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Assessing late-time singular behaviour in models of three dimensional Euler fluid flow

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11211545

Thesis submitted to University College Dublin in fulfillment of the requirements for the degree of Doctor of Philosophy in

Applied and Computational Mathematics (Major in Simulation Science)

School of Mathematics and Statistics

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March 2016
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Abstract

The open question of regularity of the fluid dynamical equations is considered one of the most fundamental challenges of mathematics and physics [36]. While the viscous Navier-Stokes equations have more physical relevance, the inviscid Euler equations present the greatest challenge and exhibit the most extreme behaviours. For this reason, the numerical study of possible finite-time blowup is typically concerned with these inviscid equations. Extensive numerical assessment of finite-time blow up of 3D Euler has been carried out, albeit with conflicting yes and no conclusions with regard to the existence of finite time singularity.

The fundamental difficulty of this important problem is the lack of analytic solutions or any a priori knowledge of asymptotic behaviour. A secondary obstacle is that the spatial collapse associated with intense vortex stretching results in numerical solutions becoming unresolved beyond a certain time. It is therefore imperative to devise a framework with nontrivial blowup dynamics and where analytic solutions are known in order to validate and compare various numerical methods, for the purposes of accurately solving the system and diagnosing blowup.

In this regard, I have proposed investigating the issue of Euler finite-time blowup using a novel approach where the original system of equations is bijectively transformed to a new mapped system which is globally regular in time [11]. Since no known analytical solution for the full 3D Euler equations exist, I have studied the robustness of the proposed novel approach using the one-dimensional Burgers equation and a proposed new one-parameter family of models of the 3D Euler equations on a 2D symmetry plane. The proposed 2D symmetry plane model equations were motivated by the work on stagnation-point-type exact solution of 3D Euler equations by Gibbon et al. [43].

I have shown that the mapped system’s numerical solution leads to more accu-
rate estimates of the blowup quantities compared with the original system. I also established that only by using the mapped system can certain late-time behaviours be observed and asymptotic trends be established.
Statement of Original Authorship

I hereby certify that the submitted work is my own, was completed while registered as a candidate for the degree stated on the Title page, and I have not obtained a degree elsewhere on the basis of the research presented in this thesis.
Sponsor

This work was supported by the University College Dublin Structured Ph.D. Programme in Simulation Science, funded under the Programme for Research in Third Level Institutions (PRTLI) Cycle 5 and co-funded by the European Regional Development Fund (ERDF).
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Last, but certainly not least, I would like to thank my family and friends for their love, support and patience.
List of Publications

Journals

During the course of this thesis the following works have been published or accepted for publication:


Conference oral abstracts


Chapter 1

Incompressible Inviscid Fluids

The three dimensional incompressible Euler fluid equations represent a triple point between the areas of engineering, physics and mathematics. Originally derived by Leonhard Euler [34], these equations have stood firm after 250 years of research, playing a pivotal role in the description of fluids of all types. This pivotal role lies in the mathematical modelling and numerical simulations of physical phenomena taking place in fluids. Quoting Gibbon [46] regarding the 3D Euler fluid equations: “...their siren song has tempted many young scientists, somewhat like Ulysses, towards the twin rocks called Frustration and Despair.” To aid Ulysses in his quest, we should make him aware of the fact that behind one of the twin rocks there lingers a big vortex, in the form of a dangerous whirlpool.

One of the main challenges these equations pose is that it is not known in detail how the energy content is transferred throughout spatial scales. Efforts towards understanding this cascade process have generated significant cross-fertilisation across disciplines of research. For real-life problems, understanding this process is needed in order to optimise industrial production of metallic alloys, gas and oil extraction and transport, and performance of turbo-machinery in general. From atmospheric science and oceanography to plasma physics, the governing
fluid equations share the same feature: nonlinear terms due to advection and pressure, which carry along transfers of energy throughout different length scales, which makes the simulation and modelling a very difficult task. The main difficulties from the practical point of view have to do with accuracy and stability of the numerical solutions. For example, numerical weather prediction relies on accurate models to improve the skill of a forecast. Interestingly, the same difficulties arise in the mathematical problem of determining whether the solution of the 3D Euler equations develops a singularity in a finite time. Also, accurate modelling of the 3D Euler and Navier-Stokes fluid equations can shed light on the unsolved problem of turbulence [65], defined as a hypothetical out-of-equilibrium nonlinear regime characterised by intermittent fluxes of energy across scales, whereby the fluid’s degrees of freedom exhibit quasi-periodic oscillations that are amenable to statistical analyses.

1.1 The 3D Euler equations

Let us consider the 3D incompressible Euler equations with uniform, unit mass density for the velocity field \( u(x, y, z, t) \in \mathbb{R}^3 \) defined for \((x, y, z) \in \mathbb{R}^3\) and in a time interval \( t \in [0, T) \):

\[
\frac{\partial u}{\partial t} + u \cdot \nabla u = -\nabla p
\]

(1.1)

with

\[
\nabla \cdot u = 0,
\]

(1.2)

where \( p(x, y, z, t) \) is the pressure. The Euler equations (1.1, 1.2) describe the motion of inviscid and incompressible fluid flow. They are used to model the behavior of real fluid flows which do not account for the effects of viscosity and is thus a good approximation for flows in which these effects are negligible. Equation (1.1)
is Newton’s second law, which states that the acceleration of a fluid particle is proportional to the pressure force acting on it. Equation (1.2) is the incompressibility condition, stating that the volume of any part of the fluid does not change under flow.

The Navier-Stokes equations are defined by

\[
\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + \nu \Delta \mathbf{u}, \quad \nabla \cdot \mathbf{u} = 0
\] (1.3)

which are the Euler equations (1.1, 1.2) with an extra term and \( \nu \) is the fluid viscosity, a measure of the fluid’s resistance to deformation by shear stress.

### 1.1.1 Vorticity formulation

A study of the rotation of the fluid is usually paramount. This is defined by the vorticity (curl of velocity) \( \omega \), which measures the amount of rotation of the flow. We define the vorticity field as the three-dimensional vector field \( \omega \equiv \nabla \times \mathbf{u} \). Inviscid fluids have the remarkable property that vorticity is transported (and sometimes stretched) along streamlines [79]. By taking the curl on both sides of equation (1.1), we obtain the following equation governing the evolution of vorticity \( \omega \):

\[
\partial_t \omega + \mathbf{u} \cdot \nabla \omega = \omega \cdot \nabla \mathbf{u}.
\] (1.4)

This is a quadratic equation because \( \nabla \mathbf{u} \) is of the same order as \( \omega \). This quadratic nonlinearity is the main difficulty in studying the dynamic stability and global regularity of the 3D Euler equations [56].

The vorticity \( \omega \) and the velocity \( \mathbf{u} \) are related by the so-called Biot-Savart law

\[
\mathbf{u}(x, t) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{x - y}{|x - y|^3} \times \omega(y, t) \, dy,
\] (1.5)
when the quantities under consideration have sufficient decay at infinity.

1.2 The open problem of Euler singularity

1.2.1 Statement of the problem

One of the open questions in fluid dynamics is whether the incompressible three-dimensional Euler equations (1.1, 1.2) develop a singularity in the vorticity field in a finite time. It is convenient to use Sobolev space $H^m(\mathbb{R}^3)$, consisting of functions whose derivatives up to order $m$ are in $L^2(\mathbb{R}^3)$ where $m$ is a positive integer and the norm of $u$ in $H^m$ is denoted by $|u|_m$. The Euler singularity problem is whether, given a smooth initial value $u_0$ in Sobolev space $H^m$ (e.g., $u_0 \in H^m$ for $m > 5/2$), will there be a finite time $T^*$ such that the solution $u(x, T^*)$ will cease to be in $H^m$? The development of singularity in Euler equations involves development of large gradients in the velocity field. Since no friction is limiting the increase in velocity gradients, infinite momentum is not mandatory for a blow up of the Euler equations. Opinion is largely divided on the matter with strong positions taken on either side. The rapid accumulation of vorticity from varied initial conditions is not disputable. Whether the accumulation is sufficiently rapid to manifest singular behaviour or whether the growth is merely exponential, has not been answered definitively.

1.2.2 The Beale-Kato-Majda (BKM) criterion

One of the most important conditional results known to date regarding regularity of classical (as opposed to ‘weak’) solutions to the 3D Euler fluid equations, is the so-called Beale-Kato-Majda (BKM) theorem introduced by Beale, Kato & Majda [5, 37, 93], which basically states that all $L^2$ Sobolev norms of the velocity
field are bounded up to time $T$ provided the time integral, up to time $T$, of the supremum norm of vorticity is finite. By definition, the BKM criterion asserts that a smooth solution of the 3D Euler equations blows up at time $T^*$ if and only if the maximum norm of vorticity $\|\omega(t)\|_\infty$ grows very fast in time so that
\[
\int_0^{T^*} \|\omega(t)\|_\infty dt = \infty.
\]
(1.6)

To apply the BKM criterion, a bound to the speed of growth of the maximum norm of vorticity $\|\omega(t)\|_\infty$ is given such that asymptotically it satisfies an inverse power law
\[
\|\omega(t)\|_\infty \approx C(T^* - t)^{-\alpha},
\]
where $C$ and $\alpha$ are scaling parameters.

Several other criteria which use the direction of vorticity were developed by Constantin et al. [26], Chae [18] and Deng et al. [30, 31]. Kuznetsov & Ruban [69] analysed a blowup scenario based on breaking of vortex lines. Because the magnitude of vorticity controls the blowup (formation of singularity), most of the research around 3D Euler singularities has concentrated at looking closely at the temporal evolution of the maximum norm of vorticity $\|\omega(t)\|_\infty$. For 3D Euler fluid equations and other ideal equations, circulation theorems such as Kelvin’s dictate that a loss of regularity must be accompanied by the collapse of vortex tubes, or regions of localised vorticity. Therefore, for a given simulation, numerical methods and diagnostics will have progressive difficulties in resolution and efforts must concentrate on resolving the spatial scales.

1.2.3 Review of Euler singularity research

Numerous numerical simulations have been carried out over the last three decades in an attempt to determine, from smooth initial data, whether the vorticity field in
the three-dimensional Euler equations develops a singularity in a finite time. Despite these endeavours, no clear understanding of the behavior of solutions of the incompressible three-dimensional Euler equations has been attained. Numerical investigations on the three-dimensional Euler equations have predicted different phenomena with opposing conclusions regarding finite-time singularity (see, for example, figure 1.1).

Kerr [61] analysed the interaction of two perturbed antiparallel vortex tubes using Chebyshev method. The results were interpreted in favour of blowup where \( \|\omega(t)\|_\infty \approx (t_0 - t)^{-1} \) with a reported increase of the maximum vorticity by a factor of 24 (figure 1.1, top). However, Hou & Li [56, 57, 58] carried out numerical simulations for the same initial conditions using pseudospectral method. They extended the solution beyond the estimated blowup time by Kerr [61] and showed that the evolution of the maximum vorticity was slower than doubly exponential (figure 1.1, bottom). Bustamante & Kerr [15] carried further studies and concluded that the qualitative results were in agreement with Kerr [61] but with a modification of the scaling laws, \( \|\omega(t)\|_\infty \approx (t_0 - t)^{-\gamma} \) for \( \gamma > 1 \), and singularity time \( t_0 \).

Pre-2008 efforts are summarised in the reviews [4, 46] and include several papers in the Proceedings of the international conference “Euler equations: 250 years on”, notably [15, 20, 42, 51, 52, 53, 60, 90, 91]. As for post-2008 efforts, we highlight:

In Orlandi et al. [89], finite difference methods are used to study numerical simulations of the 3D Euler and Navier-Stokes equations in the context of the Chaplygin-Lamb dipole initial conditions. These initial conditions have reduced regularity in the sense that only low-order \( L^2 \) Sobolev norms of the velocity field exist: the initial vorticity has bounded support. The low regularity of the solution in this case makes most of the available theorems inapplicable, so the focus is put
Figure 1.1: Isosurfaces of vorticity from two studies of finite-time blow-up with antiparallel vortex tubes. Kerr [61], top panel, argued in favour of singularity behaviour, while Hou and Li [56], bottom panel, against it.
on the power-law of the energy spectrum, $E(k, t) \sim k^{-n(t)}$, where the exponent $n(t)$ seems to tend to the value 3 at late times, consistent with a finite-time singularity in 3D Euler [61]. Cichowlas et al. [19] observed that the Euler equation has long-lasting transients which behave just as those of high-Reynolds number viscous flow; in particular they found an approximately $k^{5/3}$ inertial range followed by a dissipative range.

Kerr [62] returns to an extended geometry case of the well studied [56, 61] antiparallel vortex tube candidate initial conditions. Kerr uses new bounds on $L^p$ norms of the vorticity field introduced by Gibbon [41] and shows that the system has two distinct behaviours: an early time power-law growth of (ordered) moments and late time super-exponential growth of moments (with broken ordering). The analysis of Gibbon’s moments [41] is extended in Donzis et al. [32] via four different pseudo-spectral methods by different research groups, to simulate 3D Euler and Navier-Stokes equations, with regular (in fact, analytical) initial conditions, obtaining evidence against finite-time singularity in both Euler and Navier-Stokes.

In Grafke et al. [50], adaptive mesh refinement methods are applied to the study of depletion of nonlinearity in the simulation of 3D Euler equations, with careful analyses of the bounds introduced by Deng et al. [30] on the local behaviour of vortex-line length and curvature near the vorticity maximum, with analytical initial conditions of the Kida-Pelz type, leading to no finite-time singularity.

In Luo & Hou [76, 77], the role of boundaries was addressed in a numerical simulation of axisymmetric 3D Euler equations in a cylinder, with strong evidence for a finite-time blowup. Boundaries are also the subject of a recent work by Kiselev & Zlatos [63] who show that the normally regular 2D Euler equations can exhibit finite-time singularity in a norm of vorticity when non-smooth bounded domains are considered.
To our knowledge, a thorough study is yet to materialise about the role of initial conditions on the singularity of the 3D Euler or Navier-Stokes equations. However, important steps towards this understanding have been taken in terms of nonlinear optimisation of initial conditions, starting with Lu et al. [75] and notably by Ayala et al. [2] in the 2D context.

It is worth mentioning some approaches that have tackled other models successfully, related to but differing from 3D Euler in key technical aspects that allow for exact results. Arguably, the first example of an integrable inviscid fluid singularity was presented by Vieillefosse [96] where the self motion of a Lagrangian “free” fluid element was considered via a local expansion of velocity and a pressure ansatz which, while satisfying conservation of angular momentum, allows energy to grow (see further work done by Cantwell [16]). Finite-time singularity was demonstrated in the generalised surface quasi-geostrophic equation: see Li & Rodrigo [74] and references therein. In shell models of turbulence, Mailybaev [78] demonstrated that inertial-range cascades of energy transfers are due to the succession of intermittent coherent structures in the form of finite-time blowups, described by universal self-similar characteristics. The Hamiltonian approach introduced by Kuznetsov & Ruban [70] allows the authors to deform the 3D Euler equations to an integrable model while keeping the vortex-line structure, establishing rigorously a finite-time blowup scenario based on the breaking of vortex lines.

1.2.4 Relationship between Euler singularity problem and the onset of turbulence

The characterization of turbulence, defined as a hypothetical out-of-equilibrium nonlinear regime described by intermittent fluxes of energy across scales whereby the fluid’s degrees of freedom exhibit quasi-periodic oscillations that are amenable
to statistical analyses, is among the most important fluid mechanics problems. The present description of turbulence, for example by Frisch [38], is built on the theory of Kolmogorov [65] whose key point is that at vanishing viscosity limit, energy dissipation has to be finite. There is some suggestion [22, 71] that the Euler singularity problem may be related to turbulence onset, although no rigorous relations have been established so far. For example, the vorticity dynamics in 3D Euler flow plays an important role in the energy cascade of turbulent flows thus providing a link between the late-time singularity structures and the dynamics of inertial range [6].

Numerical simulations of turbulence have shown that turbulent flows are governed by coherent structures [99]. In fully developed, homogeneous and isotropic turbulence these structures appear as filamentary vortex tubes, similar to the spatio-temporal structures observed in pre-shock singularities of 3D Euler fluid flow. These vortex tubes then generate a swirling velocity field interacting in a nonlinear manner, leading to the complex spatio-temporal structure of turbulent flows. The dynamic mechanism of advection, vortex stretching and diffusion of vorticity is usually observed during the interaction of these structures. The formation of Euler singularities may therefore signify the onset of turbulence and may be a mechanism for energy transfers to small scale (energy cascade) which is crucial to the development of a viable theory of turbulence [22].

1.3 Conserved quantities for the 3D Euler equations

The Euler equations (1.1, 1.2) are a result of two conservation laws: the conservation of mass and the conservation of momentum. In addition, a flow governed by the Euler equations exhibits conserved quantities beyond the two conservation laws. Smooth solutions of the 3D Euler equations conserve kinetic energy, helic-
ity [80] and circulation. The total kinetic energy is proportional to the $L^2$ norm of velocity and is defined as

$$ E = \frac{1}{2} \int |u|^2 \, d^3x. $$

The kinetic energy $E = \sum_k E(k,t)$ is conserved for smooth flows, where the energy spectrum $E(k,t)$ is defined as the sum of the Fourier coefficients $\hat{u}(k,t)$ over spherical shells (see details in appendix B.1),

$$ E(k, t) = \frac{1}{2} \sum_{k - \frac{1}{2} < |k| < k + \frac{1}{2}} |\hat{u}(k, t)|^2. $$

Kolmogorov [65] predicted that in a fully developed turbulent flow, the energy spectrum $E(k)$ in the inertial range is given by a power law of the form

$$ E(k) = C \varepsilon^{2/3} k^{-5/3} $$

where $\varepsilon$, a positive number independent of viscosity, is the average of the energy dissipation rate. Motivated by Kolmogorov’s theory of turbulence, Onsager [88] conjectured that the conservation of energy in 3D incompressible Euler flows occur if and only if the solutions are smoother than the velocities supporting the Kolmogorov theory [25]. Eyink [35] also proved the energy conservation using Besov space as opposed to the Hölder space $C^\alpha$ with exponent $\alpha > 1/2$, and the case when $\alpha > 1/3$ with the Hölder norm replaced by a stronger norm

$$ \|u\|_{\alpha} = \sum_k |k|^{\alpha} |\hat{u}_k|, \quad \hat{u}_k = \int_D u(x) e^{ik\cdot x} \, d^3x. $$

In order to describe the circulation and helicity, we first discuss particle paths in relation to vorticity, equation (1.4). A vortex line, at a particular time $t$, is a curve which has the same direction as the vorticity vector $\omega$ [1]. The vortex lines which pass through some simple closed curve in space are said to form the
boundary of a vortex tube.

Let \( C \) denote a closed circuit that consists of the same fluid particles as time proceeds. Then the circulation around \( C \) is defined as the flux of vorticity through the surface enclosed by the curve \( C \):

\[
\Gamma = \oint_C \mathbf{u} \cdot d\mathbf{x} = \int_S \omega \cdot \mathbf{n} \ dS
\]

where \( S \) spans \( C \) and \( \mathbf{n} \) is the normal to the surface \( S \). The direction of the normal \( \mathbf{n} \) to \( S \) is chosen from the orientation of \( C \) by the ‘right-hand rule’. The rate of change of circulation, \( \Gamma \) must be zero for any closed curve \( C \) based on Kelvin’s circulation theorem. In other words, Kelvin’s circulation theorem states that for inviscid fluid of uniform density with conservative forces, the circulation around a closed material curve remains constant.

The helicity of a fluid flow confined to a volume \( V \) (bounded or unbounded) is the integrated scalar product of the velocity field \( \mathbf{u} \) and the vorticity field \( \omega \) [80].

\[
H = \int_V \mathbf{u} \cdot \omega \ d^3x .
\]

The physical interpretation of the integral is that it provides a measure of the degree of knottedness and/or linkage of the vortex lines of the flow [81]. It is also a measure of the lack of mirror symmetry of the flow.

1.4 Motivation for current research

Computing Euler singularities numerically is an extremely challenging task and requires huge computational resources due to the tremendous resolutions required to capture near singular behaviour. Convergence studies have to be meticulously carried out to avoid interpreting the blowup of an under-resolved computation as
an evidence of finite time singularities [58]. Numerical integration of the 3D Euler equation exhibit a spatial distribution of vorticity which tends to get localised in structures that become sharp with time. Finite memory limitations inherent in computer simulations exhibits a finite-time loss of resolution. This extreme behaviour coupled with the absence of any known analytical solution present an opportunity to devise a framework with nontrivial blowup dynamics where analytical solutions can be established so as to accurately solve the system and diagnose blowup.

This has motivated the current novel approach involving bijective mapping of the original equations to a new system of equations which we refer to as mapped system, whose solutions are globally regular [11]. To illustrate the usefulness of the mapping, we have carried out numerical simulations of a well known nonlinear wave equation known as inviscid Burgers equation which has a known analytical solution. We have also illustrated proof of concept using a new one-parameter family of models of the 3D Euler on a 2D symmetry plane which we have introduced. The one-parameter family of symmetry plane models was motivated by the work on stagnation-point type of exact solutions (with infinite energy) of 3D Euler fluid equations by Gibbon, Fokas & Doering [43] and the subsequent demonstration of finite-time blowup by Constantin [23]. Our new models are seen as a deformation of the 3D Euler equations, which still respect the structure of the original equations so that explicit solutions can be found for the supremum norms of the basic fields. In particular, the value of the model’s parameter determines whether there is a finite-time blowup, and the singularity time can be computed explicitly in terms of the initial conditions and the model’s parameter. This is essential in ensuring that a comparison of the numerical solution from the original and mapped system can be done with reference to the analytical solution.
The state of the art exploited in this thesis spins out of the Beale-Kato-Majda theorem as a set of interesting applications:

(i) The bijective mapping to regular fields introduced by Bustamante [11] is motivated by the BKM Theorem idea (1.6). This is a nonlinear mapping of both time and velocity field, that transforms the original system to a globally (in time) regular system. The solution of the mapped system is amenable to numerical simulations using the same methods as in the original system, and the evidence indicates that the numerical simulation of the mapped equations should provide more accurate results than the numerical simulation of the original equations. The applicability of this mapping has a wide range, including Burgers equations, 3D and 2D models of Euler, Navier-Stokes and magneto-hydrodynamics (MHD).

(ii) The bridge between the BKM theorem and the analyticity-strip method, developed in Bustamante & Brachet [12] for 3D Euler and applied in Brachet et al. [8] for 3D MHD. This bridge implies that, if the initial condition is analytic with analyticity-strip width $\delta_0$, then the local blowup of a quantity (say a supremum norm of some field) must be accompanied by a fast-enough loss of analyticity of the solution. In the case of a finite-time singularity this means that the instantaneous analyticity-strip width $\delta(t)$ must go to zero in a finite time, at a fast-enough rate.

1.5 Structure of the thesis

The structure of this thesis is as follows. In chapter 2 we show as a review the analytical solution of some quantities (such as the maximum velocity gradient), using known methods, for the 1D inviscid Burgers equation. We then construct an explicit mapping to a new system of equations which is globally regular [11]
and which we refer to as mapped system. We compare the analytical solution for the maximum norm of velocity gradient, with the numerical data from direct numerical simulation of the original equation and the mapped system.

In chapter 3 we introduce a one-parameter family of models of the 3D Euler fluid equations on a 2D symmetry plane. We use a representative parameter value, for which the solution blows up in finite-time, as a benchmark for the systematic study of errors in numerical solutions. We compare numerical integrations of our original model equations with a mapped version of these equations. We show that the mapped system’s numerical solution increases accuracy in estimates of supremum norms and singularity time while entailing only a small additional computational cost. We study the Fourier spectrum of the model’s numerical solution and find that the analyticity-strip width (a measure of the solution’s analyticity) tends to zero as a power law in a finite time. This is in agreement with the finite-time blowup of the supremum norms, in the light of rigorous bounds stemming from the bridge [12] between the analyticity-strip method and the BKM type of theorems.

In chapter 4 we reinvestigate the infinite-energy, stagnation-point-type solutions of the 3D Euler nonlinear fluid equations [43], which exhibit finite-time blowup that can be assessed analytically [23]. We review and formulate the analytic and asymptotic results for blowup for both the original and mapped equations. A thorough investigation of the Fourier spectra of the solution is presented, followed by error analysis of our numerics and an assessment of singularity time and proximity to it. Crucially, the singularity time $t = T^*$ maps to $\tau = \infty$ in the mapped time variable. This helps us uncover curious late-time behaviour of the Fourier spectrum such that in the mapped variables the solution remains well converged spectrally at time $t$ close to singularity time $T^*$ well within floating-point precision: $T^* - t \ll 10^{-16}$. We find that the mapped variables maintain acceptable
levels of error in the main blowup quantities such as the $L^\infty$ and $L^2$ norms of the vortex stretching rate at this extreme closeness to $T^*$. 

A summary of the results presented in this work, along with a discussion of the implications stemming from these results is given in chapter 5. The interpolation method used in this thesis is discussed in appendix A, whereas details regarding the numerical methods used in this thesis are presented in appendix B. Appendix C provides details regarding generation of new symmetries of the 2D symmetry plane model equations. This helps in finding conservation laws which allows one to solve analytically for the blowup quantities of the symmetry plane model.
Chapter 2

The inviscid Burgers equation in one dimension

In this chapter, we study the 1D inviscid Burgers equation and discuss the analytical solution of the equation along characteristics. Using the method introduced by Bustamante [11] on mapping bijectively the original variables to a globally regular system based on the existence of a BKM type of theorem [5], we map the original time and velocity field variables to a new system of variables which we refer to as mapped.

