A simple recursive numerical method for Bermudan
option pricing under Lévy processes

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A numerical method is developed that can price options, including exotic options that can be priced recursively such as Bermudan options, when the underlying process is an exponential Lévy process with closed form conditional characteristic function. The numerical method is an extension of a recent quadrature option pricing method so that it can be applied with the use of fast Fourier transforms. Thus the method possesses desirable features of both transform and quadrature option pricing techniques since it can be applied for a very general set of underlying Lévy processes and can handle certain exotic features. To illustrate the method it is applied to European and Bermudan options for a log normal process, a jump diffusion process, a variance gamma process and a normal inverse Gaussian process.

1 Introduction

Existing Fourier and fast Fourier transform option pricing models are essentially closed form option pricing formula that can be applied to a wide range of underlying stochastic processes.

*The author is indebted to comments from Roger Lord, Cornelis Oosterlee and the participants of the 2005 EFA Moscow conference.
Fourier and fast Fourier transform methods (hereafter referred to collectively as transform methods) can handle general stochastic processes and can also be applied to exotic versions of the Black Scholes (BS) option pricing formula such as barrier options, digital options etc. They cannot be applied recursively so this limits their applicability when an option includes certain exotic features, for example, if an option has a changing barrier, if an option is exercisable on a number of dates or if there is a discrete dividend paid during the life of the option. Pioneering articles on the application of transform methods to option pricing include Stein and Stein (1991), Heston (1993), Bates (1996), Duffie, Pan and Singleton (2001), Carr and Madan (1999), Backshi and Madan (2000), Lewis (2000) and Lee (2004). Existing quadrature option pricing models have been developed for a restrictive class of underlying processes, those with a closed form density function, however they are recursive models and so they can accurately price options with exotic features. Option pricing using numerical integration methods was first introduced by Parkinson (1977) but because of the discontinuities involved in option pricing this has been one of the least used option pricing methods. Sullivan (2000) successfully applied quadrature routines to price American options under geometric Brownian motion. One of the most recent of these numerical integration schemes specifically deals with the non-linearity error introduced by the discontinuities in option prices, see Andricopoulos, Widdicks, Duck and Newton (hereafter AWDN) (2003). In their article it was recognised that options could be priced accurately using a discounted integration of the payoff provided the payoff is segmented so the integral is only carried out over continuous segments of the payoff. They run a quadrature routine (QUAD) to estimate each section of the segmented integral. In this way the quadrature routine is only applied to functions that are continuous and have continuous higher order derivatives and thus the method retains its accuracy. If there is more than one time step needed to price the option, an asset price lattice is set up and the price at each point in the lattice at time $t$ is calculated recursively from the prices at time $t+1$ using the quadrature routine. Thus, for example, in a plain vanilla

