Stokes's Fundamental Contributions to Fluid Dynamics

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Introduction

George Gabriel Stokes was one of the giants of hydrodynamics in the nineteenth century. He made fundamental mathematical contributions to fluid dynamics that had profound practical consequences. The basic equations formulated by him, the Navier-Stokes equations, are capable of describing fluid flows over a vast range of magnitudes. They play a central role in numerical weather prediction, in the simulation of blood flow in the body and in countless other important applications. In this chapter we put the primary focus on the two most important areas of Stokes's work on fluid dynamics, the derivation of the Navier-Stokes equations and the theory of finite amplitude oscillatory water waves.

Stokes became an undergraduate at Cambridge in 1837. He was coached by the 'Senior Wrangler-maker', William Hopkins and, in 1841, Stokes was Senior Wrangler and first Smith's Prizeman. It was following a suggestion of Hopkins that Stokes took up the study of hydrodynamics, which was at that time a neglected area of study in Cambridge. Stokes was to make profound contributions to hydrodynamics, his most important being the rigorous establishment of the mathematical equations for fluid motions, and the theoetical explanation of a wide range of phenomena relating to wave motions in water.

Stokes's Collected Papers

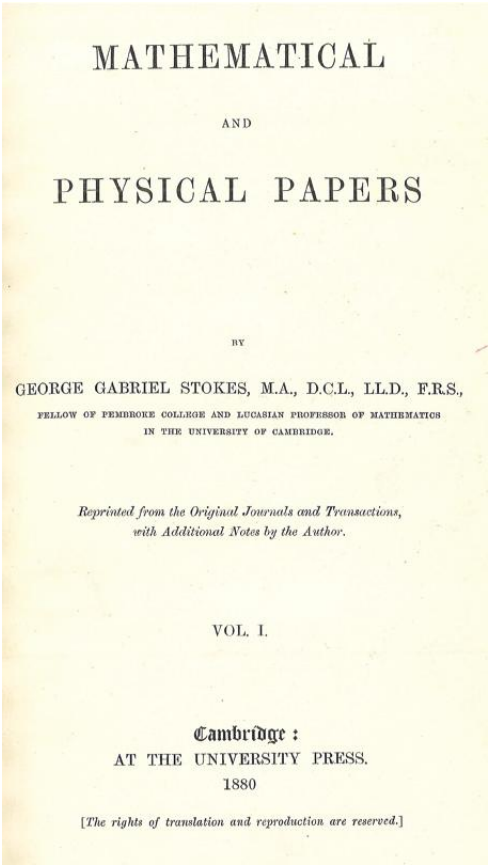
The collected mathematical and physical papers of Stokes[1] [referenced below as MPP] were published over an extended period from 1880 to 1904. They contain articles originally published in journals, additional notes prepared by Stokes and miscellaneous material such as examination papers. Stokes

published about 140 scientific papers. Of these, some 23 were on hydrodynamics. The papers are in chronological order of original publication. The first three volumes, published respectively in 1880, 1883 and 1901, were prepared by Stokes himself. Volumes IV and V were published in 1904 and 1905, edited by Joseph Larmor after Stokes had died. The final volume includes an interesting obituary of Stokes by Lord Rayleigh.

The majority of papers in Vol. I of MPP are on fluid motion. The volume contains two of Stokes's most profound papers, one on the fundamental equations of motion now known as the Navier-Stokes equations, and one on oscillatory wave motion in fluids. The first paper in the collection, *On the steady motion of incompressible fluids* is starkly mathematical in style. The paper is concerned mainly with fluid motion in two dimensions. Little is presented by way of motivation or physical background. In this work, Stokes introduced the notion of fluid flow stability. He pointed out that the existence of a solution does not imply that it can be sustained, as there may be many other motions compatible with the given boundary conditions. Stokes wrote "There may even be no steady state of motion possible, in which case the fluid would continue perpetually eddying". He was beginning to grapple with the recondite problem of turbulence. Ever since, stability of fluid flow has been a fundamental hydrodynamical concept.