A comparison of the numerical solution from the original equation and its mapped counterpart for the assessment of finite-time singularity is presented. We monitor errors arising from the two systems. With reference to the works of Sulem, Sulem and Frisch [94] and Bustamante and Brachet [12], we present a thorough study of the spectra of the spatial derivative of velocity and investigate the spatial structure of the blowup using the analyticity strip method.
2.1 The 1D Burgers equation

We consider the one dimensional Burgers partial differential equation (PDE)

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = \nu \frac{\partial^2 u}{\partial x^2},$$  \hspace{1cm} (2.1)

where $u$ is the velocity field and $\nu$ is the viscosity coefficient, a measure of a fluid’s resistance to shear stress. When $\nu = 0$, equation (2.1) becomes the inviscid Burgers equation (equation (2.2)) $\forall x \in \mathbb{R}, \forall t \in [0, T^*)$, where $T^*$ is the first time when shock occurs (singularity time):

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = 0,$$ \hspace{1cm} (2.2)

$$u(x + L, t) = u(x, t),$$

where $L$ is the periodicity, with an initial condition given by

$$u(x, 0) = u_0(x).$$

The Burgers equation is a nonlinear hyperbolic conservation law which has been used by fluid dynamicists to prove concepts due to the fact that it shares some essential difficulties with the 3D Euler equations (and Navier-Stokes equations) [59, 84]. The equation was initially used as a simplistic model for one dimensional turbulence [10] and has since attracted numerous studies both analytically and numerically [21, 48, 54, 67, 72, 84, 85, 86, 95, 97]. It is known to blow up in a finite time and possesses a BKM type of criterion for blowup. Also, all its relevant norms (supremum of gradient, Sobolev) can be found analytically in terms of the initial conditions, by the method of characteristics.
2.2 Analytical solution of inviscid Burgers equation

We now solve equation (2.2) using the method of characteristics. The method of characteristics involves reducing the PDE (equation (2.2)) to a system of ordinary differential equations (ODEs). This is accomplished by changing the coordinates from \((x, t)\) to a new coordinate system \((x, s)\) in which equation (2.2) becomes an ODE along certain curves in the \((x, t)\) plane. Such curves, \((x(s), t(s))\) along which the solution to equation (2.2) reduce to an ODE are called the characteristics curves. The variable \(s\) can be varied, whereas \(x\) changes along the line \(t = 0\) on the plane \((x, t)\) and remains constant along the characteristics. By chain rule we find

\[
\frac{d}{ds} \left[u(x(s), t(s))\right] = \frac{\partial u}{\partial x} \frac{dx}{ds} + \frac{\partial u}{\partial t} \frac{dt}{ds}.
\]  

(2.3)

Therefore to solve equation (2.2), we solve the characteristic system of ODEs as follows. Setting \(t(s)\) to satisfy \(\frac{dt}{ds} = 1\) along with the initial condition \(t(0) = 0\) we obtain \(t = s\). Using the definition of characteristics \(\frac{dx}{ds} = u\) and equation (2.2) we replace into equation (2.3) and obtain

\[
\frac{d}{ds} \left[u(x(s), t(s))\right] = \frac{\partial u}{\partial x} u - u \frac{\partial u}{\partial x} = 0,
\]

hence \(u\) is constant along characteristics. This fact allows us to integrate for \(x(s)\) and obtain

\[
x(s) = u_0(x_0)s + x_0.
\]  

(2.4)

Since \(s = t\) equation (2.4) can be rewritten as, \(x = u_0(x_0)t + x_0\). Hence the general solution of equation (2.2) is given by

\[
u(x, t) = u_0 \left(x - u_0(x_0)t\right) = u_0 \left(x - u(x, t)t\right).
\]  

(2.5)
Equation (2.5) is an implicit formulation which defines a classical solution for up to the time when the first shock singularity develops at singularity time $T^*$. After the shock singularity develops, equation (2.5) gives a multi-valued solution. An entropy condition is required to select a unique solution beyond the shock singularity [73, 97]. The characteristics are straight lines, but not all the lines have the same slope. Some of the characteristics will intersect at some time $T^*$ (see figure 2.1). The characteristics can be rewritten as

$$t = \frac{x}{u_0(x_0)} - \frac{x_0}{u_0(x_0)}.$$  \hfill (2.6)

The slope $\frac{1}{u_0(x_0)}$ of the characteristics depends on the point $x_0$ and on the initial condition $u_0$. Differentiating equation (2.5) with respect to $x$ we obtain

$$u_x = \frac{u'_0(x_0)}{1 + tu'_0(x_0)} \frac{\partial x_0}{\partial x}.$$  

Taking the spatial derivative of equation (2.4) we get

$$1 = [1 + tu'_0(x_0)] \frac{\partial x_0}{\partial x}.$$  

Hence the spatial derivative of velocity is given by

$$u_x = \frac{u'_0(x_0)}{1 + tu'_0(x_0)}.$$  

The time $T^*$ at which the characteristics cross (singularity time) and shock forms, for a given smooth initial data, can be evaluated as

$$T^* = \left[ \min_{x \in [0,2\pi]} [u'_0(x_0)] \right]^{-1}. \hfill (2.7)$$

20
We now define the supremum norm of $u_x$ for equation (2.2) as

$$
\|u_x(\cdot, t)\|_\infty = \max_{x \in [0, 2\pi]} |u_x(x, t)|. \quad (2.8)
$$

When the initial condition is well chosen such that

$$
\|u'_0(\cdot)\|_\infty = \min_{x \in [0, 2\pi]} u'_0(x_0),
$$

the analytical solution of the norm can be obtained by solving equation (2.2) along characteristics as time $t$ tends to the singularity time $T^*$ using the analytical solution

$$
\|u_x(\cdot, t)\|_\infty = \frac{1}{T^* - t} \quad (2.9)
$$

upto $t$ sufficiently close to $T^*$.

### 2.3 Mapping the inviscid Burgers equation to a globally regular system

Bustamante [11] developed a general theory to study nonlinear partial differential equations whose solutions present evidence of finite-time singularity. The method involves bijectively transforming the original variables (in this case for the inviscid Burgers equation, time, velocity and the spatial derivative of velocity) into new variables, which we shall refer to as mapped variables, which are globally regular in time. The mapping makes use of the BKM theorem [5], stating that all relevant norms of the velocity field are bounded up to time $t$ if and only if

$$
\tau(t) = \int_0^t F[u](t')dt',
$$

is bounded: where $F[u]$ is some known functional of the velocity field $u$. In the case of the inviscid Burgers system (2.2), based on the power law equation (2.9), the functional is $F[u](t') \equiv \|u_x(\cdot, t')\|_\infty$, the $L^\infty$ norm of
the spatial derivative of the velocity field. Explicitly, starting with smooth initial conditions, the boundedness of the integral

$$\tau(t) = \int_0^t ||u_s(\cdot, t')||_{\infty} dt'$$  \hspace{1cm} (2.10)$$

will ensure the continuity of the velocity field until time $t$. Using equation (2.9) the mapped time is evaluated in terms of the original time $t$ as

$$\tau(t) = \int \frac{dt}{T^* - t}.$$  

Performing the integral we obtain:

$$\tau = \ln \left( \frac{1}{T^* - t} \right) + K$$  \hspace{1cm} (2.11)$$

where $K$ is a constant of integration. At time $t = 0$, $\tau = 0$ and $K = \ln (T^*)$. Replacing for $K$ in equation (2.11) and simplifying we obtain

$$\frac{1}{T^* - t} = \frac{1}{T^*} \exp \tau.$$  \hspace{1cm} (2.12)$$

Using equation (2.9) and replacing for $1/(T^* - t)$ as in equation (2.12) we get

$$||u_s(\cdot, t(\tau))||_{\infty} = \frac{1}{T^*} \exp \tau.$$  \hspace{1cm} (2.13)$$

The bijective mapping of the inviscid Burgers classical system, equation (2.2) (which we shall refer to as the original system) to new “mapped, regular variables” consists of the time mapping $t \rightarrow \tau$ (equation (2.10)) as well as re-scaling of the velocity fields. The new mapped system is thus mapped from variables $(t, u(x, t))$ to the new variables $(\tau, v(x, \tau))$, where $v(x, \tau)$ is the mapped velocity.
From equation (2.10) the mapped time $\tau$ can also be expressed as

$$
\frac{d\tau}{dt} = \|u_\tau(\cdot, t)\|_\infty. \tag{2.14}
$$

The re-scaled velocity and spatial derivative of velocity are defined by:

$$
v(x, \tau) = \frac{u(x, t)}{\|u_\tau(\cdot, t)\|_\infty}, \tag{2.15}
$$

$$
v_\tau(x, \tau) = \frac{u_\tau(x, t)}{\|u_\tau(\cdot, t)\|_\infty}. \tag{2.16}
$$

We now derive the equations verified by this new field

$$
\frac{\partial v}{\partial \tau} = \frac{\partial u}{\partial \tau} \frac{1}{\|u_\tau(\cdot, t)\|_\infty} + u \frac{\partial}{\partial \tau} \left( \frac{1}{\|u_\tau(\cdot, t)\|_\infty} \right)
$$

$$
= \frac{\partial u}{\partial t} \frac{1}{\|u_\tau(\cdot, t)\|_\infty} \frac{dt}{d\tau} + u \frac{\partial}{\partial t} \left( \frac{1}{\|u_\tau(\cdot, t)\|_\infty} \right) \frac{d\tau}{dt} \frac{1}{\|u_\tau(\cdot, t)\|_\infty}
$$

$$
= \frac{1}{\|u_\tau(\cdot, t)\|_\infty^2} (\|u_\tau(\cdot, t)\|_\infty) + v \frac{\partial}{\partial t} \left( \frac{1}{\|u_\tau(\cdot, t)\|_\infty} \right)
$$

$$
= -v v_\tau - v \frac{\partial}{\partial t} \left( \frac{1}{\|u_\tau(\cdot, t)\|_\infty^2} \right).
$$

Simplifying we get

$$
v_\tau + vv_\tau = -v(x, \tau) \times \alpha(\tau) \tag{2.17}
$$

where,

$$
\alpha(\tau) = \frac{\partial}{\partial t} \left( \frac{1}{\|u_\tau(\cdot, t)\|_\infty^2} \right).
$$

Equation (2.17) arises in its own right in contexts such as nonlinear acoustics, see Engelbrecht et al. [33], where mapping between (2.17) and Burgers equation (2.1) are employed.
To solve for \( \alpha(\tau) \), we first differentiate equation (2.2) with respect to \( x \) and obtain

\[
\begin{align*}
    u_{xt} + u_{x}^2 + uu_x &= 0, \\
    (\partial_t + u\partial_x)|u_x| + |u_u|x &= 0 .
\end{align*}
\]

Evaluating at the point where we have maximum spatial derivative of the velocity, \( x = X(t) \), we get

\[
\frac{d}{dt}|u_x(X, t)| = -u_x(X, t)|u_x(X, t)|.
\]

Therefore,

\[
\alpha(\tau) = -\frac{u_x(X, \tau)|u_x(X, \tau)|}{||u_x(\cdot, \tau)||_\infty^2} = -v_x(X, \tau).
\]

Equation (2.17) is supplemented by the initial constraint \( ||v_x(\cdot, 0)||_{\infty} = 1 \). The equation differs from the original system (equation (2.2)) by the extra term on the right hand side which ensures that \( ||v_x(\cdot, \tau)||_{\infty} = 1 \) for all \( \tau < \infty \). Since the condition \( \tau < \infty \) implies boundedness of the original fields, the solution of the mapped system is globally regular in time \( \tau \).

### 2.3.1 Using the mapped system to assess blowup of original system

The mapping (2.15) and (2.16) is bijective as long as \( \tau < \infty \). Correspondingly, integration of the mapped equation (2.17) should give enough information to assess blowup quantities in the original variables. Unlike the original equation, the mapped equation has \( ||v_x(\cdot, \tau)||_{\infty} = 1 \) and also does not provide direct access to the original variable \( ||u_x(\cdot, t)||_{\infty} \). In numerical simulations of the mapped equation we have direct access to the mapped energy. The energy for the original inviscid
Burgers equation is defined as

\[ E = \frac{1}{2} \int_0^{2\pi} |u(x, t)|^2 \, dx, \quad (2.18) \]

and is conserved.

Similarly we define the “energy” from the mapped system \( E_{\text{map}}(\tau) \):

\[
E_{\text{map}}(\tau) = \frac{1}{2} \int_0^{2\tau} |v(x, \tau)|^2 \, dx \\
= \frac{1}{2} \int_0^{2\tau} |u(x, t)|^2 \, dx \\
= \frac{1}{2} \left( \|u(x, \tau)\|_{\infty} \right)^2,
\]

which simplifies to

\[ E_{\text{map}}(\tau) = \frac{E}{\left( \|u(x, \tau)\|_{\infty} \right)^2}. \quad (2.19) \]

Rearranging we obtain

\[ \|u(x, \tau)\|_{\infty} = \sqrt{\frac{E}{E_{\text{map}}(\tau)}}. \quad (2.20) \]

In practical applications, both the original inviscid Burgers energy \( E \) and mapped energy \( E_{\text{map}}(\tau) \) are nonzero. We reconstruct the original time \( t(\tau) \) by integrating \( dt/d\tau = 1/\|u(x, t(\tau))\|_{\infty} \) and using equation (2.20):

\[ t(\tau) = \int_0^{\tau} \frac{1}{\|u(x, t(\tau'))\|_{\infty}} \, d\tau' = \int_0^{\tau} \sqrt{\frac{E_{\text{map}}(\tau')}{E}} \, d\tau'. \quad (2.21) \]

This allows us to express the BKM criterion for the singularity of inviscid Burgers, in terms of the regular solution of the mapped equation (2.17):

\[ T' = \int_0^{\infty} \sqrt{\frac{E_{\text{map}}(\tau)}{E}} \, d\tau. \quad (2.22) \]

Correspondingly, using equations (2.13), (2.20) and (2.22) the mapped energy can
be obtained explicitly:

\[ E_{\text{map}}(\tau) = (T^*)^2 E \exp(-2\tau). \] (2.23)

### 2.4 Numerical solution of original and mapped systems and comparison with analytical solution

The pseudo-spectral method (details in appendix B) was selected to solve both the original and mapped equation numerically due to its advantage of efficiently computing the nonlinear convective term \[57\] using the Fast Fourier Transform (FFT). The discrete Fourier transform of a periodic function however introduces aliasing errors \[7, 17, 49\] partially due to the artificial periodicity of the discrete Fourier coefficient as a function of the wave number. The aliasing error distorts the accuracy of the high frequency modes. To ensure that the pseudo-spectral method does not suffer from numerical instability \[47\] due to aliasing errors, dealiasing (details in appendix B) was carried out. The \(2/3\) dealiasing rule was used in this chapter, where the last \(1/3\) of the high frequency modes are set to zero with the first \(2/3\) of the Fourier modes remaining unchanged.

In both the original and mapped system, time advancement was carried out using the fourth order Runge-Kutta (RK4) method. In the mapped system, a uniform time step \(d\tau\) is used as advised by the fact that singularity is at \(\tau = \infty\). This uniform \(d\tau\) would imply that in the original system, the time step \(dt\) should be adaptive and is given by \(dt = d\tau/\|u_s(\cdot, t)\|_\infty\). This adaptive method is normally used for reasons of accuracy near the singularity time. On the other hand, it is convenient to use adaptive time steps for sampling purposes so that a proper comparison is possible between the original and mapped system. Unlike the integration of the original system, the numerical integration of the mapped system
requires a special method: a normalisation so that the mapped system satisfies the constraint $\|v(x, \tau)\|_\infty = 1$ for all time $\tau$. The accuracy of the normalisation is essential for the performance of the mapped system and therefore accurate interpolation of $\|v(x, \tau)\|_\infty$ at every time step was carried out. The initial condition chosen for this study is $u_0(x) = 1 + \cos(x)$.

### 2.4.1 Shock profile and convergence studies

The shock behaviour of the velocity field and the spatial derivative of velocity is illustrated by the shock profile in figure 2.1 and figure 2.2 respectively at a spatial resolution $N=4096$. The dash-dotted line indicates the initial condition $1+\cos(x)$ for the velocity and $-\sin(x)$ for the spatial derivative of velocity whereas the dashed line indicates the profile at time $t = 0.95 * T^\ast$. We observe from figure 2.1 (right) the characteristics (lines at bottom of figure) having different gradients leading to a singularity (shock) at the first instant when either of the lines cross (i.e. $t = 1$ for the initial condition chosen).

![Figure 2.1: Plot of velocity profile from time t=0 to t = 0.95 * T^ast at a resolution N = 4096. The dash-dotted line (red) indicates the initial condition (1 + cos(x)) and the dashed line (blue) indicates the profile at time t = 0.95 * T^ast.](image-url)

The behaviour of the maximum norm of the spatial derivative of velocity $u_x$
which controls blowup in line with the BKM theorem is presented in figure 2.3. From figure 2.3 it is visible that the maximum norm of the spatial derivative of velocity $\|u_t\|_\infty$ from numerical solution is in agreement up to some time depending on the resolution. This suggests that $\|u_t\|_\infty$ has significant contributions coming from the high-wavenumbers modes since $\|u_t\|_\infty$ decreases at a given time $t > 0.96$ if one truncates the higher wavenumbers of the velocity field (figure 2.3, bottom).

Convergence studies of the maximum norm of spatial derivative of velocity from the numerical solution of the original equation was carried out. Figure 2.4, top panel, shows the study of the inverse of the maximum norm of the spatial derivative of velocity as a function of original time, using numerical solution from the original equation. Good convergence to the analytical solution $(T^* - t)$, equation (2.9)) is obtained. Figure 2.4, bottom panel, shows the lin-log convergence study of the maximum norm of the spatial derivative of velocity plotted as a function of the mapped time $\tau = -\ln(T^* - t)$, using the numerical solution of the original system as well. Again, good resolution convergence to the analytical solution is noted.
Figure 2.3: Plot showing $\|u_s(., t)\|_\infty$ as a function of original time $t$ at different resolutions $N = 2048, 4096, 8192, 16384$. Bottom panel details the late time behaviour, showing that $\|u_s(., t)\|_\infty$ is in agreement up to some time depending on the resolution.
Figure 2.4: Resolution studies of the data from the original system at different resolutions $N = 2048, 4096, 8192, 16384$. Top panel: Plot of $1/\|u_s(\cdot, t)\|_\infty$ giving convergence to the analytical solution $1/\|u_s(\cdot, t)\|_\infty \approx (T^* - t)$ (solid line). Bottom panel: Lin-log plot of $1/\|u_1(\cdot, t(\tau))\|_\infty$ as a function of $\tau$ giving convergence to the analytical solution (solid line).
Finally, figure 2.5 shows the lin-log convergence study of mapped Energy $E_{\text{map}}(\tau)$ which is used to compute $\|u_x(., t(\tau))\|_\infty$ using the numerical solution of the mapped system. Good convergence to the analytical equation (2.23) is verified.

![Figure 2.5: Lin-log plot of resolution studies of the numerical data from the mapped energy $E_{\text{map}}(\tau)$ at different resolutions $N = 2048, 4096, 8192, 16384$ as a function of mapped time $\tau$. Convergence to the analytic solution $E_{\text{map}}(\tau) = (T^*)^2 E \exp(-2\tau)$ equation (2.23) (solid line) is verified.](image)

### 2.4.2 Comparison of errors from the original and mapped system

We now compare the results from the numerical simulation of both the original and mapped system with the analytical solution. The error of the results from the numerical simulation of the original system was obtained by

$$e_1 = \ln(\|u_x(., t)\|_\infty) - \ln\left(\frac{1}{T^* - t}\right).$$

(2.24)
For comparison, we use the numerical data for the mapped energy $E_{\text{map}}(\tau)$ obtained from the independent integration of the mapped system.

\[ e_2 = -\ln(T^* - t) - \ln \left( \frac{E}{E_{\text{map}}(\tau)} \right) \]

\[ = \tau - \ln \left( \frac{E}{E_{\text{map}}(\tau)} \right) . \]

The error definition can be simplified to

\[ e_2 = \tau + \frac{1}{2} \left( \ln E_{\text{map}}(\tau) - \ln E \right) . \tag{2.25} \]

To compare the error from the mapped system (2.25) with the error from the original system (2.24), we require both systems to be on the same time scale. We therefore convert the original time variable $t$ to the mapped time variable $\tau$ using $\tau(t) = - \ln(T^* - t)$.

To gain an appropriately accurate estimate of these errors, we must consider the best method for approximating $\|u(x, t)\|_\infty$ from the original system. The simplest approach is to apply the maximum value across the collocation points of the discretised field. However, this leads to significant spurious oscillations at late times close to the singularity time, figure 2.6 (dashed line). To mitigate this, we located the maximum velocity gradient as well as its position accurately. In order to achieve this, the same interpolation as used in the normalisation procedure for the mapped system was carried out around the maximum of the velocity gradient. Figure 2.6 (solid line) shows the error of the numerical results incorporating this interpolation. Compared to figure 2.6 (dashed line), it clearly indicates not only a reduction of the oscillations at late times, but also a reduction of the magnitude of error. A comparison of the evolution of the errors from the original system and the mapped system with time $\tau$ shows a marginal decrease of the error from
Figure 2.6: The errors for the original system when the numerical solution has interpolation (solid line) of maximum norm of spatial derivative and when no interpolation is incorporated in the numerical solution (dashed line) at spatial resolution $N = 2048$. A significant reduction in oscillations and error magnitude is achieved.

the integration of the mapped system compared to the integration of the original system (figure 2.7).

### 2.4.3 Energy spectrum and analyticity strip method

One of the methods that has been used to gain insight into the behaviour of a typical singularity scenario in numerical simulations is the analyticity strip method [9, 12, 94]. To describe the method, the spatial coordinates are usually examined as complex variables and the temporal coordinates as real variables. The logarithmic decrement of the energy spectrum (also known as the width of the analyticity strip) $\delta(t)$ is the imaginary part of the complex-space singularity of the velocity field nearest to the real space. The temporal evolution of $\delta(t)$ is then monitored. The analyticity strip method makes use of the rigorous result [3] that a real-space
singularity occurring at time $T^*$ must be preceded by a nonzero $\delta(t)$ that vanishes at $T^*$. When using spectral methods, like in our case, $\delta(t)$ is obtained directly from the high-wavenumber exponential fall off of the spatial Fourier transform of the solution [38] effectively establishing a distance to the singularity [39].

The kinetic energy spectrum is usually obtained by summing the square of the modulus of the spatial Fourier coefficients $\hat{u}$ over circular shells

$$E(k, t) = \frac{1}{2} \sum_{k - \frac{1}{2} < \abs{k} < k + \frac{1}{2}} |\hat{u}(k, t)|^2. \quad (2.26)$$

Using our numerical results, we test the analyticity-strip method and its bridge with the BKM theorem [12] from a purely numerical point of view. The function $u(x, t)$ remains analytic in the space variables if $E(k, t)$ can be bounded by

$$E(k, t) \lesssim C(t)k^{-n(t)}e^{-2k\delta(t)}. \quad (2.27)$$
where $\delta(t)$ is the analyticity strip width and $C(t), n(t)$ are positive numbers. We assume this approximation holds for our functions. Figure 2.8 shows snapshots of the energy spectrum $E(k, t)$ in lin-log scaling, where the slopes of the straight lines are proportional to the logarithmic decrement $\delta(t)$. Figure 2.9 shows the snapshots of the energy spectrum in log-log scaling where the solid line indicates $E(t) = k^{-8/3}$ in line with the results from the working hypothesis introduced by Bustamante & Brachet [12]. This gives us the evidence of the feasibility of using this approximation, equation (2.27).

![Figure 2.8: Lin-log plot of energy spectrum at different times $t=0.8, 0.9, 0.95, 0.98, 0.99$ (curves progressing from bottom to top) and resolution $N = 8192$ corresponding to $k_{\text{max}} = 2730$ using the 2/3-rd dealiasing filter rule. The slopes of the straight line are proportional to the logarithmic decrement $\delta(t)$.](image)

Using a least-squares fit, at each time $t$, on $\ln E(k, t)$ over some interval $k_i \leq k \leq k_f$, we find the coefficients $C(t), n(t)$ and $\delta(t)$. The least-square fit procedure is carried out at each time $t$ on the logarithm of the energy spectrum $E(k, t)$, using a function of the form

$$\ln E(k, t) = \ln C(t) - n(t) \ln k - 2k\delta(t).$$  \hspace{1cm} (2.28)
Figure 2.9: Log-log plot of energy spectrum at different times $t=0.8, 0.9, 0.95, 0.98, 0.99$ (curves progressing from bottom to top) and resolution $N = 8192$ corresponding to $k_{max} = 2730$ using the $2/3$-rd dealiasing filter rule. The slopes of the straight line are proportional to the exponent $n(t)$. The solid line indicates the $E(t) = k^{-8/3}$ in line with the results from the working hypothesis introduced by Bustamante & Brachet [12].
The error on the fit interval $k_i \leq k \leq k_f$,

$$\chi^2(t) = \sum_{k_i}^{k_f} [\ln E(k, t) - \ln C(t) + n(t) \ln k + 2k\delta(t)]^2,$$

(2.29)

is minimised by solving the equations $\partial\chi^2/\partial C = 0$, $\partial\chi^2/\partial n = 0$ and $\partial\chi^2/\partial \delta = 0$. The problem becomes linear in the parameters $\ln C(t), n(t)$ and $\delta(t)$.

Figure 2.10: Plot of evolution of the constant $C(t)$ from the fit $E(t) \approx C(t)k^{-n(t)}e^{-2k\delta(t)}$ against: left panel, original system time variable $t$ and, right panel, mapped system time variable $\tau$ at resolution $N = 8192$.

Figure 2.11: Plot of evolution of the exponent $n(t)$ from the fit $E(t) \approx C(t)k^{-n(t)}e^{-2k\delta(t)}$ against: left panel, original system time variable $t$ and, right panel, mapped system time variable $\tau$ at various resolution $N = 2048, 4096, 8192$. 