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1Transform methods can also price exotic options that cannot be priced in the BS framework, for example, Asian options where the average is taken over the entire life of the option, see Backshi and Madan (2000).
call option only one time step is needed and the payoff is split into two continuous regions, the segment that is greater than zero and the segment that is zero. As another example, a single barrier at time $t$ can be included by working backwards from maturity to find the prices on the lattice at time $t$ and then applying the barrier condition. The option price prior to time $t$ is calculated recursively from the time $t$ prices which are split into two continuous regions, one on either side of the barrier. In this article the quadrature option pricing method developed by AWDN is extended so that it can be applied with the use of Fourier and fast Fourier transform (FFT) methods originated by Heston (1993) and Carr and Madan (1999) respectively. This numerical method allows transform option pricing methods to be applied recursively resulting in a model that can incorporate certain exotic features and can be applied for a wide range of stochastic processes, specifically those stochastic processes that result in a process with independent increments $^2$ and whose conditional characteristic function is known in closed form. Many Lévy processes fall into this category. Examples of finite activity Lévy processes include the jump-diffusion process of Merton (1976), and the double exponential jump-diffusion model of Kou (2002). Examples of infinite activity Lévy processes (Lévy processes whose jump arrival rate is infinite) include the variance gamma process of Madan and Seneta (1990), Madan and Milne (1991) and Madan, Carr and Chang (1998), and the normal inverse Gaussian model of Barndorff-Nielsen (1998). See Cont and Tankov (2000), Geman (2002) and Carr and Wu (2004) and references therein for more on these and other Lévy processes. The numerical method developed in this paper can be considered as a simple alternative to current numerical pricing methods for exotic options under Lévy processes, such as the finite difference method of Hirsa and Madan (2004), the lattice method of Kellizi and Webber (2003) and the multinomial tree method of Maller, Solomon and Szimayer (2004). To demonstrate the accuracy of the numerical method European options are priced and compared to existing prices under a number of different processes. These processes include $^2$The numerical method presented in this paper can be used with processes that do not have independent increments, for example options can be priced under Heston’s (1993) stochastic volatility process. However the model becomes slower than competing methods as the integrations must run over two dimensions: the asset price and the variance.
geometric Brownian motion (GBM) for benchmarking purposes, Merton’s jump-diffusion (JD) process as an example of a finite activity Lévy process, a variance gamma (VG) and a normal inverse Gaussian (NIG) process as examples of infinite activity Lévy processes. In the GBM case American options are approximately priced using extrapolation techniques and compared with prices from existing American option pricing methods to demonstrate the ability of the numerical method to price certain exotic options. For the remaining processes Bermudan options are priced for benchmarking purposes, however extrapolation techniques can be applied to approximate American prices. The remaining article is organised as follows. Section 2 covers the methodology by first examining the existing QUAD model and then extending it so that transform techniques can be applied in conjunction with the quadrature routine. Section 3 contains the implementation details for both the Fourier and the FFT approach. Section 4 contains results for European and American options priced under GBM, and European and Bermudan options priced under JD, VG and NIG processes. Section 5 concludes.

2 Methodology

2.1 Quadrature method

Let us start by considering the quadrature method of AWDN (2003). Assume the underlying process is geometric Brownian motion. The notation is as follows: \( S_t \) is the stock price, \( E \) is the exercise price, \( r \) is the constant risk free rate, \( q \) is the dividend yield and \( \sigma \) is the volatility. Let \( T = t + \Delta t \) be the time at which the payoff (or option price) is known, noting that the method is recursive. Define the scaled log price as \( x = \log (S_t/E) \) and \( y = \log (S_{t+\Delta t}/E) \). The option price at time \( t \) can be written as

\[
V(x,t) = e^{-r\Delta t} E_t^Q [V(y,t+\Delta t)]
\]  

(1)
where the expectation is a risk neutral (RN) expectation and where \( V(y, t + \Delta t) \) is the time \( t + \Delta t \) known payoff (or option price) at the scaled log price \( y \). This RN expectation can be written as

\[
V(x, t) = e^{-r\Delta t} \int_{-\infty}^{\infty} V(y, t + \Delta t) f(y|x) \, dy
\]

(2)

where \( f(y|x) \) is the conditional RN probability density of attaining \( y \) given \( x \). If the stock price process is geometric Brownian motion then \( y \) is normally distributed and

\[
f(y|x) = \frac{1}{\sqrt{2\pi\sigma^2\Delta t}} \exp\left(-\frac{(y-x-(r-q-(1/2)\sigma^2)\Delta t)^2}{2\sigma^2\Delta t}\right)
\]

(3)

and

\[
V(x, t) = \frac{e^{-r\Delta t}}{\sqrt{2\pi\sigma^2\Delta t}} \int_{-\infty}^{\infty} V(y, t + \Delta t) \exp\left(-\frac{(y-x-(r-q-(1/2)\sigma^2)\Delta t)^2}{2\sigma^2\Delta t}\right) \, dy
\]

\[
= \frac{1}{\sqrt{2\pi\sigma^2\Delta t}} A(x) \int_{-\infty}^{\infty} V(y, t + \Delta t) B(x,y) \, dy
\]

\[
= \frac{1}{\sqrt{2\pi\sigma^2\Delta t}} A(x) \int_{-\infty}^{\infty} F(x,y) \, dy
\]