The second paper, *On some cases of fluid motion*, opens with an expository section of four pages before the author launches into mathematical details. Stokes writes that "Common observation seems to show that, when a solid moves rapidly through a fluid ... it leaves behind a succession of eddies in the fluid" (MPP, Vol. I, p.54). He then presents a comprehensive account of some fourteen problems in fluid flow. In this paper, Stokes begins to consider friction in fluid flow, discussing no-slip boundary conditions and their consequences.

The issue of internal friction is central in Stokes's monumental paper, *On the theories of the internal friction of fluids in motion, and of the equilibrium and motion of elastic solids*. This paper[2] extends over 55 pages. Stokes notes that it is commonly assumed that the mutual action of two adjacent fluid elements is normal to the surface separating them. The resulting equations yield solutions agreeing with observations in a range of applications. However, there is an entire class of motions for which this theory, which makes no allowance for the tangential action between elements, is wholly inadequate. This effect arises from the "sliding of one portion of a fluid along another, or of a fluid along the surface of a solid". Stokes notes that the tangen-



MATHEMATICAL

AND

PHYSICAL PAPERS

BY

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Figure 1: Title page of Stokes's *Mathematical and Physical Papers, Vol. 1.*

tial force plays the same role in fluid motion that friction does with solids. Stokes gives the example of water flowing down a straight inclined chute. The then-current theory of fluid flow would indicate a uniform acceleration of the water, something that is completely at odds with experience.

In his masterful paper in 1847, *On the theory of oscillatory waves*[3], Stokes investigates the dynamics of surface waves in the case where the height of the waves is not assumed to be infinitesimally small. In a supplement to this paper, also included in Vol. I of MPP, Stokes showed that, for the highest possible wave capable of propagation without change of form, the surfaces at the crest enclose an angle of 120°.

Vol. II of MPP contains three sets of Notes on Hydrodynamics. These

were part of a series of notes for students prepared by William Thompson and Stokes. The three sets by Stokes are entitled "On the dynamical equations", "Demonstration of a fundamental theorem" and "On Waves". In the first of these, on the dynamical equations, Stokes gives a homely illustration of fluid viscosity: "The subsidence of the motion in a cup of tea which has been stirred may be mentioned as a familiar instance of friction ...". In the third set, on waves, he revisited his paper on oscillatory waves.

MPP, Vol. III opens with an extensive study of the effects of air friction on the motion of a pendulum. This paper[4], published in 1850, is 141 pages in length. Surprisingly, it was considered by Stokes to be one of his greatest contributions to science. The remaining papers in Vol. III are on the physics of light. Vol. IV of MPP contains little of relevance to fluid dynamics. In Vol. V we find "On the highest wave of uniform propagation"[5], published in 1883, which considers the waves of maximum steepness that can propagate without change of form. The volume also contains a second supplement to Stokes's great 1847 paper on oscillatory waves. Finally, the questions for the Mathematical Tripos and Smith's Prize for the period from 1846 to 1882 are included in Vol. V. In the Smith's Prize paper of February 1854, Question 8 asked for a proof of what is now known as Stokes's Theorem, a standard result in vector calculus.

It is abundantly clear from MPP that Stokes's greatest work was done before he had reached his 35th birthday. Stokes's work on fluid dynamics was done during two distinct periods, from 1842 to 1850 and, after a thirty year gap, between 1880 and 1898. In his obituary, Lord Rayleigh remarks that "if the activity in original research of the first fifteen years had been maintained for twenty years longer, much additional harvest might have been gathered in".

The Navier-Stokes Equations

The Navier-Stokes equations are the universal mathematical basis for fluid dynamics problems. Navier's original derivation in 1822 was not immediately accepted, and gave rise to some heated discussions and debate. An excellent review can be found in Darrigol[6]. There were several attempts, following Navier's publication in 1822, to develop a rigorous derivation of equations for viscous fluid flow. Most notable were those of the French scientists Poisson, Cauchy and Saint-Venant. George Green also made substantial contributions to the problem. Although Stokes was not the first to derive the equations in

their final form, his derivation was founded on more general and physically realistic assumptions. Stokes also found several particular solutions to the viscous equations. Euler had obtained his fluid equations in 1755:

$$\rho\left(\frac{\partial \mathbf{V}}{\partial t} + \mathbf{V}\cdot\nabla\mathbf{V}\right) = -\nabla p + \mathbf{F}$$