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Figure 2.12: Plot of evolution of the exponent $\delta(t)$ from the fit $E(t) \approx C(t)k^{-n(t)}e^{-2\delta(t)}$ (dashed line) against: left panel, original system time variable $t$ and, right panel, mapped system time variable $\tau$ at resolution $N = 8192$. The solid line indicates the finite time singularity power law $\delta(t) \approx (T^\ast - t)^\Gamma$ where $\Gamma = 3/2$.

Figure 2.10, 2.11 and 2.12 shows the evolution of the fit parameters $C$, $n$ and $\delta$ with respect to time (both original time $t$ and mapped time $\tau$) at resolution $N = 8192$ for $C$ & $\delta$ and at various resolutions for the exponent $n$. Similar to the results obtained by Bustamante & Brachet [12], when they studied the inviscid Burgers equation using their working hypothesis in equation (17) using an initial condition with $T^\ast = 1$, we obtained the $k^{-8/3}$ power law at times near singularity time $T^\ast$ (solid line in figure 2.9). The analytical bounds obtained by Bustamante & Brachet [12] from their hypothesis where $n_0(t) = 8/3$ and $\delta_0(t) \approx (T^\ast - t)^\Gamma$ where $\Gamma = 3/2$ based on inequality $\Gamma \geq \frac{2}{2-n(t)}$ (solid line in figure 2.12) were also obtained from our numerical results.

### 2.5 Conclusion

We have studied the 1D inviscid Burgers equation and evaluated the performance of the bijective mapping to a system that is globally regular [11] in improving the
diagnosis of blowup of the equation. Both the original and mapped systems were solved numerically using pseudospectral method. We have established that the numerical integration of the mapped regular system produce more accurate results compared to the numerical integration of the original system. The improvement in accuracy is not as striking as initially found by Bustamante [11] who used a second order leap-frog time advancement method in contrast to our study where we used the fourth order RK4 time advancement method. We established that in using the RK4 method, the numerical integration would get further in time and also entails up to two orders of magnitude increase in accuracy compared to the leap-frog time stepping method.

We also carried out an investigation of the evolution of the energy spectrum of the numerical solution. Using the working hypothesis introduced by Bustamante & Brachet [12], we establish an inertial range of wavenumbers at which the energy is transferred to small scales with a spectrum of the form $E(k, t) \sim k^{-8/3}$ at times close to the singularity time. The finite-time blowup of the maximum norm of the spatial derivative of velocity implies that the energy spectrum’s logarithmic decrement $\delta(t)$, which is a measure of the loss of the analyticity, must decay to zero fast enough at the singularity time. We observe a decay $\delta(t) \sim (T^* - t)^{3/2}$, consistent with rigorous bounds by Bustamante & Brachet [12] (figure 2.12).

To test the usefulness of our methodology further, we will now introduce and study a one parameter family of symmetry plane models of the 3D Euler equation (chapter 3) which, unlike the 3D Euler equations, has analytical results which we can use to evaluate our numerical results.
Chapter 3

Symmetry plane model of 3D Euler flows

In this chapter, we discuss a symmetry plane model of the 3D Euler equations. We find that the evolution on the plane is determined by two scalar fields: vorticity and stretching rate. We obtain a rigorous system of evolution equations for these fields and show that the equations are not closed: knowledge of the 3D flow is needed in order to get a pressure term on the plane. However, we demonstrate that a simple closure condition is sufficient in order to model this pressure term, thus generalising the condition on the pressure term by Gibbon et al. [43]. In this way we introduce a one-parameter family of models satisfying the closure condition.

We then review the analytical solution along characteristics of this family of models and show explicitly for some choices of the model’s parameter that the fields have a finite time singularity, for generic initial conditions. In particular, the singularity time is found analytically in terms of the initial condition and the value of the model’s parameter. We apply the method introduced by Bustamante [11] on mapping bijectively the original variables to a globally regular system, with mapped time and fields that are based on the existence of a BKM type of
Theorem [5]. The analytic and numerical advantages of working on the mapped variables are discussed. The analytical solutions for the blowup quantities are derived in terms of the mapped variables. Importantly, formulae for the original blowup quantities in terms of the mapped variables are presented. In particular, an estimate of the singularity time of the original system is obtained in terms of the mapped system’s numerical solution.

A comparison of the numerical solution of the original symmetry plane model and its mapped counterpart for the assessment of finite-time singularity is presented. We monitor errors in several quantities relative to the analytic solution and discuss a number of nuances which arise. With reference to the works of Sulem, Sulem and Frisch [94] and Bustamante and Brachet [12], we present a thorough study of the spectra of stretching rate and investigate the spatial structure of the blowup via the analyticity strip method. Finally we consider the estimation of singularity time from both systems and demonstrate a robust improvement on using the mapped system even when considering the additional computational burden it incurs.

3.1 Symmetry plane

We consider a special configuration of the three-dimensional Euler equation (equation (1.1)) fields and define a symmetry plane at $z = 0$ by the following conditions on the velocity and pressure fields:

\[
\begin{align*}
    u_x(x, y, z, t) &= u_x(x, y, -z, t) \\
    u_y(x, y, z, t) &= u_y(x, y, -z, t) \\
    u_z(x, y, z, t) &= -u_z(x, y, -z, t) \\
    p(x, y, z, t) &= p(x, y, -z, t),
\end{align*}
\]  

(3.1)
for arbitrary \((x, y, z) \in \mathbb{R}^3\) and \(t \in [0, T)\). These conditions are consistent with the evolution equations (1.1-1.2). As for the vorticity, these conditions imply
\[
\begin{align*}
\omega_x(x, y, z, t) &= -\omega_x(x, y, -z, t), \\
\omega_y(x, y, z, t) &= -\omega_y(x, y, -z, t), \\
\omega_z(x, y, z, t) &= \omega_z(x, y, -z, t),
\end{align*}
\]
for all \((x, y, z) \in \mathbb{R}^3\) and \(t \in [0, T)\).

At the symmetry plane \(z = 0\), the 3D Euler fluid equations will simplify because:

(i) \(u_z(x, y, 0, t) \equiv 0\) so the velocity field is parallel to the plane,
(ii) \(\omega_z(x, y, 0, t) = \omega_z(x, y, 0, t) = 0\) so the vorticity field is perpendicular to the plane.

This leads to a new system of equations which is “almost” 2D (except for a pressure term depending on the full 3D velocity field). Let us denote the “horizontal” component of the velocity field and the pressure at the symmetry plane by
\[
\begin{align*}
\mathbf{u}_h(x, y, t) &\equiv (u_x(x, y, 0, t), u_y(x, y, 0, t)), \\
p_h(x, y, t) &\equiv p(x, y, 0, t).
\end{align*}
\]

The horizontal components of equations (1.1-1.2) become, at \(z = 0\),
\[
\frac{\partial \mathbf{u}_h}{\partial t} + \mathbf{u}_h \cdot \nabla_h \mathbf{u}_h = -\nabla_h p_h, \tag{3.2}
\]
where \(\nabla_h\) denotes the “horizontal” gradient operator, \(\nabla_h = (\partial_x, \partial_y)\). The incompressibility condition in equation (1.2) allows us to define the stretching-rate scalar
on the symmetry plane:

\[ \gamma(x, y, t) \equiv u_{z,z}(x, y, 0, t) = -\nabla_h \cdot u_h(x, y, t). \tag{3.3} \]

Therefore, even though \( u_z = 0 \) at the symmetry plane \( z = 0 \), we have \( z \)-derivative of the \( z \)-component of velocity \( u_{z,z} \neq 0 \) at \( z = 0 \). Let us compute the \( z \)-derivative of the \( z \)-component of equation (1.1) and then evaluate at \( z = 0 \). We obtain

\[ \frac{\partial \gamma}{\partial t} + u_h \cdot \nabla_h \gamma + \gamma^2 = -p_{zz}\bigg|_{z=0}. \tag{3.4} \]

This can be simplified by noticing that the symmetry-plane conditions (3.1) imply \( u_z = u_{z,z} = 0 \) and \( u_{x,z} = u_{y,z} = 0 \) at \( z = 0 \) so that we are left with

\[ \frac{\partial \gamma}{\partial t} + u_h \cdot \nabla_h \gamma + \gamma^2 = -p_{zz}\bigg|_{z=0}. \tag{3.4} \]

Although this equation does not close (the pressure still depends on the full 3D velocity profile), it is remarkable that the equation for the vorticity at the symmetry plane does close. As mentioned before, the vorticity at the symmetry plane has no horizontal component so we can define the vorticity scalar

\[ \omega(x, y, t) \equiv \omega_z(x, y, 0, t) = \partial_x u_y - \partial_y u_x. \tag{3.5} \]

An evolution equation for vorticity is obtained by taking the curl of the 2D equation (3.2). We readily obtain

\[ \frac{\partial \omega}{\partial t} + u_h \cdot \nabla_h \omega = \gamma \omega, \tag{3.6} \]

which explains the meaning of \( \gamma \) as the stretching rate of vorticity.

Taken together, equations (3.3)–(3.6) would be a closed system in two dimen-
sions, except for the pressure term which, as usual, depends on the full 3D velocity profile. An important consistency condition on the pressure term is derived after integrating spatially equation (3.4) over the whole horizontal domain, and discarding boundary terms by assuming either periodic or vanishing boundary conditions on the horizontal velocity field. The condition reads

\[
\int \int p_{zz} \bigg|_{z=0} \, dx \, dy = -2 \int \int (\gamma(x, y, t))^2 \, dx \, dy. \tag{3.7}
\]

We propose a consistent family of *closure* models for the pressure term, based on an exact solution by Gibbon *et al.* [43]. These models will be discussed in the next subsections.

From here on, we will assume periodic boundary conditions in the two spatial directions \((x, y)\), for the three-dimensional velocity field components \(\mathbf{u} = (u_x, u_y, u_z)\) and the pressure scalar \(p\), with periodicity box \([0, 2\pi] \times [0, 2\pi]\). As for the \(z\)-direction, we do not assume yet any boundary condition.

### 3.1.1 Gibbon *et al.* exact solution of 3D Euler (and Navier-Stokes)

A general class of exact solutions of 3D Euler (and Navier-Stokes) was presented by Gibbon *et al.* [43]. In the case of the 3D Euler equations in the presence of a symmetry plane, this exact solution becomes

\[
\mathbf{u}(x, y, z, t) = (\mathbf{u}_h(x, y, t), z \gamma(x, y, t)),
\]

where \(\mathbf{u}_h\) and \(\gamma\) satisfy equations (3.3)–(3.6), along with the following condition on the pressure:

\[
p_{zz}(x, y, z, t) = f(t)
\]
which is a function of time only. Due to the periodicity of the horizontal domain, condition (3.7) implies the closure \( f(t) = -2\langle \gamma^2 \rangle \), where

\[
\langle F \rangle \equiv \frac{1}{(2\pi)^2} \int_0^{2\pi} \int_0^{2\pi} F(x, y, t) \, dx \, dy.
\]

Correspondingly, equations (3.3)–(3.6) determine the fate of the full 3D flow via the knowledge of the scalars \( \gamma \) and \( \omega \). Remarkably, along the characteristics of the horizontal velocity field \( u_h \) the equation for stretching rate \( \gamma \) is “decoupled” from the system and reads

\[
\left( \frac{\partial}{\partial t} + u_h \cdot \nabla_h \right) \gamma + \gamma^2 = 2\langle \gamma^2 \rangle
\]

(3.8)

This allowed Constantin [23, 24] to solve for \( \gamma \) along characteristics (and for vorticity \( \omega \), which can be found a posteriori), proving that the stretching rate \( \gamma \) would blow up in a finite time, with explicit formulae for the singularity time which confirmed the accuracy of the numerical blowup predictions in [87]. We also note here for completeness the work of Gibbon, Moore & Stuart [44] who also proved blow-up in the axisymmetric case.

### 3.1.2 A physically-motivated model on the symmetry plane

Starting from the result in Gibbon et al. [43], we have a closure condition \( \left. p_{xz} \right|_{z=0} = f(t) \) which, however exhibits spatial uniformity which is not observed in experiments. Therefore, we take an improved closure condition reflecting experimental observations. We propose a one-parameter family of models which achieves all this, while still keeping the analytic structure of characteristics, so blowup can be assessed analytically. Looking at equations (3.3)–(3.6), the model is defined by
the closure

\[ p_{zd}|_{z=0} = -2\langle y^2 \rangle + \lambda \left( \gamma^2 - \langle \gamma^2 \rangle \right), \]

where \( \lambda \) is a real (free) parameter. With this closure, the family of models corresponds to the following equations on the symmetry plane:

\[
\begin{align*}
\frac{\partial \gamma}{\partial t} + u_h \cdot \nabla_h \gamma &= (2 + \lambda)\langle y^2 \rangle - (1 + \lambda)\gamma^2, \\
\frac{\partial \omega}{\partial t} + u_h \cdot \nabla_h \omega &= \gamma \omega,
\end{align*}
\]

(3.9) (3.10)

where \((x, y) \in \mathbb{T}^2 \equiv [0, 2\pi] \times [0, 2\pi]\) and \(\lambda \in \mathbb{R}\). The case \(\lambda = 0\) recovers the equations of Gibbon et al. [43]. The horizontal velocity field \(u_h = (u_x, u_y)\) is defined, as usual, by equations (3.3) and (3.5), where the vorticity is recovered using constitutive equation for the stream function \(\psi\)

\[ \Delta \psi = -\omega \]

and the stretching rate using the constitutive equation for the potential \(\phi\)

\[ \Delta \phi = -\gamma. \]

The horizontal velocity components can then be computed after inverting the Laplace equations using

\[
\begin{align*}
u_x &= \frac{\partial \psi}{\partial y} + \frac{\partial \phi}{\partial x}, \\
u_y &= -\frac{\partial \psi}{\partial x} + \frac{\partial \phi}{\partial y}.
\end{align*}
\]

The evolution equations (3.9) and (3.10) are supplemented by initial conditions \(\gamma(x, y, 0) = \gamma_0(x, y)\) and \(\omega(x, y, 0) = \omega_0(x, y)\), which must have zero mean: \(\langle \gamma_0 \rangle = \langle \omega_0 \rangle = 0\).
Our family of models is motivated by the physical interpretation of the pressure term at the symmetry plane and the introduction of a tuneable parameter, which provides a range of phenomenology and asymptotic behaviours. This approach provides a valuable framework for developing methods for assessing blow-up.

### 3.2 Symmetry-plane model: Analytical solutions

Equations (3.9) and (3.10) can be solved for $\gamma$ and $\omega$ along characteristics, using a classical method of finding conservation laws of the model equations using infinitesimal symmetries [13, 14], see details in Appendix C. The conservation law is given by

$$D(x, y, t) = \frac{1}{A(t)\gamma(x, y, t) - \frac{A(0)}{1+t}} - (1 + \lambda) \int_0^t \frac{1}{[A(s)]^2} ds, \quad \lambda \neq -1, \quad (3.11)$$

$$\ddot{A} - (1 + \lambda)(2 + \lambda)(\gamma^2)A = 0. \quad (3.12)$$

By definition, the conservation law satisfies,

$$\frac{\partial D}{\partial t} + u \cdot \nabla D = 0.$$

Characteristics are two-dimensional curves $(X(t), Y(t))$ defined by the system of equations

$$\frac{dX}{dt} = u_x(X(t), Y(t), t),$$

$$\frac{dY}{dt} = u_y(X(t), Y(t), t).$$

The conservation law (3.11) allows one to solve for $\gamma$ along characteristics. Explicitly, let the characteristic have initial condition $(X(0), Y(0)) = (X_0, Y_0)$, and
noting $A$ satisfies a second order ODE, we set for simplicity $A(0) = 1$, $\dot{A}(0) = 0$.

Defining $\gamma_0(x, y) = \gamma(x, y, 0)$ we get

$$D(X(t), Y(t), t) = D(X_0, Y_0, 0) = \frac{1}{\gamma_0(X_0, Y_0)}.$$  

Defining $\dot{S} = 1/A^2$ and in the case $\lambda \neq -1$ we have

$$\frac{1}{A(t)} \left( A(t) \gamma(X(t), Y(t), t) - \frac{\dot{A}(0)}{1+\lambda} \right) - (1+\lambda)S(t) = \frac{1}{\gamma_0(X_0, Y_0)}.$$  \hspace{1cm} (3.13)

Simplifying and noting that $\dot{A}/A = -\dot{S}/2\dot{S}$, then in the case $\lambda \neq -1$ the solution is

$$\gamma(X(t), Y(t), t) = \frac{d}{dt} \left( \ln \left[ \frac{1 + (\lambda + 1) \gamma_0(X_0, Y_0) S(t)}{S(t)^{1/2}} \right]^{1/\lambda} \right),$$  \hspace{1cm} (3.14)

where the function $S(t)$ satisfies the following ordinary differential equation (ODE):

$$\dot{S}(t) = \left[ \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} [1 + (\lambda + 1)\gamma_0(X_0, Y_0)S(t)]^{\lambda/2} \, dx \, dy \right]^{-2(\lambda+1)}, \quad S(0) = 0.$$  \hspace{1cm} (3.15)

This ODE is obtained due to an identity satisfied by the Jacobian of the back-to-labels transformation:

$$J(t; X_0, Y_0) = \det \left( \frac{\partial(X(t), Y(t))}{\partial(X_0, Y_0)} \right).$$  \hspace{1cm} (3.16)

From the fact that $\nabla_h \cdot u_h = -\gamma$ we readily obtain an evolution equation for $J$ which can be solved:

$$\frac{\dot{J}(t; X_0, Y_0)}{J(t; X_0, Y_0)} = -\gamma(X(t), Y(t), t) \quad \implies \quad J(t; X_0, Y_0) = e^{-\int_0^t \gamma(X(s), Y(s), s) \, ds},$$

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and using equation (3.14) we obtain the solution

\[
J(t; X_0, Y_0) = \left[ \frac{1 + (\lambda + 1) \gamma_0(X_0, Y_0) S(t)}{S(t)^{1/2}} \right]^{-\frac{1}{\lambda + 1}}.
\]

This Jacobian satisfies the identity

\[
\left( \int_{T^2} dxdy = 4\pi^2 \right) = \int_{T^2} J(t; X_0, Y_0)dX_0dY_0,
\]

which leads to the ODE (3.15) satisfied by \( S(t) \).

Notice that Kelvin’s theorem on circulation conservation, or more accurately Cauchy’s invariants [40, 68], follows directly from the fact that

\[
J(t; X_0, Y_0)\omega(X(t), Y(t), t) = \omega_0(X_0, Y_0)
\]

for any characteristic’s initial condition \((X_0, Y_0)\). This gives us the corresponding solution for \( \omega \) along characteristics:

\[
\omega(X(t), Y(t), t) = \omega_0(X_0, Y_0) \left[ \frac{1 + (\lambda + 1) \gamma_0(X_0, Y_0) S(t)}{S(t)^{1/2}} \right]^{\frac{1}{\lambda + 1}}.
\]  

**3.2.1 Blowup solutions and asymptotics**

The solutions along characteristics for stretching rate (3.14) and vorticity (3.17) will develop a singularity if the factor \( 1 + (\lambda + 1) \gamma_0(X_0, Y_0) S(t) \) becomes zero for some time \( t \) and position \((X_0, Y_0)\). Since \( S(0) = 0 \) and \( S(t) \geq 0 \), it follows that \( S(t) \) can only grow in time and thus the singularity will occur first at the characteristic starting at the position of the infimum (if \( \lambda > -1 \)) or the supremum (if \( \lambda < -1 \)) of \( \gamma_0 \) over \( T^2 \). Consequently, the singularity time \( T^* \) is defined by the condition \( S(T^*) = S^* \), where \( S^*(>0) \) is defined by
Integrating the ODE (3.15) using separation of variables, an explicit formula for the singularity time $T^*$ is derived:

$$
-\frac{1}{S^*} = \begin{cases} 
(\lambda + 1) \sup_{(x,y)\in T^2} \gamma_0(x,y), & \lambda < -1, \\
(\lambda + 1) \inf_{(x,y)\in T^2} \gamma_0(x,y), & \lambda > -1. 
\end{cases} 
$$  \hspace{1cm} (3.18)

The blowup time $T^*$ is finite if and only if this integral converges.

The blowup structure for the stretching rate and vorticity solutions for different values of the model’s free parameter $\lambda$. There is a conditional result that helps us to classify the cases for which there is a finite time singularity. The function $\dot{S}$ is the multiplicative inverse of the integrand in equation (3.19). Let the initial condition $\gamma_0$ be continuous on $T^2$ and of zero mean. If the function $S(t)$ satisfies $0 < \lim_{t \to T^*} \dot{S}(t) < \infty$ then the singularity occurs at a finite time. In other words, if the function $\dot{S}(t)$ is bounded from below then the integral (3.19) converges. The utility of the solution to the ODE is that in order to determine the singular behaviour of the system, we just need to look at the behaviour of the ODE (3.15) near $S = S^*$. The parameter space is divided in regions of finite-time blowup alternating with regions of infinite-time blowup as illustrated in figure 3.1. The region of $\lambda$ where $\dot{S}(t) = 0$ depends on the initial conditions; if the local profile of initial stretching near the infimum is paraboloidal (a generic situation), then the regions are as depicted in figure 3.1 where:

- If $\lambda < -1$ then the singularity occurs at finite time: $T^* < \infty$ and $0 < \lim_{t \to T^*} \dot{S}(t) < \infty$.
Singular/non singular behaviour of system (3.1)–(3.3) depending on the value of the model’s parameter $\lambda$. The limiting case $\lambda = -1$ gives an infinite-time singularity.

- If $-1 < \lambda \leq -1/2$ then the singularity occurs at infinite time: $T^* = \infty$ and $\lim_{t \to T^*} \dot{S}(t) = 0$.
- If $1/2 < \lambda \leq 0$ then the singularity occurs at finite time: $T^* < \infty$ and $\lim_{t \to T^*} \dot{S}(t) = 0$.
- If $0 < \lambda$ then the singularity occurs at finite time: $T^* < \infty$ and $0 < \lim_{t \to T^*} \dot{S}(t) < \infty$.