(4)

(5)

(6)

where

\[
A(x) = \frac{1}{\sqrt{2\pi\sigma^2\Delta t}} \exp\left(-\frac{1}{2}kx - \frac{1}{8}k^2\sigma^2\Delta t - r\Delta t\right)
\]

(7)

\[F(x,y) = V(y, t + \Delta t) B(x,y)\]

(8)

\[B(x,y) = \exp\left(-\frac{(y-x)^2}{2\sigma^2\Delta t} + \frac{1}{2}ky\right)\]

(9)

\[
k = 2\left(\frac{r-q}{\sigma^2}\right) - 1
\]

(10)

Simpson’s rule, or any other quadrature routine, is simply applied to each continuous segment of \( F(x,y) \) to compute the integral \( \int_{-\infty}^{\infty} F(x,y) \, dy \). For example, Simpson’s rule is carried out as
follows:

\[
\int_a^b g(y) \, dy \approx \frac{\delta y}{6} \left[ g(a) + 4g\left(a + \frac{1}{2}\delta y\right) + 2g(a + \delta y) + 4g\left(a + \frac{3}{2}\delta y\right) \\
+ 2g(a + 2\delta y) + \cdots + 2g(b - \delta y) + 4g\left(b - \frac{1}{2}\delta y\right) + g(b) \right]
\]

for some continuous function \(g(y)\). For a plain vanilla call option Simpson’s rule is applied for \(y > 0\) (corresponding to \(S_{t+\Delta t} > E\)) to estimate the integral \(\int_0^\infty F(x,y) \, dy\). For \(y < 0\) (corresponding to \(S_{t+\Delta t} < E\)) the integral is zero everywhere i.e. \(\int_{-\infty}^0 F(x,y) \, dy = 0\). Consider an option that has a single knock-out barrier\(^3\) at time \(t\) such that the option become worthless if the asset price drops below the barrier at this time. Suppose the time \(t + \Delta t\) option prices have been calculated recursively, Simpson’s rule is applied for \(y > b\), where \(b = \ln(B/E)\) is the scaled log barrier at time \(t + \Delta t\) and for \(y < b\) the integral is zero. See AWDN for more details on this quadrature option pricing method.

### 2.2 Quadrature transform method

Writing the option price as above results in a major part of the work being carried out analytically, however there is no need to write the option price in this manner. It can simply be written as

\[
V(x,t) = e^{-r\Delta t} \int_{-\infty}^{\infty} V(y,t + \Delta t) f(y|x) \, dy
\]  

(11)

Provided the conditional RN density function \(f(y|x)\) and the time \(t + \Delta t\) option prices are known, Simpson’s rule (or any other quadrature routine) can be used to estimate the integral in (11) assuming the integral is split up into its relevant continuous segments. However \(f(y|x)\) is known for only for a small number of plausible stock price processes, including geometric Brownian

\(^3\)This option has a “once-off” barrier that only holds at time \(t\) and is different from a standard barrier option where the barrier holds over the entire life of the option. This type of option is considered for illustration purposes only.
motion, arithmetic Brownian motion and Ornstein-Uhlenbeck processes. Define the conditional characteristic function (CCF) of a process $y$ given $x$ as

$$
ccf(u, \Delta t; \Theta, x) = E_Q^\mathcal{Q}[\exp(iuy)|x] = \int_{-\infty}^{\infty} \exp(iuy) f(y|x) dy
$$

(12)

where $i$ is the imaginary number $\sqrt{-1}$, $u$ is a transform parameter ($u$ is usually a real number but can be an imaginary number, see Lewis (2000)) and $\Theta$ is the parameter vector of the underlying process. For example, a process following geometric Brownian motion has a CCF given by

$$
ccf(u, \Delta t; \Theta, x) = \exp\left\{iux + \left(r - q - \frac{1}{2}\sigma^2\right)\Delta tu - \frac{1}{2}\sigma^2\Delta t u^2\right\}
$$