Unfortunately, these equations produced absurd results in a wide range of practical situations where fluid resistance was important. This was recognised by d'Alembert and by Euler himself. D'Alembert expressed his concerns thus: "I do not see how one can satisfactorily explain, by theory, the resistance of fluids." He remarked that the theory leads to "a singular paradox which I leave to future geometers for elucidation"[6]. Thus, at the beginning of the nineteenth century, fluid dynamics was incapable of explaining a wide range of important fluid flow phenomena. Hydraulic engineers had an armory of empirical techniques, but these were not firmly based on fundamental physical principles.

Navier's equation, first written down in 1822, was freshly discovered at least five times, by Navier, Cauchy, Poisson, Saint-Venant and Stokes. Cauchy and Poisson paid no attention to Navier's work. Saint-Venant and Stokes acknowledged it but regarded it as lacking in precision and rigour. We can write the Navier-Stokes equations in modern notation as

$$\frac{\partial \mathbf{V}}{\partial t} + \mathbf{V}\cdot\nabla\mathbf{V} + \frac{1}{\rho}\nabla p - \nu\nabla^2\mathbf{V} + \mathbf{F} = 0 \tag{1}$$

with the assumption of nondivergent flow, $\nabla\cdot\mathbf{V} = 0$. These are equations (13) in Stokes's paper of 1845, and he comments that they are "applicable to the determination of the motion of water in pipes and canals, to the calculation of the effect of friction on the motions of tides and waves, and such questions."

Stokes notes in his introduction that, having derived his equations, he discovered that Poisson had arrived at the same equations, and that the same equations had also been obtained in the case of an incompressible fluid by Navier. However, both Poisson and Navier had used methods markedly different from Stokes.

In addition to the equations applying to the interior of the fluid, Stokes also considered the conditions that must be satisfied at solid boundaries. There was widespread controversy about the appropriate boundary conditions, without which problems could not be formulated, let alone solved. At

the time Stokes presented his memoir on the fluid equations (1845), he already believed that the most natural assumption for the relative velocity at a rigid boundary was that it must vanish.

Controversy

James Challis, the Plumian Professor at Cambridge, had the great misfortune to have observed Neptune on two occasions a month before Urbain Le Verrier's predictions were confirmed, but to have failed to identify it as a planet. He blamed pressure of other work for this oversight. During his undergraduate years, Stokes attended some of the lectures of Challis on fluid dynamics. He differed strongly with Challis in several important ways, and their disagreements led to the publication of several acrimonius exchanges. Challis published some fourteen papers on hydrodynamics, characterised by Craik[7] as 'mostly worthless'. Challis argued that the assumption of irrotational flow implied rectilinear motion. Of course we can easily show that there is no essential link between curvature and rotation of the flow. Linear flow with lateral shear has non-vanishing vorticity; moreover, a circular vortex with azimuthal velocity varying inversly with radial distance from the centre, as in the external region of a Rankine vortex, is irrotational. Challis also maintained that Euler's equations for incompressible fluid flow were incomplete, a view strongly contested by Stokes. Ultimately, Stokes tired of the ongoing conflict. In a letter to William Thomson in 1851, Stokes wrote about an 'awful heterodoxy' of Challis in the Philosophical Magazine[8]. He concluded 'I am half inclined to take up arms, but I fear the controversy would be endless.'

Stokes's Applications of the Equations

The solution of the full Navier-Stokes equations was quite beyond any analytical attack. However, when drastic approximations are made, systems amenable to analysis may result.