We now produce a study for real values of the parameter $\lambda \neq -1$, for the type of blowup for the stretching rate ($\gamma$) and vorticity ($\omega$) (equation (3.14) and (3.17) respectively). If $\lambda \in (-\infty, -1)$ then the stretching $\gamma$ blows up at time $T^*$ at only one point $(X(T^*), Y(T^*))$, corresponding to the unique characteristic with initial position $(X_0, Y_0) = (X_\gamma, Y_\gamma)$ at which $\gamma_0$ attains its global maximum, i.e., $\gamma_0(X_\gamma, Y_\gamma) = (S^*|\lambda + 1|)^{-1}$ (equation (3.18)). The asymptotic blowup at that characteristic is defined explicitly as:

$$
\sup_{(x, y, t)} \gamma(x, y, t) \approx \gamma(X(t), Y(t)) \approx \frac{1}{|\lambda + 1|} (T^* - t)^{-1}.
$$

(3.20)

If $\lambda \in (0, \infty)$ then the stretching rate $\gamma$ blows up at time $T^*$ at only one point $(X(T^*), Y(T^*))$, corresponding to the unique characteristic with initial position...
$(X_0, Y_0) = (X_-, Y_-)$ at which $\gamma_0$ attains its global minimum, i.e., $\gamma_0(X_-, Y_-) = -(S^*(\lambda + 1))^{-1}$ (equation (3.18)). Explicitly we have the asymptotic blowup at that characteristic:

$$\inf \gamma(x, y, t) \approx \gamma(X(t), Y(t)) \approx - \frac{1}{\lambda + 1} \left( T^* - t \right)^{-1}. \quad (3.21)$$

### 3.3 Mapping to regular fields and their evolution equations

Regardless of the availability of an analytical solution for the relevant fields, Bustamante [11] developed a general theory to study nonlinear evolution equations whose solutions present evidence of possible finite-time singularity. The main idea is to transform the original physical variables (such as velocity field) into new, so-called mapped variables which are regular (globally in time), and thus more amenable to new analytical studies and more accurate numerical studies. The transformation, or mapping, has applications in a wide variety of PDE models including 3D Euler/Navier-Stokes fluid equations, 3D and 2D magnetohydrodynamics equations, Burgers equations, etc. The key ingredient to construct this mapping is a type of Beale-Kato-Majda (BKM) theorem [5], which states that all relevant norms of the velocity field are bounded if and only if $\tau(t) = \int_0^t F[u](t')dt'$, is bounded: where $F[u]$ is a given functional of the velocity field $u$. In the case of our 2D symmetry-plane model (3.9) and (3.10), following a classical analysis analogous to that in Gibbon & Ohkitani [45] we deduce that the functional is $F[u_h](t') \equiv \|\gamma(\cdot, t')\|_\infty$, the $L^\infty$ norm of the stretching rate $\gamma(x, y, t)$ over the spatial domain $\mathbb{T}^2$. Explicitly, the boundedness of the integral

$$\tau(t) = \int_0^t \|\gamma(\cdot, t')\|_\infty \, dt' \quad (3.22)$$

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will ensure the continuity of the velocity field \( u_h \) until time \( t \) (provided the initial conditions are smooth). For example, if \( \tau(t) \) is bounded then vorticity is bounded because, from equation (3.17), it follows

\[
|\omega(X(t), Y(t), t)| = |\omega_0(X_0, Y_0)| \exp \left( \int_0^t \gamma(X(t'), Y(t'), t') \, dt' \right)
\leq |\omega_0(X_0, Y_0)| \exp \left( \int_0^t \|\gamma(\cdot, t')\|_\infty \, dt' \right)
= |\omega_0(X_0, Y_0)| \exp(\tau(t)).
\]

The bijective mapping from “original variables” to “mapped, regular variables” consists of the time mapping \( t \rightarrow \tau \), equation (3.22), along with a re-scaling of stretching rate, vorticity and velocity vector fields:

\[
\gamma_{\text{map}}(x, y, \tau) = \frac{\gamma(x, y, t)}{\|\gamma(\cdot, t)\|_\infty}, \quad (3.23)
\]

\[
\omega_{\text{map}}(x, y, \tau) = \frac{\omega(x, y, t)}{\|\gamma(\cdot, t)\|_\infty}, \quad (3.24)
\]

\[
\Longrightarrow u_{\text{map}}(x, y, \tau) = \frac{u_h(x, y, t)}{\|\gamma(\cdot, t)\|_\infty}.
\]

For this bijective mapping to lead to tractable evolution equations for the mapped variables, the “BKM” functional \( \|\gamma(\cdot, t)\|_\infty \) must have a time derivative that can be expressed in terms of the original variables. In our case, equation (3.9) implies

\[
\frac{d}{dt} (\|\gamma(\cdot, t)\|_\infty) = \sigma_\infty \left[ (2 + \lambda) (\gamma^2) - (1 + \lambda) \|\gamma(\cdot, t)\|_\infty^2 \right], \quad (3.25)
\]

where

\[
\sigma_\infty = \text{sign} \, \gamma(X_\gamma(t), t) \quad (3.26)
\]

is the sign of \( \gamma \) at the position \( X_\gamma(t) \) where the maximum of \( |\gamma(x, t)| \) is attained.
We now derive the equations verified by this new field

\[
\frac{\partial \gamma_{\text{map}}}{\partial \tau} = \frac{\partial \gamma}{\partial \tau} \frac{1}{\|y(\cdot, t)\|_\infty} + \gamma \frac{\partial}{\partial \tau} \left( \frac{1}{\|y(\cdot, t)\|_\infty} \right)
\]

\[
= \frac{\partial \gamma}{\partial t} \frac{1}{\|y(\cdot, t)\|_\infty} \frac{dt}{d\tau} + \gamma \frac{\partial}{\partial t} \left( \frac{1}{\|y(\cdot, t)\|_\infty} \right) \frac{dt}{d\tau}
\]

\[
= \frac{\partial \gamma}{\partial t} \frac{1}{\|y(\cdot, t)\|_\infty} \|y(\cdot, t)\|_\infty + \gamma \frac{\partial}{\partial t} \left( \frac{1}{\|y(\cdot, t)\|_\infty} \right) \|y(\cdot, t)\|_\infty
\]

\[
= \frac{1}{\|y(\cdot, t)\|_\infty^2} (2 + \lambda)(\gamma^2) - (1 + \lambda)\gamma^2 - u \cdot \nabla \gamma + \gamma_{\text{map}} \frac{\partial}{\partial t} \left( \frac{1}{\|y(\cdot, t)\|_\infty} \right)
\]

\[
= (2 + \lambda)(\gamma_{\text{map}}^2) - (1 + \lambda)\gamma_{\text{map}}^2 - u_{\text{map}} \cdot \nabla \gamma_{\text{map}} + \gamma_{\text{map}} \frac{\partial}{\partial t} \left( \frac{1}{\|y(\cdot, t)\|_\infty} \right).
\]

From equation (3.25), we have

\[
\frac{\partial}{\partial t} \left( \frac{1}{\|y(\cdot, t)\|_\infty} \right) = -\frac{1}{\|y(\cdot, t)\|_\infty} \frac{d}{dt} \left( \|y(\cdot, t)\|_\infty \right)
\]

\[
= -\frac{1}{\|y(\cdot, t)\|_\infty} \left( \sigma_\infty \left( (2 + \lambda)(\gamma^2) - (1 + \lambda)\gamma^2 \right) \right)
\]

\[
= \sigma_\infty \left( -2 + \lambda \gamma^2 \right) + 1 + \lambda.
\]

Similarly mapping the vorticity field

\[
\frac{\partial \omega_{\text{map}}}{\partial \tau} = \frac{\partial \omega}{\partial \tau} \frac{1}{\|y(\cdot, t)\|_\infty} + \omega \frac{\partial}{\partial \tau} \left( \frac{1}{\|y(\cdot, t)\|_\infty} \right)
\]

\[
= \frac{\partial \omega}{\partial t} \frac{1}{\|y(\cdot, t)\|_\infty} \frac{dt}{d\tau} + \omega \frac{\partial}{\partial t} \left( \frac{1}{\|y(\cdot, t)\|_\infty} \right) \frac{dt}{d\tau}
\]

\[
= \frac{\partial \omega}{\partial t} \frac{1}{\|y(\cdot, t)\|_\infty} \|y(\cdot, t)\|_\infty + \omega \frac{\partial}{\partial t} \left( \frac{1}{\|y(\cdot, t)\|_\infty} \right) \|y(\cdot, t)\|_\infty
\]

\[
= \frac{1}{\|y(\cdot, t)\|_\infty^2} (\gamma \omega - u \cdot \nabla \omega) + \omega_{\text{map}} \frac{\partial}{\partial t} \left( \frac{1}{\|y(\cdot, t)\|_\infty} \right)
\]

\[
= \gamma_{\text{map}} \omega_{\text{map}} - u_{\text{map}} \cdot \nabla \omega_{\text{map}} + \omega_{\text{map}} \frac{\partial}{\partial t} \left( \frac{1}{\|y(\cdot, t)\|_\infty} \right).
\]
The mapped variables thus satisfy the following system of evolution equations:

$$\frac{\partial \gamma_{\text{map}}}{\partial \tau} + \mathbf{u}_{\text{map}} \cdot \nabla \gamma_{\text{map}} = (2 + \lambda)(\gamma_{\text{map}}^2) - (1 + \lambda)\gamma_{\text{map}}^2$$

$$+ \sigma_{\infty} \gamma_{\text{map}} \left\{ 1 + \lambda - (2 + \lambda)(\gamma_{\text{map}}^2) \right\} , \quad (3.27)$$

$$\frac{\partial \omega_{\text{map}}}{\partial \tau} + \mathbf{u}_{\text{map}} \cdot \nabla \omega_{\text{map}} = \gamma_{\text{map}} \omega_{\text{map}}$$

$$+ \sigma_{\infty} \omega_{\text{map}} \left\{ 1 + \lambda - (2 + \lambda)(\gamma_{\text{map}}^2) \right\} . \quad (3.28)$$

These equations are supplemented with the initial constraint $\|\gamma_{\text{map}}(\cdot,0)\|_\infty = 1$. They differ from the original system simply by extra “drag” terms, which ensure that $\|\gamma_{\text{map}}(\cdot, \tau)\|_\infty = 1$ for all $\tau < \infty$. The most striking result is that, as the condition $\tau < \infty$ implies boundedness of the original fields, the solution of the mapped system (3.27) and (3.28) is globally regular in time $\tau$.

### 3.3.1 Using the mapped system to assess blowup of original system

The mapping (3.22), (3.23) and (3.24) is bijective as long as $\tau < \infty$. Correspondingly, integration of the mapped system (3.27) and (3.28) should give enough information to assess blowup quantities in the original variables.

The norm $\|\gamma(\cdot, t)\|_\infty$ satisfies the ODE (3.25). In terms of the mapped time $\tau$ and the mapped stretching rate $\gamma_{\text{map}}(x, y, \tau)$, this equation reads

$$\frac{d}{d\tau} \ln(\|\gamma(\cdot, \tau(t))\|_\infty) = \sigma_{\infty} \left[ (2 + \lambda)(\gamma_{\text{map}}^2) - (1 + \lambda) \right] ,$$

where we have used

$$\frac{d\tau}{dt} = \|\gamma(\cdot, \tau(t))\|_\infty .$$

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Correspondingly we obtain, after a simple $\tau$ integration,
\[
\|\gamma(\cdot, t(\tau))\|_\infty = \|\gamma_0\|_\infty \exp \left[-(1 + \lambda) \int_0^\tau \sigma_\infty \, d\tau' + (2 + \lambda) \int_0^\tau \sigma_\infty \langle \gamma^2_{\text{map}} \rangle \, d\tau' \right].
\]  
(3.29)

Note that the right hand side is written entirely in terms of mapped fields. The integrands $\sigma_\infty$ and $\sigma_\infty \langle \gamma^2_{\text{map}} \rangle$ are bounded by 1 so, remarkably, the blowup assessment of the original variables is done in terms of bounded quantities. In particular, this leads to the following general formula for the singularity time $T^*$:
\[
T^* = \frac{1}{\|\gamma_0\|_\infty} \int_0^\infty \exp \left[(1 + \lambda) \int_0^\tau \sigma_\infty \, d\tau' - (2 + \lambda) \int_0^\tau \sigma_\infty \langle \gamma^2_{\text{map}} \rangle \, d\tau' \right] \, d\tau.  
\]  
(3.30)

Notice that this integral converges if and only if the original problem has a finite-time singularity.

From equation (3.10) it follows that the norm $\|\omega(\cdot, t)\|_\infty$ satisfies
\[
\frac{d}{dt} \ln \|\omega(\cdot, t)\|_\infty = \gamma(X_\omega(t), t),
\]
where $X_\omega(t)$ is the position at which the maximum of $|\omega(x, t)|$ is attained. In terms of the mapped variables, this reads
\[
\frac{d}{d\tau} \ln \|\omega(\cdot, t(\tau))\|_\infty = \gamma_{\text{map}}(X_\omega(t(\tau)), \tau),
\]
which gives
\[
\|\omega(\cdot, t(\tau))\|_\infty = \|\omega_0\|_\infty \exp \left[\int_0^\tau \gamma_{\text{map}}(X_\omega(t(\tau')), \tau') \, d\tau' \right].  
\]  
(3.31)

Remarkably again, the blowup assessment of the original vorticity depends on a
bounded quantity \((-1 \leq \gamma_{\text{map}} \leq 1)\).

### 3.4 Numerical solution of original and mapped systems and comparison with analytic solution

We turn our attention to the original evolution equations (3.9) and (3.10) and the mapped evolution equations (3.27) and (3.28). Both systems were solved numerically using a standard pseudospectral method (details in Appendix B). To remove the usual aliasing errors, Hou’s high-order exponential filter [57] is used, where we multiply the spectrum at each time step by the factor \(\exp\left(-36(2k/N)^{36}\right)\), where \(k\) is the modulus of the wavevector and \(N\) is the spatial resolution. We checked that the 2/3 dealiasing rule (in which the last 1/3 of the high frequency modes are set to zero) gives similar results, but Hou’s filter provides sensible spectra for a slightly broader range of wavevectors.

In both systems time marching was carried out using the fourth-order Runge-Kutta (RK4) method (details in Appendix B). In the mapped system, a uniform time step \(d\tau\) is used since \(\tau\) stretches the temporal domain such that the singularity is at \(\tau = \infty\). This uniform \(d\tau\) would imply that in the original system the time step \(dt\) should be adaptive, via \(dt = d\tau/\|\gamma(\cdot, t)\|_{\infty}\). This adaptive method is normally used in 3D Euler blowup assessment studies for reasons of accuracy close to singularity. In the results below adaptive time stepping in original variables is used for this reason, with the added advantage that the data from the original and the mapped systems are more comparably spaced.

Finally, unlike the integration of the original system, the numerical integration of the mapped system requires a special method: a normalisation so that the mapped system satisfies the constraint \(\|\gamma_{\text{map}}(\cdot, \tau)\|_{\infty} = 1\), for all \(\tau\). The accuracy of this normalisation is essential for the performance of the mapped system’s nu-
merical integration. To apply the normalisation, \( \| \gamma_{\text{map}}(\cdot, \tau) \|_\infty \) is computed using a \( P_{6,k} \) quarter-section interpolation. This \( P_{6,k} \) interpolation is an efficient procedure to compute the field’s maximum and its position using an iterative application of cubic-splines at progressively finer resolution. This was tested against several other interpolation methods. See the discussion in appendix A.

The initial conditions for this chapter were chosen as

\[
\begin{align*}
\gamma_0(x, y) &= \sin(x) \sin(y) - \cos(y), \\
\omega_0(x, y) &= -\sin(x) - \cos(x) \cos(y).
\end{align*}
\]

(3.32) (3.33)

Figure 3.2 shows perspective and isosurface plots of the initial condition for both the stretching rate and vorticity. The discrete symmetry \( (x \rightarrow \pi + x, y \rightarrow -y, u_x \rightarrow u_x, u_y \rightarrow -u_y) \) of this initial condition is preserved under the time evolution. Thus, we can restrict the analysis to the half-plane \( \pi \leq x \leq 2\pi, \ 0 \leq y \leq 2\pi \).

### 3.4.1 Case with infinite time singularity \( \lambda = -1/2 \)

We shall briefly discuss results for a case with infinite time singularity, \(-1 < \lambda \leq -1/2 \) (see figure 3.1). Snapshots of perspective and isosurface plots for both the stretching rate and vorticity at various times in this case when \( \lambda = -1/2 \) are shown in figures 3.3 and 3.4 respectively.

We show the time evolution of the maximum norms of stretching rate \( \| \gamma(\cdot, t) \|_\infty \) and vorticity \( \| \omega(\cdot, t) \|_\infty \) in figure 3.5. From this figure we see that the maximum norm of stretching rate which controls blow up does not increase. However, we see an increase of the maximum norm of vorticity with time. At time \( t = 2 \), figure 3.4 already show the spatial distribution of vorticity getting localised in sharp structures. This increases with time leading to a finite-time loss of resolution.
Figure 3.2: Perspective and isosurface plots for stretching rate ($\gamma$) (top) and vorticity ($\omega$) (bottom) of the initial conditions (equations (3.32) and (3.33)) for the 2D Euler model.
Figure 3.3: Snapshots of perspective and isosurface plots of stretching rate ($\gamma$) for $\lambda = -1/2$ at $t = 1.0$ and $t = 2.0$ from the 2D Euler symmetry plane model.
Figure 3.4: Snapshots of perspective and isosurface plots of vorticity ($\omega$) for $\lambda = -1/2$ at $t = 1.0$ and $t = 2.0$ from the 2D Euler symmetry plane model.

Figure 3.5: Temporal evolution of maximum norm of stretching rate $\|\gamma(\cdot, t)\|_\infty$, left panel, and vorticity $\|\omega(\cdot, t)\|_\infty$, right panel, for $\lambda = -1/2$ at resolution $N = 1024$. 
3.4.2 Case with finite time singularity $\lambda = -3/2$

We now turn our attention to a case where we have finite time singularity, $\lambda < -1$. The results below in this chapter will henceforth concentrate on the case $\lambda = -3/2$.

3.4.2.1 Motivation for the choice of parameter $\lambda = -3/2$

We will focus on one particular choice of parameter: $\lambda = -3/2$. This choice has several advantages:

- In this case the evolution equation for $S(t)$ is of the form
  \[ \dot{S} = 1 + a^2 S^2, \quad S(0) = 0, \]
  which can be solved analytically for any initial condition ($a$ depends on the initial condition), leading to explicit analytical solutions in terms of trigonometric functions for all the blowup quantities. The utility of this is that we can perform direct comparisons between theory and numerics.

- This case gives finite-time singularity with $\dot{S}(T^*) < \infty$ (figure 3.6) leading to simple asymptotic expressions for the blowup quantities. The singularity is controlled by the supremum of $\gamma_0$ and leads to a blowup of the form $\sup \gamma(x, y, t) \sim 2(T^* - t)^{-1}$ (using equation 3.20) when $t$ is close to the singularity time $T^*$. This feature is analogous to what is normally expected in a 3D Euler fluid simulation in a potential singularity scenario.

- In this case there exists a special conserved quantity: $\langle \gamma^2 \rangle$, which is reminiscent of the “energy” in 2D and 3D ideal models, and provides an opportunity for a simplified analysis of the Fourier spectrum.

We will exploit analytical solutions (3.14) for stretching rate and (3.17) for vorticity in order to validate direct numerical simulations of the system. For ex-
ample, using the fact that the back-to-labels transformation is bijective for $t < T^*$, one can use equations (3.14), (3.15) and (3.17) to calculate the infimum and supremum of stretching rate and vorticity from the knowledge of the initial conditions. This gives either explicit expressions in terms of simple functions or numerically computable expressions to any desired accuracy.

We evaluate the analytical solutions for $\lambda = -3/2$, which are summarised in table 3.1. Using equation (3.15) and the initial condition for stretching rate equation (3.32) we get

$$
\dot{S}(t) = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} \left[ 1 - \frac{1}{2} (\sin x \sin y - \cos y)S \right]^2 \, dx \, dy
$$

This simplifies to

$$
\dot{S}(t) = 1 + \frac{3}{16} S^2.
$$
Integrating this equation using the method of separation of variables we obtain

\[ S = \frac{4}{\sqrt{3}} \tan \left( \frac{\sqrt{3}}{4} t \right). \]  

(3.34)

Using our initial condition (equation (3.32)), we have \( \sup_{(x,y)\in T^2} \gamma_0(x,y) = \sqrt{2} \) which using equation (3.18) gives \( S \) at singularity time \( S^* = \sqrt{2} \). We can then compute the singularity time explicitly as

\[ T^* = \frac{4}{\sqrt{3}} \arctan \left( \frac{\sqrt{6}}{4} \right) \]
\[ \approx 1.268940246686793 \]  

(3.35)

From equation (3.14) we solve for the supremum norm of \( \gamma \) and obtain

\[ \sup \gamma = -\frac{S}{2(\lambda + 1)S} + \frac{\dot{S} \gamma_{0_{\text{max}}}}{(1 + (\lambda + 1)\gamma_{0_{\text{max}}}S(t))} \]  

(3.36)

From equation (3.34) we also have

\[ \dot{S}(t) = \frac{1}{\cos^2 \left( \frac{\sqrt{3}}{4} t \right)} \]

and

\[ \frac{\ddot{S}}{S} = (\ln \dot{S})' = -2 \frac{d}{dt} \left[ \ln \cos \left( \frac{\sqrt{3}}{4} t \right) \right] \]
\[ = 2 \frac{\sin \left( \frac{\sqrt{3}}{4} t \right)}{\cos \left( \frac{\sqrt{3}}{4} t \right)} \cdot \frac{\sqrt{3}}{4} \]

which reduces to

\[ \frac{\ddot{S}}{S} = \frac{\sqrt{3}}{2} \tan \left( \frac{\sqrt{3}}{4} t \right). \]
Equation (3.36) thus simplifies to

\[
\sup \gamma = \frac{\sqrt{3}}{2} \tan \left( \frac{\sqrt{3}}{4} T \right) + \frac{\sqrt{2}}{\cos^2 \left( \frac{\sqrt{3}}{4} T \right)} \left( 1 - \frac{4}{\sqrt{3}} \tan \left( \frac{\sqrt{3}}{4} r \right) \right).
\]  

(3.37)

Since \( \sup \gamma = \| \gamma \|_{\infty} \) for the initial conditions (3.32) and (3.33), (see figure 3.7), we solve for \( \tau(t) \) using

\[
\tau(t) = \int_0^t \sup \gamma(t') dt'.
\]

\[
= \int_0^t \frac{\sqrt{3}}{2} \tan \left( \frac{\sqrt{3}}{4} t' \right) + \frac{\sqrt{2}}{\cos^2 \left( \frac{\sqrt{3}}{4} r' \right)} \left( 1 - \frac{4}{\sqrt{3}} \tan \left( \frac{\sqrt{3}}{4} r' \right) \right) dt'.
\]

Simplifying and rearranging we obtain

\[
\tau(t) = -2 \ln \left( \cos \left( \frac{\sqrt{3}}{4} t \right) \right) - 2 \ln \left( 1 - 2 \sqrt{\frac{2}{3}} \tan \left( \frac{\sqrt{3}}{4} t \right) \right).
\]

(3.38)

Figure 3.7: Maximum and minimum values of stretching rate (\( \gamma \)) from the 2D Euler model for \( \lambda = -3/2 \) using initial conditions in equations (3.32) and (3.33).
Case \( \lambda = -3/2 \)

Initial condition
\[
\begin{align*}
\gamma_0(x, y) &= \sin(x) \sin(y) - \cos(y) \\
\omega_0(x, y) &= -\sin(x) - \cos(x) \cos(y)
\end{align*}
\]

Singularity time
\[
T^* = \frac{4}{\sqrt{3}} \arctan \left( \frac{\sqrt{3}}{4} \right) \approx 1.26894
\]

Solution for \( S(t) \)
\[
S(t) = \frac{4}{\sqrt{3}} \tan \left( \frac{\sqrt{3} t}{4} \right)
\]

\( S(T^*) = S^* \) and \( \dot{S}(T^*) \)
\[
S^* = \sqrt{2}, \quad \dot{S}(T^*) = 11/8
\]

\[
\begin{align*}
||\gamma(\cdot, t)||_\infty &= \left( \sup_{(x,y) \in \mathbb{T}^2} \gamma(x, y, t) \right) \\
\sup_{(x,y) \in \mathbb{T}^2} \gamma(x, y, t) &= \frac{\sqrt{3}}{2} \tan \left( \frac{\sqrt{3} t}{4} \right) + \frac{\sqrt{3}}{\cos \left( \frac{\sqrt{3} t}{4} \right) \left( 1 - \frac{1}{\sqrt{3}} \frac{1}{\tan \left( \frac{\sqrt{3} t}{4} \right)} \right)}
\end{align*}
\]

\[
\begin{align*}
\inf_{(x,y) \in \mathbb{T}^2} \gamma(x, y, t) &= -\frac{11}{2 \sqrt{3} \cos \left( \frac{\sqrt{3} t}{4} \right) + 4 \sqrt{2}} - \sqrt{2} \quad \text{(bounded)}
\end{align*}
\]

\[
\left\langle \left| \gamma(\cdot, t) \right|^2 \right\rangle_{\mathbb{T}^2} = \frac{3}{4} \quad \text{(constant, only in the case } \lambda = -3/2 \text{)}
\]

Vorticity at position of \( ||\gamma(\cdot, t)||_\infty \)
\[
\omega(X, t) = \frac{\sec \left( \frac{\sqrt{3} t}{4} \right)}{\left( 1 - 2 \frac{1}{\sqrt{3}} \frac{1}{\tan \left( \frac{\sqrt{3} t}{4} \right)} \right)^2}
\]

Asymptotics as \( t \to T^* \)
\[
\sup \gamma \sim 2 (T^* - t)^{-1}
\]
\[
\sup \omega \sim \frac{16}{11} (T^* - t)^{-2}
\]

Table 3.1: Summary of analytical results for the case \( \lambda = -3/2 \). Supremum of vorticity is computable numerically from formula (3.17).
3.4.2.2 Analytical solution for the mapped variables in the case $\lambda = -3/2$

The analytical solution along characteristics (3.14), (3.15) and (3.17) leads to a connection with the mapped variables. Let us consider the case $\lambda = -3/2$ and the initial conditions (3.32) and (3.33). Since $\lambda < -1$, the supremum of $\gamma$ blows up and this happens along the characteristic starting at the position of the supremum of $\gamma_0$, $X_+ = \left(\frac{3\pi}{2}, \frac{5\pi}{4}\right)$ (see figure 3.2 for reference). On the other hand, the infimum of $\gamma$ (a negative quantity) remains small in size (figure 3.7). Thus, in this case the norm $\|\gamma(\cdot, t)\|_\infty$ can be identified with $\sup \gamma(\cdot, t)$ (i.e. one can set $\sigma_\infty \equiv 1$) for all times $0 \leq t < T^*$. Equation (3.14) evaluated at $X_0 = X_+$ is then compared with definition of mapped time $\tau$ (equation (3.22)) to give

$$\tau(t) = \ln \frac{1}{\cos \left(\frac{\sqrt{3} t}{4}\right) - 2 \sqrt{\frac{2}{3}} \sin \left(\frac{\sqrt{3} t}{4}\right)}^2,$$

which coincides with the explicit result (3.38). The above relation can be inverted to solve for $t$ as a function of $\tau$, provided $t < T^*$. The supremum of original stretching rate, in terms of $\tau$, can be obtained after using this inversion along with the formula in table 3.1, giving

$$\|\gamma(\cdot, t(\tau))\|_\infty = \frac{1}{2} e^{\tau/2} \sqrt{11 - 3 e^{2\tau}}, \quad 0 \leq \tau < \infty.$$  (3.39)

There is a simple analytical expression (in terms of mapped time $\tau$) for vorticity at the position where the maximum of $|\gamma(x, t)|$ is attained, $X_+(t)$. Equation (3.17) evaluated at $X_0 = X_+$ gives

$$\omega(X_+(t(\tau)), t(\tau)) = e^{\tau}, \quad 0 \leq \tau < \infty.$$  (3.40)

Notice that this is a lower bound for the $L^\infty$ norm of the vorticity. The latter norm can be obtained at all times in terms of the initial conditions, by maximising
the right hand side of equation (3.17) over the region $\pi \leq x \leq 2\pi$, $0 \leq y \leq 2\pi$. Although this rarely leads to an explicit analytical expression for $\|\omega(\cdot, t(\tau))\|_{\infty}$, it can always be computed numerically to any desired accuracy.

3.4.2.3 Numerical results from the original and mapped systems for the case $\lambda = -3/2$

Figures 3.8 and 3.9 show snapshots of perspective and isosurface plots for both the stretching rate and vorticity respectively, at various times. Figure 3.10 shows the maximum norms of stretching rate and vorticity fields increase very rapidly at late time and are resolution dependent with the values in agreement up to some resolution-dependent time.