(13)

where the parameter vector is now a scalar with $\Theta = \sigma$. The conditional characteristic function is known in closed form for many processes including Heston’s (1993) stochastic volatility process, jump-diffusion stochastic volatility processes, see Bates (1996), and Lévy processes, see Lewis (2001), Geman (2002) and Carr and Wu (2004). In many of these cases the CCF is known in closed form but the probability density (PDF) function is not known in closed form and it is more convenient to work in characteristic space. The CCF is a Fourier transform of the PDF and hence there exists a one-to-one relationship between the PDF and its CCF. If the CCF is known the cumulative density function (CDF) is given by the inverse Fourier transform

$$
Pr\{v < y|x\} = \int_{-\infty}^{y} f(v|x) dv = \frac{1}{2} - \left[\frac{1}{\pi} \int_{0}^{\infty} \text{Re}\left\{\frac{\exp(-iuy)ccf(u, \Delta t; \Theta, x)}{iu}\right\} du\right]
$$

(14)

(15)
therefore to calculate the PDF from the conditional characteristic function take the derivative w.r.t. \( y \) of the CDF in characteristic space to yield

\[
f(y|x) = \frac{1}{\pi} \int_{0}^{\infty} \Re \left[ \exp(-iuy) \ccf(u, \Delta t; \Theta, x) \right] du
\]

(16)

To apply the quadrature option pricing method using transform techniques (16) is calculated at discrete points along the density function using a transform technique and these are substituted into the option pricing formula in (11) and the option price itself is calculated using a quadrature routine. Thus there are two sources of numerical error in this numerical method: the error from calculating the density function using a transform technique and the error from running the quadrature routine to calculate the option price. However, if the CCF is known in closed form, the error from approximating the density function can be controlled, see Pan (2002) and Lee (2004). Thus one can insure that the density function calculations are at a particular level of accuracy before using them as inputs into the quadrature routine. The density function can be calculated (to within a specified error tolerance) at discrete points using either of two transform techniques, a Fourier transform or a FFT. The resulting option pricing method is recursive, unlike other existing transform option pricing methods that use closed form CCFs. Hence, if required, the time dimension can be divided into a discrete number of steps and one can work backwards from maturity applying the appropriate boundary conditions at each point, rendering this method with the ability to price certain exotic path dependent options\(^4\).

\(^4\)Specifically the method can price weakly path dependent options that can be priced using recursive methods such as Bermudan options.
2.3 Greeks

The greeks can also be calculated using this method. A plain vanilla call options sensitivity to the underlying price, delta, is given as follows

$$\Delta = \frac{\partial V}{\partial S} = \frac{\partial V}{\partial x} \frac{\partial x}{\partial S} = \frac{\partial V}{\partial x} \frac{1}{S}$$

(17)

where

$$\frac{\partial V}{\partial x} = e^{-r\Delta t} \int_{-\infty}^{\infty} V(y, t + \Delta t) \frac{\partial f(y|x)}{\partial x} dy$$

(18)

and

$$\frac{\partial f(y|x)}{\partial x} = \frac{1}{\pi} \int_{0}^{\infty} \text{Re} \left[ iu \exp(-iuy) ccf(u, \Delta t; \Theta, x) \right] du$$

(19)

Thus the delta is calculated in same manner as the option price itself. Suppose $\theta_i \in \Theta$ is a parameter of the model. The options sensitivity to $\theta_i$ is given by

$$\frac{\partial V}{\partial \theta_i} = e^{-r\Delta t} \int_{-\infty}^{\infty} V(y, t + \Delta t) \frac{\partial f(y|x)}{\partial \theta_i} dy$$

(20)

where

$$\frac{\partial f(y|x)}{\partial \theta_i} = \frac{1}{\pi} \int_{0}^{\infty} \text{Re} \left[ \exp(-iuy) \frac{\partial ccf}{\partial \theta_i} \right] du$$

(21)

and the options sensitivity to $\theta_i$ is calculated in same manner as the option price itself.