Stokes's First and Second Problems

For steady flows with parallel streamlines, the nonlinear terms vanish and a full solution is normally easily obtained. The associated initial value problems, where the motion is started impulsively, are also amenable to solution as the advection terms drop out again. The flow due to the impulsive motion

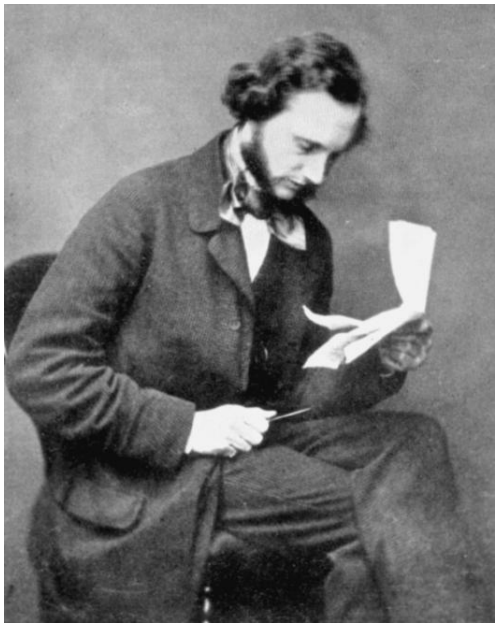

Figure 2: William Thomson, Lord Kelvin (1824–1907).

of a flat plate parallel to itself is known as Stokes's First Problem [9]. The fluid motion known as Stokes's Second Problem is the flow around an infinite flat plate that moves sinusoidally in its own plane. There is a natural time-scale here, imposed by the period of the forcing, and there is no similarity solution for this problem. Stokes found the solution that obtains after the initial transient response has decayed.

The Pendulum

Stokes's motivation for studying fluid resistance came from his interest in the use of pendulums for geodesic measurements. Friedrich Bessel had published an influential memoir taking into account the effects of atmospheric drag on the motion of the pendulum. This triggered a series of practical experiments, but a full theoretical understanding was lacking.

The pendulum has provided an invaluable scientific apparatus and has played a vital role in horological science and in geodesy. The precise measurment of time has been of crucial importance in the scientific world and also in many practical situations. A most notable example was the determination

of longitude, essential for the purposes of navigation.

Theoretical results often assume that the apparatus is *in vacuo*, so that the effects of air must be considered when comparing experimental results with theoretical values. Stokes's extensive paper addresses this question. In his study *On the effect of the internal friction of fluids on the motion of pendulums*[4], Stokes assumed that the viscosity of air is proportional to the density. It was only later that Maxwell showed that the viscosity is insensitive to density over a wide range.

Creeping flow around a sphere

In his study of the effect of air resistance on the motion of a pendulum, Stokes was led to examine the resistance on a sphere of radius a moving at speed U through a viscous fluid. He gave a solution for the creeping flow around a sphere. He considered axisymmetric laminar flow and, assuming high viscosity, neglected the inertial terms in the equation of motion, deriving an equation

$$\left[\frac{\partial^2}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right)\right]^2 \psi = 0\,.$$

He solved this with appropriate boundary conditions to find the velocity and pressure fields. The pressure maximum is at the forward stagnation point and the minimum is at the rear stagnation point. Stokes then found an expression for the drag force,

$$D = 6\pi\mu a U\,,$$

showing that the resistance is proportional to the velocity. One third of the drag is due to pressure and two thirds to skin friction. The result that, for low Reynolds number flow, the drag force varies linearly with speed is frequently referred to as Stokes's law of resistance. This result was later crucial for Millikan in designing his oil-drop experiment to measure the charge on an electron.

For a two-dimensional obstacle such as a cylinder, the Stokes balance

$$\frac{1}{\rho}\nabla p = \nu\nabla^2\mathbf{u}$$

has no solution satisfying the boundary conditions at infinity. This result is often called Stokes's Paradox. This led Stokes to conclude that a steady

slow flow around a cylinder cannot exist in nature. The explanation is that the small parameter 1/Re multiplies the highest-order term in the governing equation and the perturbation problem is singular. An improvement of Stokes's solution, using a linear approximation of the inertial term at large distances, was provided by Oseen in 1910. His result may be written

$$D = 6\pi\mu a U[1 + 3Ua/8\nu]$$

Stokes used his law of resistance to explain why tiny droplets of moisture in a cloud can remain suspended over a long time scale. Stokes remarks that "The pendulum thus, in addition to its other uses, affords us some interesting information on the department of meteorology." This is in relation to the application of his analytical results to droplets of moisture falling through the atmosphere. For the small droplets forming a cloud, the terminal velocity is so small that "the apparent suspension of the clouds does not seem to present any difficulty."