We produce standard resolution convergence studies of the quantities of the stretching rate of vorticity. Figure 3.11, top, shows the classical convergence study of the multiplicative inverse of the supremum norm of stretching rate, using the numerical solution of the original system, plotted as a function of original time $t$. Good convergence to the analytical result is obtained. This is the basis for the method of computing running estimates of singularity time $T^*$ (Method A in Section 3.4.2.7). Figure 3.11, bottom, shows the less classical (but similar in spirit) lin-log convergence study of the supremum norm of stretching rate, again using the numerical solution of the original system, plotted as a function of mapped time $\tau$. Again, good convergence to the analytical asymptote is obtained. Finally, figure 3.12 shows the lin-log convergence study of the spatial average of the square of the mapped stretching rate, $\langle \gamma_{\text{map}}^2 \rangle$, using the numerical solution of the mapped system. Good convergence to the analytical asymptote is verified.
Figure 3.8: Snapshots of perspective and isosurface plots of stretching rate ($\gamma$) for $\lambda = -3/2$ at $t = 0.5$ and $t = 1.0$ from the 2D Euler symmetry plane model.
Figure 3.9: Snapshots of perspective and isosurface plots of vorticity ($\omega$) for $\lambda = -3/2$ at $t = 0.5$ and $t = 1.0$ from the 2D Euler symmetry plane model.
Figure 3.10: Temporal evolution of maximum norm of stretching rate $\|\gamma(\cdot, t)\|_\infty$ (top) and vorticity $\|\omega(\cdot, t)\|_\infty$ (bottom) as at different resolutions $N = 256, 512$ and 1024.
Figure 3.11: Two different resolution studies of the data from original system at different resolutions $N = 256, 512, 1024, 2048$. Top: Classical plot of $1/\|\gamma(\cdot, t)\|_\infty$, giving a convergence to the analytically-obtained asymptotic regime $1/\|\gamma(\cdot, t)\|_\infty \sim (T^* - t)/2$ (solid line). Bottom: New plot, in lin-log scaling, of $\|\gamma(\cdot, t(\tau))\|_\infty$ as a function of $\tau$, giving a convergence to the analytically-obtained asymptotic regime $\|\gamma(\cdot, t(\tau))\|_\infty \sim \sqrt{1/2} \exp(\tau/2)$ (solid line).
Figure 3.12: Plots of $\langle \gamma_{\text{map}}(\cdot, \tau)^2 \rangle$ as a function of $\tau$ using data from the numerical integration of the mapped system, at different resolutions: from top to bottom, $N = 256, 512, 1024, 2048$. The curves converge to the analytically-obtained asymptotic regime $\langle \gamma_{\text{map}}(\cdot, \tau)^2 \rangle \sim (3/11) \exp(-\tau)$.

3.4.2.4 Errors in local quantities near blowup: $\|\gamma(\cdot, t)\|_{\infty}$ and $\omega(X_\gamma(t), t)$

We now compare the accuracy of the results from the original and mapped systems. A sensible definition of the error of the numerical simulation of the original 2D Euler model is the relative difference between the supremum norm of $\gamma$ obtained from the numerical simulation and the exact analytical formula, given in table 3.1. We can also define the error in $\omega$ by evaluating it at $X_\gamma(t)$, the location of the supremum of $|\gamma|$, also given analytically in table 3.1. We define the relative errors as

$$E_\gamma(t) = \left| \frac{\|\gamma_{\text{num}}(\cdot, t)\|_{\infty}}{\|\gamma_{\text{ana}}(\cdot, t)\|_{\infty}} - 1 \right|,$$

(3.41)

$$E_\omega(t) = \left| \frac{\omega_{\text{num}}(X_\gamma(t), t)}{\omega_{\text{ana}}(X_\gamma(t), t)} - 1 \right|,$$

(3.42)
where the subscripts “num” and “ana” stand for “numerical” and “analytic”. While these are useful for instantaneous monitoring purposes, given a time series of a numerical solution of $\gamma$ or $\omega$ we would like to measure the error using a single number for each variable. The naive measure in terms of the $L^2$ norm of the relative error,

$$
\|E\|_2 \equiv \sqrt{\int_0^T [E(t)]^2 \, dt},
$$

is not the best choice because it is not bounded \textit{a priori}. Given two signals $f(t)$ and $g(t)$ for comparison, we define an $L^2$ norm of the error (\textit{not} relative error) normalised with the sum of norms of the individual signals [92].

$$
Q(f, g) = \frac{\|f - g\|_2}{\|f\|_2 + \|g\|_2}
$$

which has 3 advantageous properties:

- $Q$ and $\|E\|_2$ are proportional if they are small: $\|E\|_2 \propto \sqrt{T} Q$ if $E \ll 1$;
- $Q$, being dimensionless, does not explicitly require a time scale ($T$) so it can be applied in a variety of contexts. In practice, though, and for assessment purposes, we will normally plot $Q$ as a function of the total integration time $T$;
- $Q$ is bounded: $0 \leq Q \leq 1$, with value 0 representing perfect match and value 1 representing perfect mismatch.

Thus we work with

$$
Q_\gamma = Q(\|\gamma_{num}(\cdot, t)\|_\infty, \|\gamma_{ana}(\cdot, t)\|_\infty)
$$

and

$$
Q_\omega = Q(\omega_{num}(X_\gamma, t), \omega_{ana}(X_\gamma, t)).
$$
To gain an appropriately accurate estimate of these errors we must consider the best method for approximating \( \| \gamma_{\text{num}}(\cdot, t) \|_\infty, X_\gamma \) and \( \omega_{\text{num}}(X_\gamma, t) \). The simplest approach is to apply the maximum value across the collocation points of the discretised field, however this leads to significant spurious oscillations. The more accurate procedure is to perform some post-processing interpolation. We use the same highly-accurate interpolation as in the normalisation procedure in the mapped system. The various interpolation options are described in appendix A.

The numerical solution of the mapped system does not provide direct access to the original variable \( \| \gamma(\cdot, t) \|_\infty \) so in order to compute it we employ equation (3.29), namely

\[
\| \gamma(\cdot, t(\tau)) \|_\infty = \| \gamma_0 \|_\infty \exp \left[ -(1 + \lambda) \int_0^\tau \sigma_\infty \, d\tau' + (2 + \lambda) \int_0^\tau \sigma_\infty \langle \gamma_{\text{map}}^2 \rangle \, d\tau' \right],
\]

where \( \int_0^\tau \sigma_\infty \langle \gamma_{\text{map}}^2 \rangle \, d\tau \) is computed using Simpson’s rule. We compare this against equation (3.39). To evaluate the original variable \( \omega(X_\gamma(t(\tau)), t(\tau)) \) from the mapped variables we use

\[
\omega(X_\gamma(t(\tau)), t(\tau)) = \omega_{\text{map}}(X_\gamma(t(\tau)), \tau) \| \gamma(\cdot, t(\tau)) \|_\infty.
\]

We compare this against equation (3.40).

Figure 3.13 shows a comparison of the errors \( Q_\gamma \) and \( Q_\omega \) at various resolutions and demonstrates how the analysis of the mapped system along with its numerical solution serve to improve the accuracy of the blowup quantities \( \| \gamma(\cdot, t) \|_\infty \) and \( \omega(X_\gamma(t), t) \) near the singularity time.

Note that we are using the global quantity \( \langle \gamma_{\text{map}}^2 \rangle \) for the assessment of the local quantity \( \| \gamma(\cdot, t) \|_\infty \). Therefore, errors of the global quantity \( \langle \gamma_{\text{map}}^2 \rangle \) might affect the errors of this assessment. To see at which times these errors might be important, figure 3.14 shows the ratio between the terms \((2 + \lambda) \int_0^\tau \sigma_\infty \langle \gamma_{\text{map}}^2 \rangle \, d\tau'\) and \(-(1 + \lambda) \int_0^\tau \sigma_\infty \, d\tau'\), appearing in the exponent in equation (3.29). It is clear that at early
times the global quantity has more influence on the size of the error $Q'_y$. This is addressed in section 3.4.2.5. In contrast, at late times the global quantity is not relevant and this explains why the size of the error $Q'_y$ remains small and stable.

### 3.4.2.5 Errors in global quantities near blowup: $\langle \gamma_{\text{map}}^2 \rangle$ and $\langle \gamma^2 \rangle$

Figure 3.13 shows that the numerical solution of the mapped system contains higher early-time errors in $Q'_y$ than the numerical solution of the original system. These errors do not affect the late-time (near singularity) behaviour and can be controlled by reducing the time step $d\tau$. They arise because the mapped equations contain additional terms (those proportional to $\sigma_\infty$ in equations (3.27) and (3.28)) which introduce an extra time scale in the $\tau$ variable, proportional to $\langle \gamma_{\text{map}}^2 \rangle^{-1}$. This time scale is bounded from below and goes to infinity as $\tau \to \infty$, so we are able to resolve it by reducing the time step $d\tau$ at early times. This extra time scale feeds into the error $Q'_y$ via formula (3.29) which gives the supremum norm of stretching rate in terms of the mapped variables. This entails the numerical approximation
Figure 3.14: Ratio between the terms $(2 + \lambda) \int_0^\tau \sigma_\infty \langle \gamma_{\text{map}}^2 \rangle \, d\tau'$ and $-(1 + \lambda) \int_0^\tau \sigma_\infty \, d\tau'$, case $\lambda = -3/2$, appearing in the exponent in equation (3.29).

of the integral $\int_0^\tau \langle \gamma_{\text{map}}^2 \rangle \, d\tau'$ which is sensitive to the extra time scale at early times. Figure 3.14 gives a quantitative measure of the significance of this integral term as a function of time.

We conclude that after $\tau \approx 6$ the integral term contributes less than 5% to the total exponent in equation (3.29). Therefore, at late times, the assessment of $\|\gamma\|_\infty$ using the mapped system’s numerical solution is controlled by the term $-(1 + \lambda) \int_0^\tau \sigma_\infty \, d\tau'$ in equation (3.29), which surprisingly does not depend on the numerical field $\gamma_{\text{map}}$, except through the term $\sigma_\infty$ which takes values ±1 according to its definition in equation (3.26). Recall that, analytically, in the case $\lambda = -3/2$ we have $\sigma_\infty = 1$ for all times, for the choice of initial conditions that we made (equations (3.32) and (3.33)). Hence, the dominant term in the exponent is just $-(1 + \lambda)\tau$ which, although still numerical, is a prescribed function of the timesteps.

The role of $\sigma_\infty$ computed numerically is illustrated by looking at the error $Q_y$, figure 3.13 left panel. For example, at resolution 2048, the jump observed at $\tau \approx 16$
is due to the fact that the numerical solution becomes under-resolved already at \( \tau \approx 6 \), and consequently the quantity \( \sigma_\infty(\tau) \) becomes noisy. We will discuss in detail the loss of spectral resolution of the numerical solutions in section 3.4.2.6.

We now perform a direct analysis of the errors in the global quantities \( \langle \gamma_{\text{map}}(\cdot, \tau) \rangle^2 \) and \( \langle \gamma(\cdot, t) \rangle^2 \), computed respectively from the numerical integrations of the mapped and original systems. We compute the error in the function

\[
\langle \gamma_{\text{map}}(\cdot, \tau) \rangle^2 = \frac{\langle \gamma(\cdot, t(\tau)) \rangle^2}{\| \gamma(\cdot, t(\tau)) \|_{L^\infty}^2}
\]

in two different ways:

- **Using the numerical solution of the original system:**

  \[
  Q_{\langle \gamma_{\text{map}} \rangle} = Q \left( \frac{\langle \gamma_{\text{num}}(\cdot, t(\tau)) \rangle^2}{\| \gamma_{\text{num}}(\cdot, t(\tau)) \|_{L^\infty}^2}, \frac{\langle \gamma_{\text{ana}}(\cdot, t(\tau)) \rangle^2}{\| \gamma_{\text{ana}}(\cdot, t(\tau)) \|_{L^\infty}^2} \right)
  \tag{3.43}
  \]

- **Using the numerical solution of the mapped system:**

  \[
  Q_{\langle \gamma_{\text{map}} \rangle} = Q \left( \frac{\langle \gamma_{\text{map, num}}(\cdot, \tau) \rangle^2}{\| \gamma_{\text{map, num}}(\cdot, \tau) \|_{L^\infty}^2}, \frac{\langle \gamma_{\text{ana}}(\cdot, t(\tau)) \rangle^2}{\| \gamma_{\text{ana}}(\cdot, t(\tau)) \|_{L^\infty}^2} \right)
  \tag{3.44}
  \]

Figure 3.15 shows the evolution of these errors. It is evident that they have a comparable size and behaviour. To understand this we also plot in the same figure the error

\[
Q_{\langle \gamma^2 \rangle} = Q(\langle \gamma_{\text{num}}^2(\cdot, t(\tau)) \rangle, \langle \gamma_{\text{ana}}(\cdot, t(\tau)) \rangle^2),
\]

computed directly from the numerical solution of the original system. This latter error is surprisingly small at all times, which implies that the error \( Q_{\langle \gamma_{\text{map}} \rangle} \) is dominated by the error in \( \| \gamma_{\text{num}}(\cdot, t(\tau)) \|_{L^\infty}^2 \). On the other hand, the error \( Q_{\langle \gamma_{\text{map}} \rangle} \) includes accumulated errors due to repeated normalisations of the field \( \gamma_{\text{map}} \) in the mapped
system. The fact that the two errors $Q_{\langle \gamma^2 \rangle}$ and $Q^*_{\langle \gamma^2 \rangle}$ are comparable indicates that they have the same origin: $\| \gamma_{\text{num}}(\cdot, t(\tau)) \|_{\infty}$.

A last comment about the global quantity $\langle \gamma(\cdot, t)^2 \rangle$. In the case $\lambda = -3/2$ an interesting coincidence occurs whereby this quantity is a constant of motion. This can be verified directly from the evolution equation (3.27). One may think of this situation as resembling 3D Euler, where energy is conserved numerically even at late times when resolution has been lost. The fact that our error $Q_{\langle \gamma^2 \rangle}$ remains consistently small for all times is thus not a surprise, rather a consequence of the pseudo-spectral method. For other values of $\lambda$, the error $Q_{\langle \gamma^2 \rangle}$ grows without bound due to loss of resolution (see e.g. chapter 4).

![Figure 3.15: Evolution of errors of global quantities $\langle \gamma^2 \rangle$ and $\langle \gamma^2_{\text{map}} \rangle$ at resolution $N = 1024$, as a function of the total integration time $\tau$, obtained from equations (3.43), (3.44) and (3.45). Notice that the curves $Q^*_{\langle \gamma^2 \rangle}$ (obtained from original system numerical data) and $Q^*_{\langle \gamma^2_{\text{map}} \rangle}$ (obtained from mapped system numerical data) are nearly the same, illustrating that their source of errors is the same.](image-url)

One should be careful about drawing too strong conclusions from these error

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measures. It should be remembered that the mapping does nothing to improve the spatial resolution of a calculation, so any small scale structures present in the flow may be expected to suffer from a loss of resolution at roughly the same time. How this contributes to the errors in certain measures of the flow will vary from one case to the next and from one measure to the next. To investigate this we consider the spectra of $\gamma$ in the following section.

3.4.2.6 Spectra, analyticity strip and Beale-Kato-Majda (BKM) theorem

An effective diagnostic for the spatial collapse associated with a typical finite-time blow-up scenario presents itself in the form of the analyticity strip method [12, 94]. Given the spectrum of spatial Fourier coefficients provided in our numerical simulation, the $L^2$ spectrum of $\gamma$ is defined as the sum of the squares of modulus of the Fourier coefficients over circular shells, or in short, the $L^2$ stretching-rate spectrum:

$$E(k, t) = \sum_{k-\frac{1}{2} \leq |k| < k+\frac{1}{2}} |\hat{\gamma}(k, t)|^2.$$  

While we know analytical solutions for the blowup quantities, so far we have not found a method to obtain analytical expressions regarding the stretching-rate spectrum, other than the formula valid for $\lambda = -3/2$, stating $\sum_{k=1}^{\infty} E(k, t) = 3/4$ (a constant).

The lack of analytical results for spectra is seen as an advantage since it allows us to test the analyticity-strip method and its bridge with the BKM theorem [12] from a purely numerical point of view. The results presented in this section can therefore be contrasted against future analytical developments.

The function $\gamma(x, t)$ remains analytic in the space variables if $E(k, t)$ can be bounded by

$$E(k, t) \leq C_E(t) k^{-n_E(t)} e^{-2k\delta_E(t)},$$
where $\delta_E(t)$ is the analyticity strip width, also known as the logarithmic decrement, and $C_E(t), n_E(t)$ are positive numbers. We assume this approximation holds for our functions. Figure 3.16 shows snapshots of the $L^2$ spectrum $E(k,t)$, in log-log as well as lin-log scaling, to provide evidence of the feasibility of this approximation. The common procedure is to find the coefficients $C_E(t), n_E(t), \delta_E(t)$ by performing a least-squares fit, at each time $t$, on $\ln E(k,t)$ over some interval $k_i \leq k \leq k_f$. The problem becomes linear in the parameters $\ln C_E(t), n_E(t)$ and $\delta_E(t)$. More details can be found in, e.g., Bustamante & Brachet [12], equation (5).

It is customary to define a “reliability time barrier” $t_{rel}$ by the condition

$$\delta_E(t) \leq dx \Leftrightarrow t \leq t_{rel},$$

where $dx$ is the grid spacing of the numerical simulation. This barrier represents the obvious requirement that the smallest scales available in the numerical simulation are well resolved. Figure 3.17 shows the results at resolution $N = 2048$ for the fit parameters at several times (in mapped time $\tau$). Figure 3.17 bottom right panel indicates a reliability time $t_{rel} \approx 4.2$, corresponding to $t_{rel} \approx 1.121$. In figure 3.17 top left panel, the + symbol shows $n_E(t)$ with the remarkable convergence to $n_E = 5/3 \pm 0.08$ (dotted horizontal line) near reliability time. The error bar 0.08 stands for the 5% error of the least-squares fit procedure.

A classical method used in Bustamante & Brachet [12] gives rise to the following rigorous inequality:

$$\|\gamma(\cdot, t)\|_\infty \leq \sum_{k=1}^{\infty} \sum_{k-\frac{1}{2} \leq |k| \leq k + \frac{1}{2}} |\hat{\gamma}(k, t)|.$$

This inequality is saturated if and only if there is alignment of the phases of the Fourier components $\hat{\gamma}(k, t)$ that carry a significant amplitude. This alignment is
Figure 3.16: Snapshots of stretching-rate 1D shell spectra $E(k, t)$ at mapped times $\tau = 1, 2, 3, 4, 5$ (curves progressing from bottom to top) in log-log scale (top) and lin-log scale (bottom). The slopes of the straight lines on the top panel are proportional to the exponent $n_E(t)$. The slopes of the straight lines on the bottom panel are proportional to the logarithmic decrement $\delta_E(t)$. Resolution: $N = 2048$. 

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Figure 3.17: Results at resolution $N = 2048$ (except top left panel) for the fit parameters of the $L^2$ spectra $E(k, t)$ and $L^1$ spectra $F(k, t)$ at several times (as function of mapped time $\tau$). Top left panel: Resolution study ($N = 256, 512, 1024, 2048$) for $n_E(t(\tau))$. Progressive convergence is observed towards $n_E \approx 5/3$ (within a 5% error) near reliability time. Top right panel: Results for $n_F(t(\tau))$ (circles). Solid and dashed curves represent the upper and lower bounds $n_E/2$ and $(n_E - 1)/2$, respectively, cf. inequality (3.47). The curve $n_F$ is consistently between these bounds. Bottom left panel: Results for $C_E(t(\tau))$ (+ symbols) and $C_F(t(\tau))$ (circles). At late times these coincide, thus confirming the isotropy of the 2D spectrum. Bottom right panel: Results for $\delta_E(t(\tau))$ (+ symbols) and $\delta_F(t(\tau))$ (circles). These coincide, in agreement with inequalities (3.47). The horizontal line is the smallest resolved scale $\Delta x = 2\pi/2048$ and the dotted line corresponds to the numerical fit $\delta = \exp(0.53 - 1.33\tau)$ obtained using data in the range $3 \leq \tau \leq 4.34$. 
expected to happen near the singularity time, as learned from the 1D inviscid Burgers equation. The fact that a $L^1$ norm (rather than the more familiar $L^2$ norm $E(k,t)$) appears in this rigorous inequality, motivates the introduction of a new type of spectrum, the $L^1$ stretching-rate spectrum

$$F(k,t) = \sum_{k-\frac{1}{2} < |k| < k+\frac{1}{2}} |\hat{\gamma}(k,t)|.$$ 

In terms of this $L^1$ spectrum the above inequality becomes

$$\|\hat{\gamma}(\cdot,t)\|_\infty \leq \sum_{k=1}^{\infty} F(k,t).$$  

(3.46)

We now make a connection between this new $L^1$ spectrum and the more familiar $L^2$ spectrum, via a rigorous equivalence:

$$\sqrt{E(k,t)} \leq F(k,t) \leq \sqrt{S_k} \sqrt{E(k,t)},$$  

(3.47)

where

$$S_k = \sum_{k-\frac{1}{2} < |k| < k+\frac{1}{2}} 1.$$ 

We have the approximate result valid as $k \to \infty$:

$$S_k \approx 2\pi k.$$ 

Inequalities (3.47) allow us to work with the $L^1$ spectrum $F(k,t)$ in the same way as we would work with the more familiar $L^2$ spectrum $E(k,t)$. In particular, a fit of the form

$$E(k,t) \approx C_E(t) k^{-\eta_E(t)} \exp(-2k\delta_E(t))$$
implies a fit of the form

\[ F(k, t) \leq C_F(t) k^{-n_F(t)} \exp(-k\delta_F(t)). \]

One could interpret these two fits as working hypotheses, as in Bustamante & Brachet [12]. In the limit \( k \to \infty \) the inequalities (3.47) imply a sandwich so \( \delta_E(t) = \delta_F(t) \) is necessary. This is verified numerically in figure 3.17, bottom right panel.

In the intermediate-\( k \) range, the inequalities imply the following bounds:

\[ \frac{n_E(t) - 1}{2} \leq n_F(t) \leq \frac{n_E(t)}{2}. \]  

(3.48)

These bounds are verified in figure 3.17, top right panel.

Finally, the proportionality factors \( C_F(t) \) and \( C_E(t) \) cannot be easily related unless the above bounds (3.48) for \( n_F(t) \) become saturated. For example, if the upper bound was saturated, this would correspond rigorously to a very anisotropic 2D spectrum \( |\hat{\gamma}(k, t)| \) in terms of orientation of \( k \). In this case we would have \( C_F \approx \sqrt{C_E} \). On the other hand, if the lower bound was saturated, as it happens at late times (see figure 3.17, top right panel) this would correspond rigorously to a very isotropic 2D spectrum \( |\hat{\gamma}(k, t)| \). In this case we would have \( C_F \approx \sqrt{2\pi C_E} \).

Figure 3.17, bottom left panel, shows the curves \( C_F \) and \( \sqrt{2\pi C_E} \) as functions of time, showing equality at late times. So we can conclude that the 2D spectrum becomes nearly isotropic at late times. To support this analysis we provide in figure 3.18 two snapshots of the stretching-rate 2D spectrum, at mapped times \( \tau = 2 \) (left panel) and \( \tau = 4 \) (right panel). It is evident that the spectrum is strongly anisotropic at early times and evolves towards isotropy at late times, in agreement with the saturation of inequalities found.

Following an analogous discussion to that in Bustamante & Brachet [12], we
Figure 3.18: Snapshots of stretching-rate 2D spectra at mapped times $\tau = 2$ (left panel) and $\tau = 4$ (right panel). Resolution: $N = 2048$.

see that the left-hand-side of inequality (3.46) has a singular behaviour. In fact, a BKM-type of theorem can be demonstrated for the left-hand side, namely we can assume

$$\int_0^T \|y(\cdot, t)\|_\infty dt = \infty.$$  

This is obvious from the analytical solution presented in table 3.1. This will imply a singular behaviour of the right-hand side of inequality (3.46):

$$\int_0^T \sum_{k=1}^{\infty} F(k, t) dt = \infty.$$  

Using the above fit for $F(k, t)$ we get the result

$$\int_0^T \sum_{k=1}^{\infty} k^{-\eta_F(t)} \exp(-k\delta_F(t)) dt = \infty.$$  

We recognise the Jonquiere’s function $\text{Li}(\eta_F(t), e^{-\delta_F(t)})$. We get

$$\int_0^T \text{Li}(\eta_F(t), e^{-\delta_F(t)}) dt = \infty.$$  

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Now, in the limit as $t \to T^*$ we have $\delta_F \to 0$ so we can approximate the Jonquiere’s function to get
\[
\int_0^{T^*} (\delta_F(t))^{n_0 - 1} \, dt = \infty,
\]
where $n_0 = \liminf_{t \to T^*} n_F(t)$. So the asymptotic behaviour $\delta_F(t) \sim (T^* - t)^F$ would be consistent with singularity behaviour if and only if
\[
\Gamma \geq \frac{1}{1 - n_0}.
\]

From figure 3.17, top right panel, we see that $n_0 \approx 0.36$ if we consider the data near reliability time ($\tau_{rel} \approx 4.34$). This gives $\Gamma \geq 1.56$, which is consistent with the fits obtained from figure 3.17, bottom right panel, that produce $\Gamma \approx 2.66$ by virtue of the analytical result $(T^* - t) \approx \exp(-\tau/2)$. At the same time this shows that the inequality (3.46) is not saturated by the field $\gamma(x, t)$. The interpretation of this lack of saturation is that the Fourier phases, $\{\arg \hat{\gamma}(k, t)\}_{k=1}^{\infty}$, do not all align near the singularity time. In fact this is evident from the physical-space snapshots in figure 3.8, where the singular structure is between a filament and a point.

### 3.4.2.7 Estimating the singularity time $T^*$ efficiently

It would be meaningless to provide results comparing the accuracy of particular numerical methods, without including some quantification of their relative computational expense.