3 Implementation

In this section the implementation details are outlined for setting up the lattice, and for applying the two different methods of calculating the density function using transform techniques.
3.1 Lattice method

The lattice method is implemented in a similar fashion to AWDN. First the asset price lattice is constructed. Suppose \( x_0 \) is the initial scaled log price at time \( t \). The scaled log prices at time \( t + \Delta t \) are given by the vector

\[
\mathbf{y} = \{y_0, y_1, \ldots, y_{N-1}\}'
\]

where \( \Delta y = 2q^*/N, q^* = L \times vol(\Delta t) \) and \( vol(\Delta t) \) is the volatility of the process \(^5\) over the time interval \( \Delta t \), so in the geometric Brownian motion case \( vol(\Delta t) = \sigma \sqrt{\Delta t} \). This means that the PDF is calculated \( L \) standard deviations either side of \( x_0 \) and should result in sufficient accuracy if \( L = 10 \) (this can be adjusted upwards or downwards accordingly). For \( i = 0, \ldots, N - 1 \) each discrete point of the density function \( f(y_i|x_0) \) can be calculated using a Fourier transform or the entire vector of discrete points \( f(y|x_0) \) can be calculated using one FFT. Suppose there are two time steps needed to price the option. Let \( x_0 \) be the initial scaled log price at time \( t \), \( \mathbf{y}_1 \) be the scaled log price vector at time \( t + \Delta t \) and \( \mathbf{y}_2 \) be the scaled log price vector at time \( t + 2\Delta t \). Define \( \mathbf{y}_1 \) and \( \mathbf{y}_2 \) as follows

\[
\mathbf{y}_1 = \{x_0 - q^*, x_0 - q^* + \Delta y, \ldots, x_0 + q^* - \Delta y, x_0 + q^*\}'
\]

\[
\mathbf{y}_2 = \{x_0 - 2q^*, x_0 - 2q^* + \Delta y, \ldots, x_0 + 2q^* - \Delta y, x_0 + 2q^*\}'
\]

Calculate \( f(y_1|x_0) \) using either transform technique. If the returns process is a proportional returns process, rather than calculating \( f(y_2_j|y_1_i) \) for each \( y_2_j \in \mathbf{y}_2 \) given each \( y_1_i \in \mathbf{y}_1 \), one can simply use the PDF estimated for \( \mathbf{y}_1 \) given \( x_0 \) and change the location of this PDF so that it is

\(^5\)This volatility can be computed for stochastic processes whose CCF is known with \( vol(\Delta t) = \sqrt{var(\Delta t)}, var(\Delta t) = \frac{1}{\tau} \left\{ \frac{\partial^2 cc f}{\partial u^2} - \left( \frac{\partial cc f}{\partial u} \right)^2 \right\}_{u=0} \)
centered at each $y_{1i} \in y_1$ with a range of $L$ standard deviations either side of $y_{1i}$. This will result in a reduction in computing time as the PDF only needs to be calculated once. This can only be done if the time step between successive lattice periods is equal. The method can also be easily applied to “piecewise Levy processes” where, for example, the density function has increments that are stationary on each time interval $[t, t_1]$, $[t_1, t_2]$ and $[t_2, T]$, but the density function can change from one interval to the next, thereby increasing calibration performance with respect to market option prices over both strike price and maturity. See Eberlein and Kluge (2004) for an example of a piecewise Lévy process applied to a term structure model. Such a process would provide a better fit to implied volatility surfaces than a single Lévy process and overcome the difficulties that are encountered when using Lévy processes to price options across different maturities. Suppose there is a discontinuity in the option price, such as a barrier, along the transformed price vector $y$ and denote this discontinuity as $b$. If $b = y_i$ for some $i = 0, \ldots, N - 1$ the integration is split in two by running it over the two ranges $[x_0 - q^*, b]$ and $[b, x_0 + q^*]$. If $b \neq y_i$ but $y_{i-1} < b < y_i$ for some $i = 1, \ldots, N - 1$, then the integration must be handled with care because the integration over the range $[x_0 - q^*, b]$ now includes a step (that from $y_{i-1}$ to $b$) that is smaller in length than $\Delta y$, the same holds for the first step in the integration over the range $[b, x_0 + q^*]$. In the results presented in this paper for Bermudan options the exercise boundary $b$ usually falls between two grid points i.e. $y_{i-1} < b < y_i$. The approach taken in the paper is to evaluate the integral from $y_0$ up to point $y_{i-1}$ using the exercise value and evaluate the integral from $y_i$ to $y_{N-1}$ using the continuation value leaving the interval from $y_{i-1} < b < y_i$ out altogether. This causes a downward correction to the upward bias caused by the integration scheme around the kink at the early exercise boundary and experimental work seems to suggest it works extremely well.