Modern Applications of Navier-Stokes Equations

Climate change and its consequences are amongst the most pressing problems facing humanity today. There are enormous uncertainties concerning the future climate, and the best means we have for reducing these is by means of predictions based on computer simulations. The computer models for simulating weather and climate are known as Earth System Models. They are of great complexity, embracing a wide range of physical phenomena, with components for the atmosphere, the oceans, the land surface and sub-surface and the cryosphere. There are strong interactions between all these subsystems. At the heart of every Earth System Model lies a dynamical core. The 'kernel of the core' comprises the Navier-Stokes equations. The same models are used regularly for short and medium range weather forecasts. Over recent decades, there has been a dramatic improvement in the accuracy and scope of computer forecasts, with enormous benefits for human society[10]. Thus, the fundamental work of Stokes underlies one of the greatest scientific advances of the twentieth century.

Of course, the Navier-Stokes equations have far wider applicability. They are used by aeronautical engineers to optimise aircraft design, by ship-builders to improve safety and minimise energy loss, by hydraulic engineers and by biologists studying blood flow in the body. Scientists use the Navier-Stokes

equations in fundamental studies of turbulence, and the properties of the solutions of these equations are amongst the great unsolved problems of mathematics.

Stokes's Theorem

Students' first encounter with Stokes's name is usually through a fundamental theorem in vector calculus. Stokes's theorem relates the surface integral of a vector field over an open surface to the line integral of the field around the boundary. In modern notation, it may be written

$$\int_A \nabla \times \mathbf{V} \cdot \mathbf{n} \, \mathrm{d}A = \oint_{\partial A} \mathbf{V} \cdot \mathrm{d}\mathbf{s}$$

Taking $\mathbf{V}$ to be the flow velocity, this result states that the areal integral of vorticity over a surface is equal to the circulation around the boundary. It also expresses the fact that the circulation around the closed boundary curve $C = \partial A$ is equal to the flux of vorticity across the surface. Thus, it implies that for irrotational flow the circulation vanishes. In more old-fashioned notation, Stokes would have written that, for irrotational flow, $u \, \mathrm{d}x + v \, \mathrm{d}y + w \, \mathrm{d}z$ is an exact differential.

Stokes's theorem is a generalization of the fundamental theorem of calculus, which states that the integral of a function f over a closed interval $[a, b]$ can be evaluated as the difference between the values of the antiderivative of f at the ends of the interval. In turn, Stokes's theorem itself has been generalized to become an important principle in differential geometry: the integral of a differential form over the boundary of an orientable manifold is equal to the integral of its exterior derivative over the manifold; symbolically,

$$\int_{\partial\Omega} \omega = \int_{\Omega} \mathrm{d}\omega \, .$$

The theorem has an intersting history. The basic result was contained in a letter from William Thomson to Stokes in 1850. Stokes set the theorem as a question on the Smith's Prize exam for 1854, which led to his name becoming attached to the result. In a footnote in Vol. V of MPP, Larmor mentions earlier researchers who had integrated the curl of a vector field over a surface. Neither Stokes nor Thomson published a proof of the theorem. The first proof appeared in an 1861 publication of Hermann Hankel. The general result, in modern form, was formulated Élie Cartan.

The Theory of Oscillatory Waves

The linear theory of water waves was developed in the eighteenth and early nineteenth century by French mathematicians, most notably Laplace, Lagrange, Poisson and Cauchy. Nonlinear waves were studied in Germany by Franz Joseph von Gerstner, who found the first exact solution for finite amplitude waves in deep water. In the 1830s and 1840s, several British physicists helped to advance the theory of waves. These included James Challis, George Green, John Scott Russell, Philip Kelland, Samuel Earnshaw and George Biddel Airy. The origins of water wave theory are reviewed comprehensively by Craik[7] and Darrigol [11].

In 1837, the year Stokes went up to Cambridge, the British Association for the Advancement of Science set up a Committee on Waves, to carry out observations and conduct experiments. John Scott Russell and Sir John Robinson were the directors of the committee. Within a few years, they had produced several reports. A topic that has attracted great attention was the "solitary wave" observed on a canal by Scott Russell. This defied theoretical explanation until 1876, when Lord Rayleigh derived a solution by retaining both dispersion and nonlinearity.