It is possible to obtain two independent estimates of the singularity time $T^*$ by using the numerical solutions from either the original system or the mapped system:

**Method A.** For the original system, we fit the stretching rate norm $||\gamma(\cdot, t)||_\infty$ as a power law locally in time, using a method introduced in Bustamante & Kerr [15]

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to determine an estimate of singularity time $T^*$. The power-law fits are of the form

$$f(t) \propto (T^*_A - t)^\alpha,$$

(3.49)

where $f(t)$ stands for $\|\gamma(\cdot, t)\|_\infty$ in this case. This ansatz is justified in this case by the analytically obtainable asymptotic formulae in table 3.1. The local fits are achieved using the function

$$g(t) = \left(\frac{d \ln f(t)}{dt}\right)^{-1} = \frac{f}{f'} = -\frac{1}{\alpha} (T^*_A - t).$$

(3.50)

Instantaneous running estimates $\alpha$ and $T^*_A$ are then computed by linear-fitting the function $g(t)$ instantaneously using adjacent data points, or more generally over a small time window (of size $\sim 0.2$) containing a good number of data points, in order to eliminate spurious oscillations in the running estimates. Note that this method provides an extra quantity: the exponent $\alpha$, which serves as an extra measure of validation. In the case $\lambda = -3/2$ one should get $\alpha = -1$ (see table 3.1). This validation is consistently held throughout the computation (figure 3.19).
**Method B.** For the mapped system, the situation relies on the explicit formula (3.30). There, the only relevant numerical quantity is $\int_0^\tau \langle y_{\text{map}}(\cdot, \tau')^2 \rangle \, d\tau'$. In analogy to the previous case, we will fit the integrand. But, unlike the previous case, we cannot rely on local fits because doing this would lead to accumulation of errors in the estimation of the integral.

Using equations (3.29) and (3.30) we obtain the estimate

$$T^*_B(\tau) = \int_0^\tau \frac{1}{\|y(\cdot, t(\tau'))\|_\infty} \, d\tau'$$

$$= \left( \int_0^\tau \frac{1}{\|y(\cdot, t(\tau'))\|_\infty} \, d\tau' \right)_{\text{num}} + \left( \int_\tau^\infty \frac{1}{\|y(\cdot, t(\tau'))\|_\infty} \, d\tau' \right)_{\text{extrap}}$$

(3.51)

where in both terms we compute the integrand using equation (3.29). The subscript “num” means that we use the numerical solution of the mapped system to compute the time integrals, using Simpson’s rule. As for the subscript “extrap”, $\bar{\tau}$ is a very big number chosen so that the numerical integral converges ($\sim 1000$ in practice) and we compute the integrand, using (3.29), as follows:

$$\frac{1}{\|y(\cdot, t(\tau'))\|_\infty} = \frac{1}{\|y(\cdot, t(\tau))\|_\infty} \times \exp \left[ (1 + \lambda) \int_\tau^{\tau'} \sigma_\infty \, d\tau'' - (2 + \lambda) \int_\tau^{\tau'} \sigma_\infty \langle y_{\text{map}}^2 \rangle \, d\tau'' \right]$$

where we set $\sigma_\infty(\tau'') = \sigma_\infty(\tau)$ in the above exponent, and we model the functions appearing in the above exponent using the following fit ansatz that is motivated by the generic asymptotic behaviour of $\langle y_{\text{map}}^2 \rangle$ as $\tau'' \to \infty$:

$$\langle y_{\text{map}}^2(\cdot, \tau'') \rangle \approx \frac{\beta}{2 + \lambda} \left[ c \exp(\beta \tau'') - 1 \right]^{-1},$$

where $\beta$ and $c$ are two positive fit parameters. In order to obtain these fit parameters we use the numerical data for $\langle y_{\text{map}}^2(\cdot, \tau'') \rangle$ in the range $0 \leq \tau'' \leq \tau$ and use a
least-squares fit. The integral $\int_{\tau}^{\tau'} \sigma_{\infty}(\gamma_{\text{map}}^2) d\tau'$ is done analytically in terms of the fit parameters. Finally, the resulting integral

$$
\int_{\tau}^{\tau'} \frac{1}{\|\gamma(\cdot, t(\tau'))\|_\infty} d\tau'
$$

is done using Simpson’s rule.

It is important to stress that in both original and mapped systems the estimation of the singularity time $T^*$ depends on two fit parameters. In general, we tried to lever as much accuracy as possible from both methods. For example we used adaptive time stepping in the original system to get a distribution of data points that is comparable to that of the mapped system, so that a more accurate estimate for $T^*$ could be obtained in the original system.

We present results from original and mapped systems regarding the assessment of estimates of singularity times. First, in figure 3.20 we plot the relative error of the running estimates

$$
E_{T_{A,B}}(\tau) = \left| \frac{T_{A,B}^*}{T_{\text{ana}}^*} - 1 \right|
$$

where $T_{\text{ana}}^* = \frac{4}{\sqrt{3}} \arctan \left( \frac{\sqrt{3}}{4} \right) \approx 1.26894$ is the analytically computed singularity time and $T_{A,B}^*$ stands for the running estimate obtained either from Method A (for the original system) or from Method B (for the mapped system). It is observed from figure 3.20 that:

(i) For each method, there is good resolution convergence in the assessment of $T^*$; 
(ii) Method B (for the mapped system) produces much better results as compared to Method A (for the original system), with an improvement of about three orders of magnitude at any given resolution.

The second set of results is the following. We produce, from each method, a single estimate (not a running estimate) $T_{A,B}^0$ of the singularity time, computed using the running estimates already obtained. It is important to stress the perhaps
Figure 3.20: Errors in the running estimates of singularity time $T^*$ using data from original system’s numerical integration (lines) and mapped system’s numerical integration (symbols) at different resolutions: from top to bottom in each case, $N = 256, 512, 1024, 2048$. 
obvious fact that the procedure to find this single estimate is completely independent of any previous knowledge of the singularity time $T^*$. The procedure is simple: since we have a reliability time $t_{\text{rel}}$ (or, in mapped time, $\tau_{\text{rel}}$) well defined for each resolution, in terms of the analysis of spectra done in section 3.4.2.6, we evaluate our running estimates at the reliability time. So we define, for the mapped system, $T^0_B = T^*_B|_{\tau_{\text{rel}}}$. As for the original system, setting the single estimate to $T^*_A|_{t_{\text{rel}}}$ would be possible, however we found that a better estimate is obtained by averaging the running estimates between $t_{\text{rel}}$ and $t_{\text{min}}$, where $t_{\text{min}}(> t_{\text{rel}})$ depends on the resolution and is defined by the time at which the running estimate has a global minimum.

Figure 3.21 shows the CPU time versus relative error of the estimated singularity time,

$$E_{A,B} = \left| \frac{T^0_{A,B}}{T^*_A} - 1 \right|,$$

for the numerical solutions of both the original (Method A) and mapped systems (Method B) at various resolutions. It is clear that while the mapping incurs some additional expense in evaluating the extra terms, computing the interpolated supremum and applying the normalisation, it is far outweighed by the positive effect on the errors (three orders of magnitude in this study). In these measures one can make a significant improvement, saving not only CPU time, but also the memory cost of high resolution runs.

We present in table 3.2 a useful summary of the reliability times and relative errors $E_{A,B}$ of the estimated singularity time, for a range of resolutions, showing that the errors are dramatically reduced in the case of the mapped equations.
Figure 3.21: Figure showing the singularity-time error $\mathcal{E}_{A,B}$ [error between estimated singularity time and analytically computed singularity time $T^*$, equation (3.52)], as a function of CPU time for both the original system, Method A (filled symbols) and the mapped system, Method B (open symbols) at various resolutions: $N = 256$ (red squares), $N = 512$ (green circles), $N = 1024$ (blue triangles), $N = 2048$ (magenta diamonds). The CPU overhead of applying the mapping is shown to be more than covered by the accuracy gain.

Table 3.2: Summary of results for a range of resolutions: reliability times, obtained using the stretching-rate spectra (original or mapped system give the same reliability times); relative error of the estimated singularity time $T^0$ stemming from original system’s numerical data (Method A); relative error of the estimated singularity time $T^0_B$ stemming from mapped system’s numerical data (Method B).
3.5 Conclusion and Discussion

We have introduced a new family of symmetry plane models of the 3D Euler equations and presented numerical and analytical solutions exhibiting finite-time blowup. In this way we have presented a simplified and tuneable setting for the study and assessment of finite-time singularity in an idealised fluid. We make use of this example to evaluate the performance of the mapping to regular systems of Bustamante [11] in improving the diagnosis of singular behaviour. We simulate both systems using the same pseudospectral methods and find that direct determination of blowup quantities from the numerical integration of the mapped regular system produces more accurate and reliable results compared with the integration of the original system.

We present a thorough investigation of the evolution of the Fourier spectrum of the numerical solution. However, unlike the case of the supremum norms of the fields, there is no available explicit solution for the Fourier components. We validated the numerical solution by checking rigorous bounds on the Sobolev norms, using the working hypotheses introduced by Bustamante & Brachet [12]. These hypotheses were designed in order to bridge the study of the loss of analyticity of solutions with the classical BKM type of theorems. The main results here are as follows.

(i) The finite-time blowup of the supremum norm of the stretching rate implies that the Fourier spectrum’s logarithmic decrement $\delta(t)$ (a measure of the loss of analyticity) must decay to zero fast enough at the singularity time. We observe a decay $\delta(t) \sim (T^* - t)^3$, consistent with the rigorous bounds.

(ii) An “inertial range” of wavenumbers at which the conserved quantity $\langle y^2 \rangle$ is transferred to small scales is found, with a 1D spectrum (i.e. shell-integrated) of the form $E(k,t) \sim k^{-5/3}$ at times close to the singularity time.
(iii) The 2D spectrum of the Fourier amplitude $|\hat{\gamma}(k, t)|^2$ becomes isotropic at late times, in agreement with the saturation of our rigorous bounds for the $L^1$ shell spectrum in terms of the $L^2$ spectrum $E(k, t)$. Figures 3.17 and 3.18 complement the above results.

We discuss, in the interest of fairness, a technicality that arises in the mapped system. Recovering the original system’s supremum norm from the mapped variables has a subtlety. At late times the dominant contribution comes from a term that depends explicitly on $\tau$, which is therefore independent of the numerical simulations. This explains the strong and robust late-time convergence we observe in our comparisons. This behaviour is relaxed for the case $\lambda \in [-1, 0]$.

On the other hand, the coincidence occurring at $\lambda = -3/2$ where $\langle \gamma^2 \rangle$ is an invariant of the motion, reduces the errors in the original system with respect to the mapped system. In chapter 4 we have confirmed that for any other value of $\lambda$ this invariance does not hold and as a result the original system accumulates significant additional errors. This scenario favours the mapped system even more than in the case $\lambda = -3/2$ studied in detail in this chapter.

To estimate the singularity time we perform a two parameter fit for both the original and mapped system in order to extrapolate a running estimate for the singularity time. The result is up to three orders of magnitude increase in accuracy when employing the mapping over the original system. It should be emphasised that this gain in performance stems from a number of sources, for example, the form of the extrapolation to compute $T^*$, the global quantity $\langle \gamma_{\text{map}}^2 \rangle$ being used to “unmap” the variables, the redistribution of numerical error within the simulation via the normalisation procedure and finally the mapping of time to distribute data appropriately near $T^*$. On this final point, it is tempting to assume that this is the main advantage of the mapping and it would be equivalent to simply employ an
adaptive time step on the original system’s numerical integration. But we have shown that, not only are the accuracy gains unrelated to time step convergence, but also that the manner of recomputing the original variables has important consequences. For these reasons, and the observations summarised earlier in this section, we show errors (figures 3.13 and 3.15), singularity time estimates $T^*$ (figure 3.20), etc. at values of $\tau$ far beyond the usual reliability time cut-off.
Chapter 4

Stagnation-point-type solutions of 3D Euler flows

In this chapter, we investigate both numerically and analytically the case $\lambda = 0$ from chapter 3 which recovers the finite time blow up of the infinite-energy stagnation-point-type solutions of the 3D Euler fluid equations introduced by Gibbon et al. [43]. We employ the method of mapping introduced by Bustamante [11] that maps nonlinearly the time and fields to a globally regular system. Using the new mapped variables, we establish a curious property of this solution that was not observed in early studies: before but near singularity time ($T^*$), the blowup goes from a fast transient to a slower regime that is well resolved spectrally, even at mid-resolutions of $512^2$. This late-time regime has an atypical spectrum which is Gaussian rather than exponential in the wavenumbers. The analyticity-strip width decays to zero in a finite time, albeit so slowly that it remains well above the collocation-point for all simulation times $t < T^* - 10^{-9000}$. Reaching such a proximity to singularity time is not possible in the original temporal variable, because floating point double precision ($\approx 10^{-16}$) creates a ‘machine-epsilon’ barrier. Due to this limitation on the original independent variable, the mapped variables
now provide an improved assessment of the relevant blowup quantities, crucially with acceptable accuracy at an unprecedented closeness to the singularity time: $T^* - t \approx 10^{-140}$.

4.1 Exact solution of the 3D Euler fluid equations

We consider a class of exact solutions of the 3D Euler equations presented by Gibbon et al. [43]. Writing $u(x, y, z, t) = (u_x(x, y, t), u_y(x, y, t), u_z(x, y, t), \gamma(x, y, t))$ and using equations (3.9) and (3.10) from chapter 3 for the case when $\lambda = 0$ we obtain

$$\frac{\partial \gamma}{\partial t} + u_h \cdot \nabla_h \gamma = 2(\gamma^2) - \gamma^2,$$

(4.1)

$$\frac{\partial \omega}{\partial t} + u_h \cdot \nabla_h \omega = \gamma \omega,$$

(4.2)

where $u_h(x, y, t) \equiv (u_x(x, y, t), u_y(x, y, t))$ denotes the “horizontal” component of the velocity field at the symmetry plane $(z = 0)$, $\nabla_h = (\partial_x, \partial_y)$ denotes the “horizontal” gradient operator, $\omega$ is the vorticity defined as

$$\omega(x, y, t) = \partial_x u_y - \partial_y u_x,$$

$\gamma$ is the stretching-rate of vorticity, which using the incompressibility condition in equation (1.2) can be defined as:

$$\gamma(x, y, t) = -\nabla_h \cdot u_h(x, y, t),$$

(4.3)

and

$$\langle f(\cdot, t) \rangle \equiv \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} f(x, y, t) \, dx \, dy$$

denotes the spatial average over the periodic 2D domain.

Constantin [23] solved for $\gamma$ along characteristics (and for vorticity $\omega$, which
can be found a posteriori), proving that the stretching rate $\gamma$ would blow up in a finite time, with explicit formulae for the singularity time which confirmed the accuracy of the numerical blowup predictions by Ohkitani & Gibbon [87]. A BKM [5] type of theorem was established by Gibbon & Ohkitani [45] where the blowup time $T^*$ is defined as the smallest time at which

$$\int_0^{T^*} \|\gamma(\cdot, t')\|_\infty dt' = \infty,$$

where $\|\gamma(\cdot, t)\|_\infty$ is the supremum norm of the stretching rate of vorticity field.

Chapter 3 of this thesis introduced the following ‘mapped’ time and ‘mapped’ fields:

$$\tau(t) = \int_0^t \|\gamma(\cdot, t')\|_\infty dt',$$

$$\gamma_{\text{map}}(x, y, \tau) = \frac{\gamma(x, y, t)}{\|\gamma(\cdot, t)\|_\infty},$$

$$\omega_{\text{map}}(x, y, \tau) = \frac{\omega(x, y, t)}{\|\gamma(\cdot, t)\|_\infty},$$

$$u_{\text{map}}(x, y, \tau) = \frac{u_h(x, y, t)}{\|\gamma(\cdot, t)\|_\infty}.$$  \hspace{1cm} (4.4)

This transformation is bijective for $t < T^*$. The mapped fields satisfy the following PDE system:

$$\frac{\partial \gamma_{\text{map}}}{\partial \tau} + u_{\text{map}} \cdot \nabla \gamma_{\text{map}} = 2\gamma_{\text{map}}^2 - \gamma_{\text{map}}^2 + \sigma_{\infty} \gamma_{\text{map}} \left( 1 - 2\gamma_{\text{map}}^2 \right)$$  \hspace{1cm} (4.5)

$$\frac{\partial \omega_{\text{map}}}{\partial \tau} + u_{\text{map}} \cdot \nabla \omega_{\text{map}} = \gamma_{\text{map}} \omega_{\text{map}} + \sigma_{\infty} \omega_{\text{map}} \left( 1 - 2\gamma_{\text{map}}^2 \right)$$  \hspace{1cm} (4.6)

where

$$\sigma_{\infty} \equiv \text{sign} \gamma(X_\gamma(t), t)$$  \hspace{1cm} (4.7)

is the sign of $\gamma$ at the position $X_\gamma(t)$ where the maximum of $|\gamma(x, t)|$ is attained.
4.2 Analytical solution of the stagnation-point-type 3D Euler flows

Solutions of (4.1) and (4.2) are exact solutions of 3D Euler equations (albeit with infinite energy), as derived originally by Gibbon et al. [43]. Ohkitani & Gibbon [87] performed a numerical study at resolution $256^2$, supported with simulations at resolution $1024^2$, which provided evidence of a finite-time singularity at $t \approx 1.4$. Higher resolution was not needed due to the fact that spectral convergence was observed during most of the simulation time.

Constantin [23] introduced a method for finding analytically the blowup quantities corresponding to the above numerical study (e.g. $\|\gamma(\cdot, t)\|_{\infty}$, $\langle \gamma^2 \rangle$) and established that there is a finite-time singularity. While it is possible to obtain the asymptotic behaviour of the blowup quantities using the method of Constantin [23], we will discuss this in the context of our method detailed in chapter 3 for the purposes of simplicity of presentation.

The initial conditions used in this study are the same as those used in the study by Ohkitani and Gibbon [87]:

$$\gamma_0(x, y) = \sin(x) \sin(y), \quad (4.8)$$
$$\omega_0(x, y) = \sin(x) \sin(y). \quad (4.9)$$

Figure 4.1 shows perspective and isosurface plots of the initial condition for both the stretching rate $\gamma$ (4.8) and vorticity $\omega$ (4.9).

The equation of motion for $S(t)$, the function upon which provides the solution for all fields along characteristics, (see chapter 3 for more details) is:

$$\dot{S}(t) = \left\{ \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} \left[ 1 + \gamma_0(x, y)S(t) \right]^{-1} \, dx \, dy \right\}^{-2}, \quad S(0) = 0. \quad (4.10)$$
Figure 4.1: Perspective and isosurface plots of stretching rate \( \gamma_0 \) (top) and vorticity \( \omega_0 \) (bottom) for the initial conditions, equations (4.8) and (4.9) respectively, of the stagnation-point-type 3D Euler flow.
Performing spatial integration over the initial condition (equation (4.8)) leads to the following:

\[ \dot{S} = \frac{\pi^2}{4 \left[ K(S^2) \right]^2}, \quad S(0) = 0, \quad (4.11) \]

where \( K(\mu^2) \) is the complete elliptic function of the first kind, written in terms of the so-called modulus \( \mu \). Thus, our variable \( S \) can be interpreted in this case as the modulus of the elliptic function. The singularity occurs when \( S = S^* \), where \( S^* = -1/\inf \gamma_0 = 1 \). Thus, \( S(t) \) goes from 0 at \( t = 0 \) to 1 at \( t = T^* \) (figure 4.2), where \( T^* \) is the singularity time defined as

\[ T^* = \frac{4}{\pi^2} \int_0^1 \left[ K(S^2) \right]^2 dS \]

\[ \approx 1.41800273492385887506223956597972945766172112 \quad (4.12) \]

Ohkitani & Gibbon [87] estimated \( T^* \approx 1.417 \) which appears to be a good estimate, but as we will show is not close enough to capture the true late time dynamics.

Recalling the results from chapter 3, where the singularity will occur first at the characteristic starting at the position of the infimum (if \( \lambda > -1 \)) of \( \gamma_0 \) over \( T^2 \), the singularity in this case when \( \lambda = 0 \) is dominated by the infimum of \( \gamma(x, t) \) and for these initial conditions (equation (4.8) and (4.9)) we can identify \( ||\gamma||_{\infty} = -\inf \gamma \), so \( \sigma_{\infty} = -1 \) will be used throughout.

### 4.2.1 Exact results

Solving the above ODE for \( S(t) \) (equation (4.11)) is in principle feasible numerically to any desired accuracy. It is also possible to obtain some exact formulae for
\[ S(t) \] goes from 0 at \( t = 0 \) to 1 at \( t = T^\ast \). Notably, \( \dot{S}(T^\ast) = 0 \).

\[ \|\gamma(\cdot, t)\|_{\infty} \text{ and } \tau(t) \text{ in implicit form through } S(t): \]

\[ \tau(t) = -\ln\left( \frac{2}{\pi} (1 - S) K(S^2) \right), \tag{4.13} \]

\[ -\inf_{x \in \mathbb{T}^2} \gamma(x, t) = \|\gamma(\cdot, t)\|_{\infty} = \frac{\pi^2 \left[ (S + 1) K(S^2) - E(S^2) \right]}{4S (1 - S^2) K(S^2)^3}, \tag{4.14} \]

\[ \sup_{x \in \mathbb{T}^2} \gamma(x, t) = \frac{\pi^2 \left[ E(S^2) - (1 - S) K(S^2) \right]}{4S (1 - S^2) K(S^2)^3}, \tag{4.15} \]

\[ \langle \gamma^3 \rangle = \frac{\pi^4 \left[ (3 - S^2) K(S^2) E(S^2) - (1 - S^2) K(S^2)^2 - 2E(S^2)^2 \right]}{32S^2 (1 - S^2)^2 K(S^2)^6}, \tag{4.16} \]
where $E(\mu^2)$ is the complete elliptic function of the second kind, related to $K(\mu^2)$ by the equation

$$E(\mu^2) = (1 - \mu^2) K(\mu^2) + \mu \frac{d}{d\mu} K(\mu^2).$$

See Whittaker et al. [98] for details.

Figure 4.3: Plot of $S(t(\tau))$ against mapped time $\tau$ for the initial condition, equation 4.8. At values of $\tau$ greater than about 33, this requires the use of arbitrary-precision arithmetic because double floating-point precision is lost.

With regard to the mapped system, note that it is possible to find $S(t(\tau))$ and $\|\gamma(\cdot, t(\tau))\|_\infty$ as functions of $\tau$ via first inverting equation (4.13) to obtain $S(t(\tau))$ (figure 4.3), and then using this on equation (4.14) to obtain $\|\gamma(\cdot, t(\tau))\|_\infty$. For the error analysis of section 4.4 we will use a numerical solution of ODE (4.11) for $S(t)$ (in principle feasible to any desired accuracy), as a closed form is not available. For this reason we term this solution quasi-analytic. At values of $\tau$ greater than about 33, this requires the use of arbitrary-precision arithmetic, provided by commercial packages such as Mathematica. The reason for the need
of arbitrary precision numbers is that $S(t)$ becomes too close to one (within the double-precision machine epsilon), i.e. double floating-point precision is lost.

### 4.2.2 Asymptotic Formulae.

At late times the solution $S(t)$ can be constructed using asymptotic formulae accurate to double precision. Defining $Z \equiv -\ln\left(\frac{1-S}{8}\right)$, the following formulae are valid asymptotically as $S \to 1$ (i.e. as $\tau \to \infty$):

$$T^* - t \approx \frac{8e^{-Z}}{\pi^2} \left( Z^2 + 2Z + 2 \right),$$  \hspace{1cm} (4.17)

$$\|\gamma(.,t)\|_{\infty} \approx \frac{\pi^2 e^{Z}}{8} \left( \frac{Z - 1}{Z^3} \right),$$  \hspace{1cm} (4.18)

$$\langle \gamma^2 \rangle \approx \frac{\pi^4 e^{2Z}}{128} \left( \frac{Z - 2}{Z^6} \right).$$  \hspace{1cm} (4.19)

Also, we obtain $\tau \approx Z - \ln\left(\frac{6Z}{\pi}\right)$. This latter formula can be inverted in the asymptotic region of interest, giving

$$Z \approx -W_{-1}\left(-\frac{1}{8\pi e^{-\tau}}\right),$$

where $W_{-1}$ is a branch of the Lambert function (also known as the Product Log function). By combining the above it is possible to obtain explicit asymptotic expressions for $T^* - t$ and $\|\gamma(.,t)\|_{\infty}$ in terms of the mapped time $\tau$.

These asymptotic formulae are very useful in practice. For $\tau = 5$ the above asymptotic formula for $t(\tau)$ has a relative error of about $10^{-9}$ and for $\|\gamma(.,t)\|_{\infty}$, a relative error of about $10^{-7}$.

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4.3 Numerical solution of original and mapped systems and comparison with analytic solution

We solve the evolution equations for both the original system (equation (4.1) and (4.2)) and the mapped system (equation (4.5) and (4.6)) numerically using a standard pseudospectral method similar to that used in chapter 3. Dealiasing is carried out using Hou’s exponential filter \( \exp\left(-36(2k/N)^4\right) \) [56], for a given spatial resolution \( N \) with a wavevector modulus \( k \), and a fourth-order Runge-Kutta scheme solves in time. Adaptive time-stepping, \( dt = d\tau/\|\gamma(\cdot, t)\|_\infty \), is used for the original equations and uniform steps of \( d\tau \) are used in the mapped system with the resulting distribution of temporal data roughly equivalent. Snapshots of perspective and isosurface plots for both stretching rate \( \gamma \) and vorticity \( \omega \) at various times \( t = 0.6, t = 1.2 \) and \( t = 1.4 \) are shown in figures 4.4 and 4.5 respectively.

