zero. Hence,  $\underline{s}\left(\left(\frac{a,b}{F}\right)\right) \leq n$ . ■

We may employ function field techniques to construct an example of a field of level  $2^k$  for all  $k$ :

**Example 4.2.**  $s(F_0((2^k + 1) \times \langle 1 \rangle)) = 2^k$ .

This result follows as an immediate consequence of [H1, Theorem 1], since  $s(F) \leq n \iff (n+1) \times \langle 1 \rangle$  is isotropic.

The following proposition provides a simple method of constructing quaternion and octonion algebras of level  $2^k$  for all  $k$ . Moreover, for  $k \geq 1$  these algebras are division, since  $D$  not division implies that  $s(D) = 1$ .

**Proposition 4.3.**  $s(C) = s(F)$ , for  $C = \left(\frac{x,y}{F(x,y)}\right)$  or  $\left(\frac{x,y,z}{F(x,y,z)}\right)$ .

*Proof.* Clearly  $s(C) \leq s(F)$ . Conversely,  $s(C) = n \implies (n+1) \times \langle 1 \rangle \perp n \times T_P$  is isotropic. Repeated applications of Springer's Theorem imply that  $(n+1) \times \langle 1 \rangle$  is isotropic over  $F$ , and hence that  $s(F) \leq n$ . ■

**Example 4.4.**  $\left(\frac{x,y}{F_0(x,y)}\right) \otimes_{F_0(x,y)} F_0(x,y)((2^k + 1) \times \langle 1 \rangle)$  is a quaternion division algebra of level  $2^k$ , for  $k \geq 1$ .

**Example 4.5.**  $\left(\frac{x,y,z}{F_0(x,y,z)}\right) \otimes_{F_0(x,y,z)} F_0(x,y,z)((2^k + 1) \times \langle 1 \rangle)$  is an octonion division algebra of level  $2^k$ , for  $k \geq 1$ .

Lewis constructed a quaternion division algebra of level  $2^k + 1$  in [Lew2]. In [KW], Krüskemper and Wadsworth constructed a quaternion algebra of sublevel 3. The following proposition extends these constructions to octonion algebras:

**Proposition 4.6.** Let  $D = \left(\frac{a,b}{F}\right)$  and  $O = \left(\frac{a,b,c}{F(c)}\right)$ , where  $c$  is transcendental over  $F$ .

- (i) If  $D$  is of level  $2^k + 1$ , then  $O$  is also of level  $2^k + 1$ .
- (ii) If  $D$  is of sublevel 3, then  $O$  is also of sublevel 3.

*Proof.* (i) Clearly  $s(O) \leq s(D) = 2^k + 1$ . Now suppose  $s(O) \leq 2^k$ . By the argument used in [Lew2, Lemma 1], either  $(2^k + 1) \times \langle 1 \rangle \perp (2^k - 1) \times T_P$  or  $\langle 1 \rangle \perp 2^k \times T_P$  is isotropic over  $F(c)$ . By Springer's Theorem, either  $(2^k + 1) \times \langle 1 \rangle \perp (2^k - 1) \times \langle a, b, -ab \rangle$ ,  $\langle 1 \rangle \perp 2^k \times \langle a, b, -ab \rangle$  or  $2^k \times \langle 1, -a, -b, ab \rangle$  is isotropic over  $F$ . If  $(2^k + 1) \times \langle 1 \rangle \perp (2^k - 1) \times \langle a, b, -ab \rangle$  is isotropic over  $F$ , then  $s(D) \leq 2^k$  by [L, Theorem 2.2], which is a contradiction. Clearly  $\langle 1 \rangle \perp 2^k \times \langle a, b, -ab \rangle$  isotropic over  $F$  also implies that  $s(D) \leq 2^k$ . Hence  $2^k \times \langle 1, -a, -b, ab \rangle$  is isotropic over  $F$  and hence hyperbolic, since it is a Pfister form. Thus,  $(1 + [\frac{2}{3} \cdot 2^k]) \times T_P$  is isotropic (see Lemma 3.4), implying that  $2^k \times T_P$  is isotropic and hence that  $s(D) \leq 2^k$ , which is a contradiction.

(ii) Clearly  $\underline{s}(O) \leq \underline{s}(D) = 3$ . Now suppose  $\underline{s}(O) \leq 2$ . Hence  $3 \times T_O$  is isotropic. By Proposition 4.1, for  $\sum_{i=1}^3 x_i^2 = -1$ , there must exist  $x_i \in O \setminus D$  for some  $i$ . Hence the subform  $3 \times c\langle 1, -a, -b, ab \rangle$  of  $3 \times T_O$  is isotropic over  $F(c)$ . Thus  $3 \times \langle 1, -a, -b, ab \rangle$  is isotropic over  $F$ , implying that  $4 \times \langle 1, -a, -b, ab \rangle$  is isotropic and hence hyperbolic, since it is a Pfister form. Thus,  $3 \times \langle -a, -b, ab \rangle$  is isotropic over  $F$ , implying that  $\underline{s}(D) \leq 2$ , contrary to our hypothesis. ■

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