The monumental *Report on Waves* published by Russell and Robinson in 1841 proved invaluable to scientists grappling with the theory of waves. Prominent amongst these was Stokes. The intriguing observations and experimental results of Scott Russell provided an impetus to Stokes to study wave dynamics, and in 1847, just ten years after the establishment of the BAAS committee, Stokes had completed his monumental work, "On the theory of oscillatory waves". This is one of the great classical papers of hydrodynamics.

In a dispersive medium, different wave components travel at different speeds, moving in and out of phase with each other. Therefore, a small amplitude disturbance of unchanging form must be sinusoidal: if there are two or more wave components, they will will travel at different speeds, changing the shape of the wave form. This led Stokes to believe that Russell's solitary wave was a mathematical impossibility.

In a nondispersive medium, where the phase speed is independent of wavenumber, nonlinear interactions can lead to unbounded growth of amplitude if there is no counteracting effect. There is an opportunity for nonlinear steepening to be attenuated by dispersion, and for for the dispersive effects to be balanced by nonlinear steepening. As a result, finite amplitude waves of constant form become possible.

Stokes was influenced by the earlier work on waves of George Biddell Airy[17], and also by the researches of George Green. Airy's survey article 'Tides and Waves' appeared just as Stokes was setting out on his hydrodynamical researches. This survey contains the now-standard linear theory of water waves, including the dispersion relation which may be written in modern form as

$$c^2 = \frac{g}{k} \tanh kh$$

where k is the wavenumber and h the mean depth. A similar result had been obtained much earlier by Laplace.

Stokes Waves

In his 1847 paper[3], Stokes studied wave motions in the case where the amplitude was sufficiently large that the nonlinear interactions could not be neglected. This was the first comprehensive analysis of waves of finite amplitude. He showed that periodic waves of finite amplitude are possible in deep water. Stokes considered weakly nonlinear periodic waves in water of intermediate or large depth. He devised a perturbation approach and derived solutions to third order in a small quantity, the product of amplitude and wavenumber or wave *steepness*, $\varepsilon = ka$. Thus, the amplitude was assumed to be small relative to the length of the waves.

Stokes's solution for the free surface elevation is

$$y = a\left[\cos k(x-ct) + \tfrac{1}{2}\varepsilon \cos 2k(x-ct) + \tfrac{3}{8}\varepsilon^3 \cos 3k(x-ct) + O(\varepsilon^3)\right]$$

All the Fourier components propagate at the same speed c, given by

$$c = (1 + \tfrac{1}{2}\varepsilon^2)\sqrt{\frac{g}{k}}$$

so that the wave profile is unchanging in time. It is noteworthy that the phase speed c depends upon the amplitude a. As there are components of different scales, the wave form is not longer a pure sinusoid. The ridges are steeper and narrower than the troughs. This wave profile might have been noticed by any keen observer of waves, but it took the genius of Stokes to provide a theoretical explanation.

One limitation of Stokes's weakly nonlinear analysis of 1847 was its inadequacy in describing the solitary waves observed by Scott Russell. Indeed,

this gave rise to ongoing controversy leading to doubt being cast upon Scott Russell's results. Neither Airy nor Stokes was convinced about the importance that Russell ascribed to his 'Great Primary Wave'. It is regrettable that, as a consequence of the growing authority of Stokes, recognition of the value of Russell's work took many decades. It was only much later that the work of Joseph Boussinesq and Lord Rayleigh, which took account of both dispersion and nonlinearity, provided a solid analysis of solitary waves.

Some fifty years after Stokes's finite amplitude wave solution was found, it was shown by Korteweg and de Vries[16] in 1895 that the Stokes wave is a large-depth approximation to the cnoidal wave solutions of the equation formulated by them. Russell's solitary waves correspond to the infinite period limit of these solutions[12p425] Many investigations following Stokes have shown that periodic wave trains of unchanging profile, such as the one discovered by him, are found in a wide range of physical systems and indeed are typical in nonlinear dispersive systems.