4.4 Errors in blowup quantities \( \|\gamma(\cdot, t)\|_\infty, \langle \gamma^2 \rangle \) and \( \langle \gamma_{\text{map}}^2 \rangle \)

Here we assess the errors between the direct numerical simulation (DNS) and analytical formulae for the blowup quantities, in order to establish superiority of the mapped system’s solution accuracy over the original system’s. Using the definition in section 3.4, the ‘normalised’ \( L^2 \) norm of the error (not relative error) is given by,

\[
Q(f, g) = \frac{\|f - g\|_2}{\|f\|_2 + \|g\|_2}.
\]
Figure 4.4: Snapshots of perspective and isosurface plots of stretching rate ($\gamma$) at $t = 0.6$, $t = 1.2$ and $t = 1.4$ from the stagnation-point-type 3D Euler flow starting with initial condition in equations (4.8) and (4.9).
Figure 4.5: Snapshots of perspective and isosurface plots of vorticity ($\omega$) at $t = 0.6$, $t = 1.2$ and $t = 1.4$ from the stagnation-point-type 3D Euler flow starting with initial condition in equations (4.8) and (4.9).
We consider the errors associated with the local quantity \( \| \gamma(\cdot, t) \|_\infty \), and the global quantities \( \langle \gamma^2 \rangle \) and \( \langle \gamma_{\text{map}}^2 \rangle \) via

\[
Q_\gamma = Q(\| \gamma_{\text{num}}(\cdot, t) \|_\infty, \| \gamma_{\text{ana}}(\cdot, t) \|_\infty),
\]

\[
Q_{\langle \gamma \rangle} = Q(\langle \gamma_{\text{num}}^2 \rangle, \langle \gamma_{\text{ana}}^2 \rangle),
\]

\[
Q_{\langle \gamma_{\text{map}} \rangle} = Q(\langle \gamma_{\text{map, num}}(\cdot, \tau)^2 \rangle, \frac{\langle \gamma_{\text{ana}}(\cdot, t(\tau))^2 \rangle}{\| \gamma_{\text{ana}}(\cdot, t(\tau)) \|_\infty^2}),
\]

respectively where the subscripts “num” and “ana” stand for “numerical” and “analytic”. As described in section 4.2.1 the “analytic” solution is the quasi-analytic form, equations (4.13) - (4.16).

The numerical solution of the mapped system does not provide direct access to the original variable \( \| \gamma(\cdot, t) \|_\infty \) therefore the following expression is required (see section 3.3 for further details):

\[
\| \gamma(\cdot, t(\tau)) \|_\infty = \| \gamma_0 \|_\infty \exp \left[ \tau - 2 \int_0^\tau \sigma_{\infty}(\gamma_{\text{map}}^2) \, d\tau' \right]
\]

(4.20)

where \( \int_0^\tau \sigma_{\infty}(\gamma_{\text{map}}^2) \, d\tau \) is computed using Simpson’s rule. We compare both this mapped estimate and the direct maximum norm from the original system against the solution of equation (4.14). Care is taken in solving equation (4.11) so that the time steps from the original system are used to solve on intervals which coincide with the data points of the mapped system and that arbitrary precision of the required level is used.

Figure 4.6 (top) shows the temporal evolution of the maximum norm of stretching rate using numerical data from the original system, where the figure shows the results are essentially the same at different resolution. We produce standard resolution convergence studies of the maximum norm of stretching rate of vorticity \( \gamma \). Figure 4.6 (bottom) shows a classic plot of the temporal evolution of the in-
verse of the maximum norm of stretching rate of vorticity ($\gamma$) indicating that the solution remains well resolved, even at relatively modest resolutions $N = 512$ to very late times. Figure 4.7 shows a lin-log plot with respect to mapped time $\tau$ of the maximum norm of stretching rate, using numerical data from the mapped system also showing the solution remains well resolved at moderate resolution.

We show convergence with respect to timestep $d\tau$ using figure 4.8 which shows a comparison of the errors $Q_\gamma$ and figure 4.9 showing a comparison of the errors $Q_{\langle \gamma^2 \rangle}$ and $Q_{\langle \gamma_{\text{map}}^2 \rangle}$ at various timesteps $d\tau$ and resolution $N = 1024$. Overall we observe an exponential growth (in $\tau$) of error $Q_\gamma$ from the original system, compared to an almost uniform error ($\sim 10^{-10}$) from the mapped version at converged $d\tau$. The figures (4.8 and 4.9) also show that the mapped system along with its numerical solution serve to improve the accuracy of the blowup quantities at late time.

The mapped system requires computation of the stretching rate norm $||\gamma(\cdot, t(\tau))||_\infty$ from equation (4.20). This entails the numerical approximation of the integral $\int_0^\tau \langle \gamma_{\text{map}}^2 \rangle d\tau'$ which is sensitive to the extra time scale at early times. Figure 4.10 gives a quantitative measure of the significance of this integral term relative to the first term in equation (4.20) as a function of time. For similar reasons to those discussed in chapter 3 we find lower late time error in the mapped system but higher early time error on comparison with the original system. This notwithstanding, we show a significant advantage when using these mapped variables to assess blowup behaviour as will be established in the following sections, in particular when considering proximity to $T^*$ (section 4.6).

In contrast to the results from chapter 3, we find that the error in the maximum norm $Q_\gamma$ from the mapped system and that in the global average $Q_{\langle \gamma_{\text{map}}^2 \rangle}$ behave in a similar manner. This implies that in this case (where $\langle \gamma^2 \rangle$ is not an invariant), the error in the maximum norm $Q_\gamma$ is simply slaved to that in the global average
Figure 4.6: Resolution studies of the data from original system at different resolutions $N = 512, 1024$ and $2048$. Top plot shows the temporal evolution of the maximum norm of stretching rate $\|\gamma(\cdot, t)\|_\infty$ with bottom plot showing the classical plot of $1/\|\gamma(\cdot, t)\|_\infty$. The analytical solution $\|\gamma\|_\infty$ is given by equation (4.14).
Figure 4.7: Lin-log plot of the temporal evolution of $\|\gamma(\cdot, t(\tau))\|_\infty$ rule of thumb from the mapped systems at different resolutions $N=512, 1024$ and $2048$. The analytical solution $\|\gamma\|_\infty$ is given by equation (4.14). As a rule of thumb, original time proximity to $T^*$ can be inferred by using $1/\|\gamma\|_\infty \sim T^*-t$, valid at large values of $\tau$.

Figure 4.8: Time evolution of the error measure $Q_\gamma$ at resolution $N=1024$ showing convergence with timestep $d\tau$. 
Figure 4.9: Time evolution of the error measure $Q_{(y^2)}$ and $Q_{(\gamma_{map}^2)}$ at resolution $N = 1024$ showing convergence with timestep $d\tau$.

Figure 4.10: Ratio between the terms $2 \int_0^\tau \sigma_\infty (\gamma_{map}^2) \, d\tau'$ and $\tau$, appearing in the exponent in equation (4.20).
Chapter 3 contains a detailed discussion on error sources in the mapped and original variables, and highlights some subtleties surrounding the behaviour of \( \langle \gamma^2 \rangle \). Here the situation is somewhat more straightforward: the original system contains unbounded error growth due to a fundamental loss of precision in the independent variable. This is explained by the earlier convergence and the ‘saturation’ of error near the double-precision limit \( (\tau \approx 37) \).

### 4.5 Spectra and analyticity strip

To investigate the spatial collapse associated with the blowup, we consider a detailed analysis of the one-dimensional spectra of stretching rate \( \hat{\gamma} \) constructed from circular shells,

\[
E(k, t) = \sum_{k-\frac{1}{2} < |k| < k+\frac{1}{2}} |\hat{\gamma}(k, t)|^2.
\]

Our first observation of the evolution of the spectrum is that there are two timescales in evidence. An initial burst can be observed with a flux towards intermediate \( k \) which is redistributed across the modes. Provided \( N > 256 \) this initial phase remains well resolved and lasts only until \( \tau \approx 25 \). Thereafter there is a slow cascade from small \( k \). In fact in original variables the initial phase is until \( T^* - t \approx 10^{-10} \). As will be shown, this is too early to establish certain asymptotic trends.

We also found that, due to the lack of direct energy cascade to large \( k \), an accumulation of round-off error propagates up-scale. The result is a small quantity of spurious energy between the large scales and the truncation wavenumber. The amount of this spurious energy is resolution dependent leading to an ill-converged spectrum. This issue was remedied by applying a small amount of hyperviscosity on the large wavenumbers, namely adding the term

\[
\nu(-1)^{2h+1} |k|^{2h} \hat{\gamma} \quad \text{for } |k| > 200, \tag{4.21}
\]
with \( h = 2 \), to the right hand side of the Fourier transform of equation (4.5) and an equivalent term for \( \hat{\omega} \) in equation (4.6). Numerically a Crank-Nicolson scheme was used on this term for stability. Figure 4.11 shows the profile of the spectra at \( \tau = 5, 10, 15 \) and 25 for \( N = 1024 \) and 4096, each with \( \nu = 10^{-10} \) and \( \nu = 0 \) in lin-log scale with figure 4.12 showing the same in log-log scale. This demonstrates that the hyperviscosity gives a well converged spectrum while leaving the large scale modes unaffected. The error in the bulk quantity \( \langle \gamma_{\text{map}}^2 \rangle \) is unchanged, however applying hyperviscosity to all modes leads to a significant error increase.

Interestingly the late time profile does not have the typical shape we might expect [12] and in this thesis chapter 3 or that which is assumed previously in this system by Ohkitani & Gibbon [87], namely

\[
E(k, t) \propto C(t) k^{-\eta(t)} e^{-2\delta(t)k}. \tag{4.22}
\]

In fact, as can be seen in figure 4.13, the profile assumes a more Gaussian shape,

\[
E(k, t) \propto C(t) k^{-\eta(t)} e^{-\delta(t)k^2}. \tag{4.23}
\]

This late time spatial form has been missed in previous work [87] on this system as it only arises after the initial burst, which does have the \( e^{-2\delta k} \) shape, and persists sufficiently close to \( T^* \) to render it next to inaccessible without the mapped variables. To ensure the convergence of the initial burst phase we first perform a least-squares fitting procedure to the spectrum with ansatz (4.22). Figure 4.14 (top) shows the fitted \( \delta_1 \) for the early burst phase. The plot shows two resolutions (\( N = 1024 \) and 2048) which are essentially indistinguishable. From figure 4.11 and 4.13 the exponential part of the profile of \( E(k, t) \) is preceded (in \( k \)) by the Gaussian shape. The downscale flux associated with the slackening of the exponential part (\( \delta_1 \) decreasing) establishes the Gaussian profile in its wake.
Figure 4.11: Snapshots of spectra for $\tau = 5, 10, 15$ and 25, on a lin-log scale. The figures show two resolutions ($N = 1024$ and 4096), with and without hyperviscosity demonstrating the need to control floating point round-off error at small scales. $\tau = 5, 10$ and 15 shows the full spectrum, until dealiasing filtered modes to show the small scale error; $\tau = 25$ plot shows only the first 500 modes to make clear the initial burst and the onset of the slow Gaussian spectrum.
Figure 4.12: Snapshots of spectra for $\tau = 5, 10, 15$ and $25$, on a log-log scale. The figures show two resolutions ($N = 1024$ and 4096), with and without hyperviscosity demonstrating the need to control floating point round-off error at small scales.
The result is that, while the trend in $\delta_1$ suggests an exponential decay in $\tau$ at early times, in reality the ansatz (4.22) ceases to be a valid analyticity measure due to the addition of a large scale Gaussian spectrum. The cross over regime is indicated by negative values of $\delta_1$ for $13 \leq \tau \leq 23$.

To analyse the true late time behaviour ($\tau > 25$) we fit with ansatz (4.23). Figure 4.14 (bottom) shows the behaviour of $\delta_2$ as a function of $\tau$. The striking observation is that the decay is now very slow (equation 4.28).

Using a classical method, it is possible to obtain a rigorous upper bound for the supremum norm of stretching rate in terms of the spectrum:

$$\|\gamma(\cdot, t)\|_\infty \leq \sum_{k=1}^{\infty} \sqrt{p_k} \sqrt{E(k, t)},$$

(4.24)
Figure 4.14: Time evolution of the analyticity distance $\delta(\tau)$ of the Fourier spectrum of $\gamma^2$. Top: profile at early times based on equation (4.22), showing the initial burst. Bottom: late time profile based on equation (4.23) showing the slow Gaussian cascade along with the estimate $\delta \approx \sqrt{\pi/\tau}$, which saturates inequality (4.28).
where

\[ p_k \equiv \#\{k \in \mathbb{Z}_2^\text{odd} \cup \mathbb{Z}_2^\text{even} : k - 1/2 < |k| < k + 1/2\} \]
\[ \approx \pi k, \quad k \to \infty. \]  \quad (4.25)

The special condition on odd-odd or even-even modes is due to the discrete symmetry of our initial condition.

Replacing the fit (4.23) into equation (4.24) leads to a bound involving an infinite sum over \( k \) with an ultraviolet divergence in the limit of small \( \delta(t) \). We can approximate this as follows (Bustamante & Brachet [12]):

\[ \|\gamma(\cdot, t)\|_\infty \leq \frac{1}{2} \sqrt{\pi C(t)} \Gamma\left(1 - \frac{n(t) + 1}{4}\right) \left[\frac{1}{2} \delta(t)^2\right]^{\frac{n(t) - 1}{2}}, \]

where \( \Gamma \) is the gamma (factorial) function. A further improvement is obtained by noticing the behaviour of \( n(t) \) at late times from figure 4.15, where it is clear that \( n(t) \to -1 \). Therefore we obtain, in this limit,

\[ \|\gamma(\cdot, t)\|_\infty \leq \sqrt{\pi C(t)} \delta(t)^{-2}. \]  \quad (4.26)

This inequality alone cannot be used to estimate the behaviour of \( \delta(t) \), since the independent factor \( C(t) \) is involved as well, so an extra equation is needed. This extra equation is provided by combining the asymptotic formulae (4.18) and (4.19):

\[ \langle \gamma(\cdot, t)^2 \rangle \approx \frac{\|\gamma(\cdot, t)\|_\infty^2}{2 \tau}. \]

The left-hand-side of this equation can be written in terms of the energy spectrum,
Figure 4.15: Time evolution of the exponent, $n(t(\tau))$, and constant factor, $C(t(\tau))$ (normalised to coincide with the mapped variables), of the fit ansatz equation (4.23) of the Fourier spectrum of $\gamma^2$. Along side $C(t(\tau))$ is the saturated estimate $\pi \tau^{-2}$ from inequalities (4.24) and (4.28).
so if we follow similar steps as in the derivation of inequality (4.26) we obtain

\[ \langle \gamma(\cdot, t)^2 \rangle \approx \frac{C(t)}{2} \delta(t)^{-2}. \]

Therefore we get

\[ C(t) \approx \frac{\|\gamma(\cdot, t)\|_\infty^2 \delta(t)^2}{\tau}. \]  (4.27)

Using this we can go back to inequality (4.26) and show that it is equivalent to:

\[ \delta(t) \leq \sqrt{\frac{\pi}{\tau}}. \]  (4.28)

This inequality is in fact saturated, as confirmed by our numerical simulation (figure 4.14). In terms of original time variable we get

\[ \delta(t) \leq \sqrt{-\frac{\pi}{W_1(-\pi(T_s - t))}} \approx \sqrt{-\ln(\pi(T_s - t))}, \]

which illustrates that the loss of regularity is very slow in the original time variable. In fact if one were to consider the reliability time with the saturated spectrum one would find that \( \tau_{rel} \approx \frac{N^2}{4\pi} \), so that for \( N = 512 \), \( \tau_{rel} \approx 2 \times 10^4 \) or \( T^* - t \approx 10^{-9000} \).

## 4.6 Assessing blowup time \( T^* \) and proximity to it, \( T^* - t \)

Previous methods for assessing the value of \( T^* \), e.g. fitting the behaviour of \( \|\gamma(\cdot, t)\|_\infty \) to a power law \( (T^* - t)^\alpha \), are based on the assumption that the solution is incurring significant errors at intermediate time and \( T^* \) requires careful extrapolation. Here the solution is remaining well resolved until late times and we find that using the original system with the adaptive timestep given above, \( t \) converges to \( T^* \) to within \( \sim 10^{-14} \). This accuracy is surprising given it arises from a simple sum.
\[ t_n = \sum_{i=0}^{n} d\tau/\|\gamma(\cdot, t_i)\|_\infty, \] 
and it cannot be improved by fitting or even by arbitrary precision arithmetic to sum \( dt \) which are below the machine precision threshold. Understanding this accuracy is aided by attempting the comparable exercise for the mapped system. Here the recovery of \( t \) is given by

\[ t(\tau) = \int_{0}^{\tau} \frac{1}{\|\gamma(\cdot, t(\tau'))\|_\infty} \, d\tau', \]

where \( \|\gamma(\cdot, t(\tau'))\|_\infty \) is obtained from formula (4.20). This integral should converge to \( T^* \) as \( \tau \to \infty \). Numerically computing it results in a saturation of error \( \sim 10^{-9} \) (slaved to the error in \( \|\gamma(\cdot, \tau)\|_\infty \)) when \( d\tau = 10^{-4} \) for sufficiently large \( \tau \). This almost leads to a paradox: why should a quantity with lower late time error produce a poorer estimate for \( T^* \) when the procedure for the estimate is qualitatively the same. The reason is that it is the early-time errors which pollute the estimate for \( t(\tau) \) as these are larger in the mapped system and occur at a point where they will contribute more significantly to the final integral (\( d\tau/\|\gamma(\cdot, t)\|_\infty \) is largest).

However, one should proceed with caution when dismissing the ability of the mapped system at assessing its original temporal position: error in the assessment of \( T^* \) is not to be confused with error in the proximity to \( T^* \). Although the original system can integrate to within \( 10^{-14} \) of \( T^* \), it is impossible to assess any behaviour beyond this point: convergence of \( t \) to this value means this is a solid barrier for the method. For the mapped system this is not the case: through \( t(\tau) \) we can produce an estimate for \( T^* - t \) as a function of \( \tau \) by considering the following ‘proximity’ integral

\[ T^* - t \approx P(\tau) = \int_{\tau}^{\infty} \frac{1}{\|\gamma(\cdot, t(\tau'))\|_\infty} \, d\tau'. \]

Recalling that \( \|\gamma(\cdot, t)\|_\infty \) is recovered from \( \langle \gamma_{\text{map}}^2 \rangle \) in the mapped system via formula (4.20), we fit the behaviour of this global measure in the preceding \((\tau - 10)\) window.
via the ansatz

$$\langle \gamma^2_{\text{map}} \rangle \sim \kappa - \frac{m}{\tau}.$$ 

This provides a running estimate for $T^* - t$ which we can validate against the asymptotic formula, equation (4.17). Figure 4.16 shows the relative error in $P(\tau)$ as a function of $T^* - t$ in order to demonstrate how the error depends on the absolute proximity to $T^*$. We find relative errors of the order of $10^{-7}$ persisting far beyond the machine precision limit meaning the mapping provides confidence in the solution at unprecedented closeness to the singularity time.

![Figure 4.16](image_url)

Figure 4.16: Relative error of proximity $P(\tau)$ compared to the asymptotic formula (4.17) plotted against $T^* - t$ (bottom axis) and $\tau$ (top axis). Curves show convergence in $d\tau$ and an error $\sim 10^{-7}$ persistent to exceptionally small values of $T^* - t$. 

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4.7 Conclusion and Discussion

In this chapter we have shown that only by mapping the singular system (4.1) and (4.2) to a regular one (4.5) and (4.6), can certain unconventional late-time behaviours be observed and asymptotic trends be established. The first unusual feature shown is the slow spatial collapse and unusual (Gaussian) Fourier spectrum very near singularity time. This means that the solution will remain well spectrally converged until extraordinarily close to singularity time for even modest resolutions. In turn this implies a fundamental constraint on the original system: in the original variables one can only hope to approach $T^*$ to the precision of the floating point arithmetic being used, usually double-precision, $\approx 10^{-16}$. Because of this lack of digits in the independent temporal variable, assessing any quantities from the original system is a hazardous undertaking as errors grow exponentially. In other words, not only does the proximity to $T^*$ present a floating point barrier: it also harms the accurate assessment of the late time behaviour of the system before the barrier is reached. On the other hand, the mapped system has no floating point arithmetic barrier as the singularity time is now at infinity and we observe uniform errors until $T^* - \tau$ is exceptionally small ($10^{-140}$ in the figures shown).

Another floating point barrier also becomes apparent, namely that $\|\gamma\|_{\infty}$ will eventually overflow, i.e. exceed $\sim 10^{308}$ at $\tau \approx 715$. Luckily the mapping allows us to postpone this barrier further by simply computing $\log \|\gamma\|_{\infty}$ (i.e. outputting just the exponent of the right hand side of equation (4.20)) and use an arbitrary precision exponential in post-processing if required.
Chapter 5

Summary and future work

The work done in this thesis has served to benchmark and validate, on 2D model, a novel method for the analysis of the outstanding question of 3D Euler finite time blowup where the original system of equations is bijectively transformed to a new mapped system which is globally regular in time. In the 3D case, we do not have at hand any analytical solution to compare with. However, the fact that the mapped system’s numerical solution leads to a more accurate estimation of singularity time than the original system in the models studied gives us hope that the mapped approach will be useful on the 3D Euler equations.

The results presented in this thesis have relevance to the regularity problem of the 3D Euler equations. The mapping to regular system for 3D Euler equations was originally derived by Bustamante [11] and is detailed below.

The function that satisfies the BKM theorem [5] for the 3D Euler equations is \( \|\omega(\cdot, t)\|_\infty \), the \( L^\infty \) norm of vorticity \( \omega(x, y, z, t) \) over the spatial domain \( \mathbb{T}^3 \). The mapping from original variables \( (t, u(x, t), \omega(x, t)) \) to mapped variables \( (\tau, u_{\text{map}}(x, t), \omega_{\text{map}}(x, t)) \) is defined by

\[
\tau(t) = \int_0^t \|\omega(\cdot, t')\|_\infty \, dt',
\] (5.1)
with mapped velocity and vorticity vector fields

\[ u_{\text{map}}(x, \tau) = \frac{u(x, t)}{\|\omega(\cdot, t)\|_{\infty}}, \]

\[ \omega_{\text{map}}(x, \tau) = \frac{\omega(x, t)}{\|\omega(\cdot, t)\|_{\infty}}. \]

We now derive the equations verified by this new field

\[ \frac{\partial u_{\text{map}}}{\partial \tau} = \frac{\partial u}{\partial \tau} \frac{1}{\|\omega(\cdot, t)\|_{\infty}} + u \frac{\partial}{\partial \tau} \left( \frac{1}{\|\omega(\cdot, t)\|_{\infty}} \right) \]

\[ = \frac{\partial u}{\partial \tau} \frac{1}{\|\omega(\cdot, t)\|_{\infty}} \frac{dt}{\|\omega(\cdot, t)\|_{\infty}} + u \frac{\partial}{\partial \tau} \left( \frac{1}{\|\omega(\cdot, t)\|_{\infty}} \right) \frac{1}{\|\omega(\cdot, t)\|_{\infty}} \]

\[ = \frac{1}{\|\omega(\cdot, t)\|_{\infty}} \left( -u \cdot \nabla u + \nabla p \right) + u_{\text{map}} \frac{\partial}{\partial t} \left( \frac{1}{\|\omega(\cdot, t)\|_{\infty}} \right) \]

\[ = -u_{\text{map}} \cdot \nabla u_{\text{map}} - \nabla p_{\text{map}} + u_{\text{map}} \frac{\partial}{\partial t} \left( \frac{1}{\|\omega(\cdot, t)\|_{\infty}} \right). \]

To evaluate \( \frac{d}{dt} \left( \frac{1}{\|\omega(\cdot, t)\|_{\infty}} \right) \), we consider the evolution of the maximum norm,

\[ \frac{d}{dt} (\|\omega(\cdot, t)\|_{\infty}) = \omega \cdot \nabla u. \]

\[ \frac{\partial}{\partial t} \left( \frac{1}{\|\omega(\cdot, t)\|_{\infty}} \right) = -\frac{1}{\|\omega(\cdot, t)\|_{\infty}^2} \frac{d}{dt} (\|\omega(\cdot, t)\|_{\infty}) \]

\[ = -\frac{\omega \cdot \nabla u}{\|\omega(\cdot, t)\|_{\infty}^2} \]

\[ = -\omega_{\text{map}}(Y(\tau), \tau) \cdot \nabla u_{\text{map}}(Y(\tau), \tau) \]

where \( Y(\tau) \) is the position of the mapped vorticity’s maximum with \( |\omega(x, \tau)| \leq \|\omega(\cdot, t)\|_{\infty} = |\omega(Y(\tau), \tau)| = 1 \quad \forall \tau, \quad \forall x \in \mathbb{T}^3. \)
Let $\beta(\tau) = \omega_{\text{map}}(Y(\tau), \tau) \cdot \nabla u_{\text{map}}(Y(\tau), \tau)$.