The analysis of nonlinear waves is complicated because boundary conditions must be specified at the free surface, the position of which is unknown until the problem is solved. Stokes circumvented this obstacle by using a perturbation approach — the Stokes expansion — that enabled him to express the boundary condition in terms of quantities at the known mean surface elevation. To avoid spurious 'secular variations', Stokes also expanded the dispersion relationship as a perturbation series. This approach, now known as the Lindstedt-Poincaré method[13], is widely applied.

Waves of Maximum Height

In 1866 Stokes was appointed to the newly created Meteorological Council. This led him to consider several practical problems involving ocean waves. He investigated methods of accurately observing and measuring wave heights and periods from ships. He also studied ways to determine the location of distant storms using measurements, recorded in ships' logs, of the swells generated by them.

Stokes, who grew up on Ireland's western shore, was a skilled swimmer and a keen observer of nature. During his many holidays in Ireland, he undertook his own observational studies of waves and swell. In his mathematical study of surface waves he recalled his youth: "In watching many years ago a grand surf which came rolling in on a sandy beach near the Giant's Causeway, without any storm at the place itself, I recollect being struck with the

blunt wedge-like form of the waves where they first lost their flowing outline, and began to show a little broken water at the very summit. It is only I imagine on an oceanic coast, and even there on somewhat rare occasions, that the form of the waves of this kind, of nearly the maximum height, can be studied to full advantage."

It was during the second period of study of sea waves, in the 1880s, that Stokes examined the question of the highest possible periodic wave. He concluded that the wave of maximum height had a sharp crest, with the water surfaces ahead of and behind the peak meeting at an obtuse angle. This also accorded with observations that he had made during his holidays in Ireland.

Stokes showed that the maximum wave steepness is $H/\lambda \approx 0.1412$ or $\sqrt{2}-1$. He returned to this question in research described in a supplement in MPP[1] (Vol. 1, pp. 314-326) and showed that the angle at the crest of these waves of maximum steepness is 120°. Stokes's solutions had, and continue to have, application to practical problems in coastal and off-shore engineering. For larger waves or shallower water, cnoidal theory, where the solutions are expressed in terms of Jacobi elliptic functions, may yield more accurate results.

The reference frame chosen by Stokes has the Oy-axis pointing downwards. This has led to some confusion. On page 211 of Vol. I of MPP, Stokes's formula for the surface height appears as

$$y = a\cos mx - \tfrac{1}{2}ma^2\cos 2mx + \tfrac{3}{8}m^2a^3\cos 3mx \tag{2}$$

(he writes m for the wavenumber k). Stokes remarks that the term of third order is almost insensible. The form of the equation is identical to that in the original paper in the *Transactions*[3]. The profile drawn by Stokes is for an amplitude $a = 7\lambda/80$, where λ is the wavelength. This corresponds to wave steepness $\varepsilon \approx 0.55$. Fig. 3, which is consistent with Stokes's illustration, shows plots of the second and third order expressions. The thick curve is the second-order approximation, the thin curve includes the third-order term. The figure confirms the fact that the effect of the third-order term is quite small or, as Stokes put it, "almost insensible".

Expressions consistent with (2) are reproduced in Darrigol[6p62] and in Craik[14p30]. The correct form of the equation is found in many texts, but often with the Oy-axis pointing upwards. This choice is made by, for example, Lamb[12] (1932, p 417, his (3)) and Whitham[15] (1999, p. 12, his (1.33).

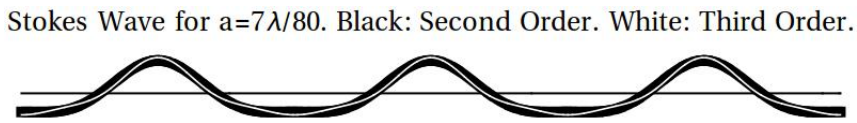

Figure 3: Form of the Stokes wave, with sharpened crests and flattened troughs. Thick curve: second-order approximation. Thin curve: third-order approximation.

However, on his equation (17) on page 419, Lamb[12] also gives the fourth-order approximation with the negative sign. One may wonder whether this error has found its way into computer codes. It may be noted that Stokes likened the wave profile to a prolate cycloid. The wave that he drew closely resembles an inverted curtate cycloid.

Stokes Drift

For nonlinear waves, there is an ambiguity in the partitioning of the solution into wave and mean-flow parts. Stokes identified two ways of defining wave speed or *celerity*. In the first approach, the wave is considered in a frame moving with the mean horizontal velocity. In the second approach, the mean horizontal mass transport in the reference frame is zero.

If we consider small amplitudes, linear wave theory indicates that fluid particles move vertically up and down as a wave travels horizontally. However, observations show that an object floating on the sea surface in the absence of wind moves slowly in the direction of the waves. This is a finite-amplitude effect, now known as *Stokes drift*.

The trajectory of the floating object is not a closed curve but has the form of an epicycloid. The mean velocity at a fixed point is zero, but the mean Lagrangian velocity of a fluid parcel is non-vanshing: the parcel's forward movement at the top of the trajectory is greater than its backward movement at the bottom. Although this is a second-order effect, it is often significant and has important practical consequences. For deep water gravity waves with amplitude a, frequency ω and wavenumber k, the mean Lagrangian speed is

$$\bar{U}_{\mathrm{L}} = a^2 \omega k \exp 2kz_0$$

where the initial coordinates (x_0, z_0) serve to label the particle. This is also called the mass transport velocity.

Group Velocity

Keen wave observers will have noticed how difficult it is to follow the movement of an individual wave crest in a deep pond. Waves occur in groups or bunches and wave crests seems to appear from nowhere at the rear of the group, move through it and vanish somewhere ahead. The first report on this phenomenon may have been by Scott Russell around 1844[11]. Russell's observations generated little interest at the time. Independently, William Rowan Hamilton considered a similar phenomenon in the context of optics[12p381].

Stokes had been studying swells in calm conditions and argued that the wave period could be used to determine the location of the storm that gave rise to the swell. He was aware that longer waves travel faster than shorter ones and that, as a consequence, the observed period of waves from a distant storm decreased with time. William Froude also noted the distinction between the speed of individual crests and that of the wave group. Froude pointed out that the relevant speed for this estimate should be that of the group, not the phase speed of the waves.

In 1876, Stokes wrote to Airy about what he believed to be an original result: the overall speed of the wave group in deep water is only half the speed of the individual waves. This is easily shown, taking the phase speed in deep water to be $c = \omega/k = \sqrt{g/k}$. The group velocity is then $c_g = d\omega/dk = \frac{1}{2}\sqrt{g/k} = \frac{1}{2}c$. Stokes posed this problem as a question for the Smith's Prize that same year. The theory of group velocity was further advanced by Lord Rayleigh, who was also inspired by experiments of Osborne Reynolds. Rayleigh demonstrated the important relationship between group velocity and energy propagation.

Group velocity is of immense importance in weather forecasting. The large wave-like disturbances in the atmosphere at middle latitudes, known as Rossby waves, travel at an approximate speed of $c = U - \beta/k^2$, where U is the mean zonal flow and β is a constant. The group velocity is easily shown to be $c_g = U + \beta/k^2$, which is *greater than the phase speed*. Wave minima or troughs are commonly linked to stormy weather. Through the action of group velocity, a new storm can appear "spontaneously" downstream of an existing chain of storms. The propagation of energy is more rapid than the movement of the individual storms.

Conclusion

Stokes's study of oscillatory waves, which initiated the nonlinear theory of dispersive waves, was far in advance of contemporary developments. He showed that periodic wave trains are possible in nonlinear systems and that their speed of propogation varies with the amplitude. This had deep influence on subsequent research. Stokes's work on waves, in addition to his other achievements, led to his appointment in 1849 as Lucasian Professor, a position he held for more than fifty years. In 1854, he became Secretary of the Royal Society and, from that time, heavy administrative responsibilities had the consequence that his scientific output was greatly diminished. Stokes was President of the Royal Society from 1885 to 1890. Through his position as Secretary and, later, President, Stokes was able to provide substantial assistance and support to a large number of younger scientists, as is evident in acknowledgments in the "Proceedings" and "Transactions".

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