This gives us
\[
\frac{\partial u_{\text{map}}}{\partial \tau} + u_{\text{map}} \cdot \nabla u_{\text{map}} = -\nabla p_{\text{map}} - \beta(\tau)u_{\text{map}}.
\]

The mapped variables thus give us the following evolution equation:
\[
\frac{\partial u_{\text{map}}}{\partial \tau} + u_{\text{map}} \cdot \nabla u_{\text{map}} = -\nabla p_{\text{map}} - \beta(\tau)u_{\text{map}},
\tag{5.2}
\]

which in terms of vorticity is expressed by
\[
\frac{\partial \omega_{\text{map}}}{\partial \tau} + u_{\text{map}} \cdot \nabla \omega_{\text{map}} = \omega_{\text{map}} \cdot \nabla u_{\text{map}} - \beta(\tau)\omega_{\text{map}}.
\tag{5.3}
\]

However, the performance of the mapped system against the original 3D Euler equations is difficult to validate because analytical solutions are not available. That is why the results presented in this thesis are so relevant: the mapping should allow us to simulate the 3D problem further in time and with better accuracy, particularly in cases when the sudden regime changes occur, which are barely resolved at current state-of-the-art resolutions. In fact, there is already some evidence that, depending on the type of initial conditions, the 3D problem has a changing late-time regime where either a depletion of nonlinearity slows vorticity growth [56] or the collision of two vortex sheets accelerates the loss of regularity [12]. It is therefore hoped that future studies of the mapped full 3D problem will give renewed confidence in the late-time behaviour of the next generation of simulations of 3D Euler.
Appendix A

Interpolation of supremum

Polynomial interpolation is the de facto standard for problems of the type where the data is regularly spaced and the region of interest is local. In particular spline interpolants, piecewise polynomials constructed to maintain continuity of derivatives, are known to be able to reconstruct a function with high accuracy while using lower-order polynomials. The primary advantage of this is to reduce the required support of the contributing data. A number of examples exist for spline interpolants, we will focus on the cubic Hermite spline and a variant of the cubic B-spline. There is a considerable body of literature on spline interpolation, we refer the reader to the texts of de Boor [29] and Knott [64]. The cubic Hermite spline over an interval of uniform data can be computed for $N$ data points by solving an $N \times N$ tridiagonal system for the slopes at the knots. This yields the following expressions for the interpolating polynomial at the interval $k$ when considering four and six points, respectively:

$$P_{4,k}(s) = \frac{y_{k-1}}{6}s(1 - s)(s - 2) + \frac{y_k}{2}(1 - s^2)(2 - s) + \frac{y_{k+1}}{2}s(2 - s)(s + 1)$$

$$+ \frac{y_{k+2}}{6}s(s^2 - 1), \quad (A.1)$$
where the $y_k$ are the values on the collocation points, and $s = ((x_p - x_k)/dx)$ is the distance of the point of interest, $x_p$, from the collocation point $x_k$ divided by the spacing, $dx$. One can also perform the global $N \times N$ problem, i.e. $P_{N,k}$, allowing each collocation point to contribute, however this incurs additional computational expense and the influence of the far away points is minimal.

The second interpolation kernel is a variant of the cubic B-spline, developed by Monaghan [82] for use in smoothed particle hydrodynamics (SPH). The order of accuracy is increased via Richardson extrapolation and it has become a popular method in SPH and vortex methods [28, 27, 66]. The four point interpolant is

\[
M_{4,k}(s) = \frac{y_{k-1}}{2}(-s)(1 - s)^2 + \frac{y_k}{2}(2 - 5s + 3s^2) + \frac{y_{k+1}}{2}(s - 4s^2 - 3s^3) + \frac{y_{k+2}}{2}s^2(s - 1) \tag{A.3}
\]

Note that these are 1D kernels, their 2D counterparts (bicubic splines) are simply their convolution in each direction. This leads to a computationally simple algorithm; weights for each collocation point in each direction are simply computed as above and combined to form the full interpolant.

Given these choices for cubic splines for computing the value of a variable at a given point, a difficulty still remains as the position of the maximum is unknown and must be located (accurately) as part of the solution. Maximising the bicubic representation is impractical analytically, computationally a more efficient and reliable strategy is a numerical approach, either a Newton method or steepest ascent.
given a starting point near the collocation point. Unfortunately the robustness of such algorithms is still problematic here, especially where the profile is steepening and values approaching infinity. For instance, given a maximum collocation point we will not know which of the four adjacent cells contains the true maximum which will result in four attempts solving the maximisation problem, three of which are likely to diverge. A short test was carried out to solve the Newton method for the roots of the derivatives of $P_{4,k}$ and converge on the maximum, results are surprisingly poor compared with the other strategies attempted.

The most straightforward robust approach is to populate the grid cells adjacent to the collocation maximum with a refined grid of interpolated points, and pick from them a new maximum. However, this will entail a large number of interpolations and accuracy is limited to the level of refinement chosen. A more efficient method is to develop a 2D ‘bisection’ method, whereby interpolation points are included on a grid at midpoints (i.e. eight points surrounding the collocation maximum) and from them a new maximum is found (or collocation maximum is retained if it is still largest) and a new set of midpoints (now spaced at $dx/4$ intervals) is populated about the new point. In practice we discovered that in fact a quarter-section method outperforms a bisection; at each iteration we populate the $7 \times 7$ grid of $dx/4$ spaced points, giving a slightly broader support at each step (see figure A.1). Note we always use the collocation data points for the interpolation onto the new points, the iterative method is simply an efficient strategy for placing points to search for the maximum.

Note one may also consider employing a more intuitive method whereby a particular function is fit through the collocation points about the collocation maximum and an interpolated maximum extracted from this local representation. An attempt was made to this end using an elliptical gaussian profile and a Newton method to converge on the fit parameters. The method is appealing as the max-
Figure A.1: Schematic of point stencil for bisection and quarter-section interpolation centred about the current maximum. Unfilled circles are the adjacent collocation points (at the first iteration, neighbouring interpolated points at following steps), larger filled circles the mid point stencil, small filled circles (plus the midpoints) the $7 \times 7$ interpolated grid.

The maximum and its position are immediately given as fit parameters. Accuracy and efficiency could be equivalent to the polynomial methods outlined above, however at late times where the profile steepens the Jacobian matrix of the Newton method becomes ill-conditioned and the method fails. Several work-arounds were explored but none resulted in the accuracy and robustness of the polynomial interpolation where matrix inversions are not required and interpolation weights are readily expressible \textit{a priori} (equations (A.1)-(A.3)).

To assess the performance of each method we compute the errors $E_{\nu}(t)$ and $E_{\omega}(t)$ as defined in section 3.4 over a simulation of the $\lambda = -3/2$ model system (unmapped) at $512^2$ resolution for $T = 1.0$ (see reliability time estimate in table 3.2). In addition to the error measures, we determine an estimate of the average CPU time by computing 1000 executions of the interpolation algorithm. Table A.1 and A.2 shows the errors of each interpolation method and CPU times. It was found that the accumulation of numerical error at late times, i.e. as the singularity time is approached, dominates the integral measures so we show the error measures of a truncated time series ($T = 0.8$) which gives a more reliable measure of the errors throughout the simulation (see figure A.2).

Figure A.2 and table A.1 and A.2 show that errors are consistently smallest
Figure A.2: Plot of errors for a selection of interpolation methods investigated at resolution $N = 512$. Black thick line, no interpolation; blue thick dashed line, $P_{6,k}$ quarter-section; thin red dashed line, $P_{4,k}$ quarter-section; thin grey line, $P_{6,k}$ on $10^2$ points; green circles, $P_{4,k}$ maximisation; magenta stars, Gaussian fit.
| Interpolation       | CPU (s) | \( Q_p(T = 0.8) \)     | \( Q_{\omega}(T = 0.8) \) | \( \frac{|E_{\omega}|}{T} \) (T = 0.8) | \( \frac{|E_{\omega}|}{T} \) (T = 0.8) |
|---------------------|---------|------------------------|-----------------------------|----------------------------------------|----------------------------------------|
| None                | 7.8     | 2.8E-9                 | 1.5E-5                      | 0.14                                   | 1.28                                   |
| \( P_{N,k} \) quarter-section | 374     | 7.3E-15                 | 8.8E-11                     | 0.0034                                 | 0.036                                   |
| \( P_{6,k} \) quarter-section | 17.1    | 4.4E-15                 | 5.2E-11                     | 0.0029                                 | 0.028                                   |
| \( P_{4,k} \) quarter-section | 16.7    | 5.1E-13                 | 5.1E-9                      | 0.011                                  | 0.097                                   |
| \( P_{N,k} \) on 100\(^2\) points | 74.5    | 7.3E-15                 | 1.3E-9                      | 0.0037                                 | 0.13                                    |
| \( P_{N,k} \) on 10\(^2\) points | 72.0    | 4.1E-13                 | 1.5E-7                      | 0.014                                  | 0.42                                    |
| \( P_{6,k} \) on 100\(^2\) points | 71.2    | 3.4E-15                 | 1.25E-9                     | 0.0028                                 | 0.13                                    |
| \( P_{6,k} \) on 10\(^2\) points | 16.8    | 3.1E-13                 | 1.5E-7                      | 0.0135                                 | 0.42                                    |
| \( P_{4,k} \) on 100\(^2\) points | 43.4    | 5.3E-13                 | 7.6E-9                      | 0.011                                  | 0.15                                    |
| \( P_{4,k} \) on 10\(^2\) points | 16.3    | 1.38E-12                | 1.5E-7                      | 0.017                                  | 0.42                                    |
| \( M_4'k \) on 100\(^2\) points | 34.2    | 1.3E-12                 | 8.4E-8                      | 0.015                                  | 0.25                                    |
| \( M_4'k \) on 10\(^2\) points | 16.4    | 2.1E-12                 | 2.9E-7                      | 0.018                                  | 0.44                                    |
| Gaussian fit        | 16.9    | 2.3E-9                  | 1.2E-5                      | 0.094                                  | 0.73                                    |
| \( P_{4,k} \) maximisation | 16.4    | 1.3E-10                 | 5.0E-6                      | 0.042                                  | 0.59                                    |

Table A.1: Table showing CPU and error measures for the interpolation methods investigated at resolution \( N = 512 \) and \( T = 0.8 \). Note CPU time is evaluated by 1000 executions of the interpolation subroutine. No interpolation simply means we take the maximal collocation value. Gaussian fails after \( t = 0.65 \) and reverts to collocation points thereafter. \( \omega_{num}(X_c(t), Y_c(t), t) \) is evaluated in each case using \( P_{N,k} \) but \( (X_c(t), Y_c(t), t) \) found from the method indicated.
Table A.2: Table showing CPU and error measures for the interpolation methods investigated at resolution $N = 512$ and $T = 1.0$. Note CPU time is evaluated by 1000 executions of the interpolation subroutine. No interpolation simply means we take the maximal collocation value. Gaussian fails after $t = 0.65$ and reverts to collocation points thereafter. $\omega_{num}(X_\ast(t), Y_\ast(t), t)$ is evaluated in each case using $P_{N,k}$ but $(X_\ast(t), Y_\ast(t), t)$ found from the method indicated.

<table>
<thead>
<tr>
<th>Interpolation</th>
<th>CPU (s)</th>
<th>$Q_y(T = 1.0)$</th>
<th>$Q_{\omega}(T = 1.0)$</th>
<th>$\frac{| \omega - \tilde{\omega} |}{T}$</th>
<th>$\frac{| \omega - \hat{\omega} |}{T}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>None</td>
<td>7.8</td>
<td>6.2E-8</td>
<td>6.0E-5</td>
<td>0.19</td>
<td>1.32</td>
</tr>
<tr>
<td>$P_{N,k}$ quarter-section</td>
<td>374</td>
<td>2.1E-10</td>
<td>7.1E-7</td>
<td>0.03</td>
<td>0.19</td>
</tr>
<tr>
<td>$P_{6,k}$ quarter-section</td>
<td>17.1</td>
<td>6.7E-10</td>
<td>1.1E-6</td>
<td>0.038</td>
<td>0.20</td>
</tr>
<tr>
<td>$P_{4,k}$ quarter-section</td>
<td>16.7</td>
<td>5.1E-9</td>
<td>1.2E-5</td>
<td>0.07</td>
<td>0.44</td>
</tr>
<tr>
<td>$P_{N,k}$ on 100$^2$ points</td>
<td>74.5</td>
<td>2.7E-10</td>
<td>3.4E-6</td>
<td>0.032</td>
<td>0.30</td>
</tr>
<tr>
<td>$P_{N,k}$ on 10$^2$ points</td>
<td>72.0</td>
<td>4.0E-10</td>
<td>4.4E-6</td>
<td>0.038</td>
<td>0.49</td>
</tr>
<tr>
<td>$P_{6,k}$ on 100$^2$ points</td>
<td>71.2</td>
<td>4.8E-10</td>
<td>4.0E-6</td>
<td>0.037</td>
<td>0.31</td>
</tr>
<tr>
<td>$P_{6,k}$ on 10$^2$ points</td>
<td>16.8</td>
<td>6.0E-10</td>
<td>6.5E-6</td>
<td>0.039</td>
<td>0.51</td>
</tr>
<tr>
<td>$P_{4,k}$ on 100$^2$ points</td>
<td>43.4</td>
<td>5.4E-10</td>
<td>1.3E-5</td>
<td>0.071</td>
<td>0.46</td>
</tr>
<tr>
<td>$P_{4,k}$ on 10$^2$ points</td>
<td>16.3</td>
<td>5.9E-10</td>
<td>1.4E-5</td>
<td>0.074</td>
<td>0.57</td>
</tr>
<tr>
<td>$M_{4,k}$ on 100$^2$ points</td>
<td>34.2</td>
<td>6.9E-9</td>
<td>1.9E-5</td>
<td>0.078</td>
<td>0.53</td>
</tr>
<tr>
<td>$M_{4,k}$ on 10$^2$ points</td>
<td>16.4</td>
<td>7.0E-9</td>
<td>2.0E-5</td>
<td>0.078</td>
<td>0.62</td>
</tr>
<tr>
<td>Gaussian fit</td>
<td>16.9</td>
<td>6.2E-8</td>
<td>6.0E-5</td>
<td>0.18</td>
<td>1.0</td>
</tr>
<tr>
<td>$P_{4,k}$ maximisation</td>
<td>16.4</td>
<td>2.1E-8</td>
<td>2.9E-5</td>
<td>0.11</td>
<td>0.77</td>
</tr>
</tbody>
</table>
for the $P_{N,k}$ and $P_{b,k}$ interpolants, better for the cases where 100 × 100 points are used or the quarter-section method. Considering the computational cost of each method it becomes quite clear that the most efficient method is the $P_{b,k}$ quarter-section. The quarter-section method will entail a far fewer interpolations to reach a similar accuracy than, for example the 100 × 100 equivalent. We find the Gaussian fit to be poor, primarily due to it failing at $t = 0.65$ (see figure A.2). The $P_{4,k}$ maximisation (root finding) is also found to be inaccurate compared to the ‘point-search’ methods. We suspect this to be where the Newton method converges (reaches machine precision tolerance in the residual) at a point which is not the true maximum. It might be possible to tune this method by using a coarse sweep (i.e. the first quarter-section grid) before commencing a Newton solve from a closer initial guess, however the quarter-section method proves so fast and accurate that for our purposes we will retain it as our default interpolation.
Appendix B

Numerical Methods

The numerical methods used in this thesis are briefly described in this appendix.

B.1 Fourier transform

Periodic boundary conditions on the domain $[0, 2\pi]^d$ have been considered for the numerical simulations performed in this thesis, where $d$ stands for the dimension of the equation under consideration. In this section, we will consider $d = 1$ where generalization to $d$ dimension is done. A field $f$ can be represented by its Fourier series

$$f(x) = \sum_{k=-\infty}^{\infty} \tilde{f}(k)e^{ikx} \quad (B.1)$$

where the Fourier transform is defined by

$$\tilde{f}(k) = \frac{1}{2\pi} \int_{0}^{2\pi} f(x)e^{-ikx} \, dx. \quad (B.2)$$

Considering the domain $[0, 2\pi]$ with $N$ collocation points $x_0, x_1, ..., x_N$ with $x_j = j\Delta x$ the discrete Fourier transform is defined by
\[ f_N(k) = \sum_{j=-N/2}^{N/2} f(x_j)e^{i\frac{2\pi j k}{N}}. \]  

The wavevectors \( k \) thus take the values \(-\frac{N}{2}, -\frac{N}{2} + 1, \ldots, \frac{N}{2} - 1\). The Fourier transform requires \( O(N^2) \) operations to be computed. In practice, Fourier transforms are performed by using the Fast Fourier Transform (FFT) algorithm that reduce the number of operations from \( O(N^2) \) to \( O(N \log_2 N) \).

### B.2 Solving Partial Differential Equations (PDE)

Partial differential equations (PDEs) form the basis of many mathematical models which are used in many real world situations to describe the behaviour of a variable usually representing some physical phenomena [83]. To investigate predictions of these models, the solution of the PDE’s is approximated numerically in combination with the analysis of simple cases. The PDE’s considered in this thesis can be written in the general form

\[ \frac{\partial u}{\partial t} = Lu + NL[u] \]  

where \( L \) is a linear operator and \( NL[u] \) is the non-linear term of the equation.

#### B.2.1 Pseudo-spectral methods

In Fourier space the nonlinear term \( NL[u] \) in equation (B.4) involves convolutions that are expensive to compute numerically. We therefore solve the equations which are of the type (B.4) using standard pseudo-spectral codes [49], that consist in evaluating the non-linear term in physical space using FFT. This procedure reduces the computational time for the convolution in the case of quadratic non-
linearity from $O(N^{2d})$ to $O(N^d \log_2 N)$ for pseudo-spectral method.

### B.2.2 Dealiasing

The gain in computational time and precision obtained by using pseudo-spectral codes generates a problem called aliasing. Suppose that $NL[u]$ in equation (B.4) has a nonlinearity of order 2. The interaction of two Fourier modes $k_1$ and $k_2$ generates contribution to the wavenumber $k_1 - k_2$ and $k_1 + k_2$. Using the definition of the discrete Fourier transform (equation (B.3)) the wavenumbers are modulated by $N$. Therefore wavenumbers satisfying $|k_1 \pm k_2| > N/2$ are considered as small wavenumbers. This can induce problems with the conservation laws of the PDE.

The commonly known $2/3$ dealiasing rule and the Hous exponential filter are used in this thesis to dealias the system. For the $2/3$ rule, suppose a nonlinearity of order 2 is present in the system. The dealiasing consists in reducing the Fourier space to a sub-set $(-k_{\text{max}}, k_{\text{max}})$ and to eliminate at each time-step the Fourier modes outside this interval. The condition that determines $k_{\text{max}}$ is given by $2k_{\text{max}} - N < -k_{\text{max}}$, i.e $k_{\text{max}} < \frac{N}{3}$.

Let $F(k/N)$ be the filter at wavenumber $k$

$$F(k/N) = \begin{cases} 1 & \text{if } |k/N| \leq 2/3, \\ 0 & \text{if } |k/N| > 2/3. \end{cases}$$

For the Hou’s high-order exponential Fourier smoothing filter [57], $F(k/N)$ is computed as follows

$$F(k/N) = e^{-\alpha |k/N|^m}$$

where in this thesis we have used $\alpha = 36$ and $m = 36$. 

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B.2.3 Time-Stepping

Suppose that initially at time $t = 0$ the value of $u_0 = u(0)$ is known. We denote $u_n = u(t_n)$ as the time discretised approximation at time $t_n = ndt$. Crank-Nicolson and fourth Order Runge-Kutta (RK4) time-stepping schemes are used in this thesis and are described below. The Crank-Nicolson method is evaluated by

$$u_{n+1} = u_n + \frac{1}{2}dt \left( (Lu_{n+1} + NL[u_{n+1}]) + (Lu_n + NL[u_n]) \right). \quad (B.5)$$

The RK4 method is evaluated by

$$u_{n+1} = u_n + \frac{1}{6} \left( k_1 + 2 * (k_2 + k_3) + k_4 \right) \quad (B.6)$$

where

$$k_1 = dt (Lu_n + NL[u_n])$$
$$k_2 = dt \left( Lu_n + NL[u_n] + \frac{k_1}{2} \right)$$
$$k_3 = dt \left( Lu_n + NL[u_n] + \frac{k_2}{2} \right)$$
$$k_4 = dt (Lu_n + NL[u_n] + k_3).$$
Appendix C

Symmetries and conservation laws in Euler flows

The evolution equation of the position of particles (also known as characteristics) are of interest in Euler fluid flow because its associated geometrical structures play a role in the solution of the Euler fluid equations. For a given velocity field \( \mathbf{u}(x, y, z, t) \), a particle’s position with local coordinates \( (X(t), Y(t), Z(t)) \) and subject to the initial conditions \( X(0) = X_0, Y(0) = Y_0, Z(0) = Z_0 \), satisfies a nonlinear, non-autonomous system of ordinary differential equations:

\[
\begin{align*}
\dot{X} &= u_x(X(t), Y(t), Z(t), t) \\
\dot{Y} &= u_y(X(t), Y(t), Z(t), t) \\
\dot{Z} &= u_z(X(t), Y(t), Z(t), t).
\end{align*}
\]  

(C.1)

We define an infinitesimal symmetry of the system (C.1) as a vector field \( \eta(x, y, z, t) \) that, via infinitesimal local displacements, takes any solution of system (C.1) to another solution of system (C.1) for all times when the solutions are defined and at all spatial points where the symmetry is defined. Explicitly, suppose
$(X(t), Y(t), Z(t))$ is a solution of (C.1), and suppose that the functions:

\[
\begin{align*}
\tilde{X}(t) &= X(t) + \epsilon \eta^x (X(t), Y(t), Z(t), t) \\
\tilde{Y}(t) &= Y(t) + \epsilon \eta^y (X(t), Y(t), Z(t), t) \\
\tilde{Z}(t) &= Z(t) + \epsilon \eta^z (X(t), Y(t), Z(t), t)
\end{align*}
\]

satisfy the same equation up to and including $O(\epsilon)$. The vector field $\eta$ satisfies the partial differential equation (PDE):

\[
\frac{\partial \eta}{\partial t} + u \cdot \nabla \eta - \eta \cdot \nabla u = 0. \tag{C.3}
\]

We also define the Lie derivative of a vector field $\eta_2$ with respect to (or along) the vector field $\eta_1$ by the vector field:

\[\mathcal{L}_{\eta_1} \eta_2 \equiv \eta_1 \cdot \nabla \eta_2 - \eta_2 \cdot \nabla \eta_1.\]

We say that two vector fields $\eta_1, \eta_2$ commute if $\mathcal{L}_{\eta_1} \eta_2 = 0$.

## C.1 Generating new symmetries from known symmetries

The vorticity field of the 3D Euler equations, defined by $\omega \equiv \nabla \times u$, satisfies equation (C.3). Therefore the vorticity field is a particular case of an infinitesimal symmetry $\eta$. If the flow admits another symmetry different from the vorticity, then it is possible to generate new symmetries based on the property of the solutions of infinitesimal symmetry (C.3) [55]:

The Lie derivative of an infinitesimal symmetry with respect to another infinitesimal symmetry is an infinitesimal symmetry. This means that if one has two dif-
different infinitesimal symmetries, in principle one can construct a hierarchy of new
infinitesimal symmetries by repeated application of Lie derivatives. The exception
of this hierarchy is the trivial case when the original two symmetries commute.

We consider the evolution equations for the 3D Euler fluid equations at the
symmetry plane \(z = 0\) as derived in chapter 3:

\[
\frac{\partial \gamma}{\partial t} + u_h \cdot \nabla_h \gamma = (2 + \lambda)(\gamma^2) - (1 + \lambda)\gamma^2, \quad (C.4)
\]

\[
\frac{\partial \omega}{\partial t} + u_h \cdot \nabla_h \omega = \gamma \omega. \quad (C.5)
\]

We now find a symmetry of the form \(\eta = (0, 0, \eta^z(x, y, t))\), where \(\eta^z\) is unknown.
Replacing this into equation (C.5) we get

\[
\frac{\partial \eta^z}{\partial t} + u_h \cdot \nabla_h \eta^z - \gamma \eta^z = 0. \quad (C.6)
\]

We then find solutions for equation (C.6) of the form

\[
\eta^z(x, y, t) = [A(t) \gamma(x, y, t) + B(t)]^k, \quad (C.7)
\]

where \(k\) is a constant to be determined and \(A(t), B(t)\) are functions of time to be
determined.

Replacing (C.7) into equation (C.6) we get:

\[
k \dot{A} \gamma + k \dot{B} + \left[ \partial_t \gamma + u_h \cdot \nabla_h \gamma \right] k A [A \gamma + B] \gamma = 0. \quad (C.8)
\]

Using equation (C.4) we obtain

\[
k \dot{A} \gamma + k \dot{B} + \left[ (2 + \lambda)(\gamma^2) - (1 + \lambda)\gamma^2 \right] k A [A \gamma + B] \gamma = 0, \quad (C.9)
\]
which we can solve by eliminating the coefficients of 1, $\gamma$ and $\gamma^2$ to get the system

$$-[1+\lambda]k + 1]A = 0, \quad k \dot{A} - B = 0, \quad k \dot{B} + (2+\lambda)(\gamma^2)kA = 0,$$

which gives the solution

$$k = \frac{-1}{(1+\lambda)},$$

$$B = \frac{-\dot{A}}{(1+\lambda)},$$

$$\dot{A} - (1+\lambda)(2+\lambda)(\gamma^2)A = 0.$$

Thus the new symmetry is

$$\eta_\nu(x,y,t) = \left(0,0, A(t) \gamma(x,y,t) - \frac{\dot{A}(t)}{1+\lambda} \right)^{-\frac{1}{\lambda+1}}, \quad \lambda \neq 1. \quad \text{(C.10)}$$

An important symmetry which is a sort of “perpendicular gradient” of $\gamma$ is given as:

$$\eta_{per}(x,y,z,t) = \left(z^{2,\lambda+1} \frac{\partial \gamma}{\partial y}, -z^{2,\lambda+1} \frac{\partial \gamma}{\partial x}, 0 \right). \quad \text{(C.11)}$$

### C.2 Finding conservation laws of the model equation using new symmetries

The symmetries (C.10) and (C.11) have an interesting commutation relation that can be exploited to generate conservation using another property of the solutions of infinitesimal symmetry: If two infinitesimal symmetries commute and are not proportional by a simple constant factor, then their vectorial cross product is either zero everywhere or is the gradient of a non-trivial conservation law of system (C.1). If their vectorial cross product is zero, then their proportionality factor is a
non-trivial conservation law of (C.1). We have $\eta_y \times \eta_{per} = \nabla D(x, y, t)$, where $D$ satisfies the conservation law:

$$D(x, y, t) = \frac{1}{A(t) \left( A(t) \gamma(x, y, t) - \frac{A(t)}{1 + t} \right)} - (1 + \lambda) \int_0^t \frac{1}{[A(s)]^2} ds, \quad \lambda \neq -1. \quad \text{(C.12)}$$
Bibliography