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6 Thanks to Roger Lord for pointing this out.
7 Initially in this paper and then conducted by others using more advanced methods, for example see Fang, Lord and Oosterlee (2006)
3.2 Fourier transform method

This section covers the details required to calculate the density function at discrete points of \( y \) using a Fourier transform. To do this \( y \) is discretised into \( N \) points with \( y = \{y_0, y_1, \ldots, y_{N-1}\}' \). Then we approximate \( f(y_i | x) \) using a Fourier transform for \( i = 0, \ldots, N - 1 \). The PDF in 16 can be discretised as follows

\[
f(y_i | x) = \frac{1}{\pi} \int_{0}^{\infty} \text{Re} \left[ \exp \left( -iuy_i \right) ccf \left( u, \Delta t; \Theta, x \right) \right] du \tag{26}
\]

\[
\simeq \frac{1}{\pi} \sum_{j=0}^{T-1} \text{Re} \left[ \exp \left( -iu_j y_i \right) ccf \left( u_j, \Delta t; \Theta, x \right) \right] \Delta u \tag{27}
\]

where the transform parameter \( u \) is discretised into a vector with \( u = \{u_0, u_1, \ldots, u_{T-1}\}' \), \( T \) is the truncation point and \( \Delta u \) is the discretisation length. The integral can be approximated as shown in (27) using the trapezium rule or any quadrature routine such as Simpson’s rule can be used to compute the integral. There will be a truncation and discretisation error in the approximation of the integral but these can be controlled given the closed form nature of the CCF, see Pan (2002). The option pricing method is denoted Q-FT when the density function is calculated in this manner.

3.3 Fast Fourier transform method

This section covers the details required to calculate the vector whose entries are discrete points on the density function, \( f(y | x) \), with one application of a fast Fourier transform (FFT). The FFT improves computational speed and error control relative to the FT methods, see Carr and Madan (1999) and Lee (2004), thus it is logical to want to use its considerable power in this setting. An FFT can be applied to compute the density function in (16) and furthermore the density function is an integrable function so there is no need to damp the function in this method as required in
previous FFT option pricing methods. To run the method proceed as follows:

\[
f(y|x) = \frac{1}{\pi} \int_0^\infty \text{Re} \left[ \exp(-iuy) cc f(u, \Delta t; \Theta, x) \right] du
\]

(28)

\[
= \frac{1}{\pi} \text{Re} \left[ \int_0^\infty \exp(-iuy) cc f(u, \Delta t; \Theta, x) du \right]
\]

(29)

An FFT of a vector \( v \) of dimension \( N \times 1 \) computes the following sum

\[
\text{FFT}(v(p)) = \sum_{n=0}^{N-1} \exp \left( -i \frac{2\pi}{N} np \right) v(n) \text{ for } p = 0, \ldots, N-1
\]

(30)

To apply this algorithm in the current setting divide \( \phi \) and \( y \) into \( N \) points with

\[
\phi_n = (n - N/2) \Delta \phi, \ y_p = (p - N/2) \Delta y
\]

(31)

for \( n, p = 0, 1, \ldots, N-1 \). Write the density function as

\[
f(y|x) \approx \frac{1}{\pi} \text{Re} \left[ \sum_{n=0}^{N-1} \exp(-i\phi_n y_p) cc f(\phi_n, \Delta t; \Theta, x) \Delta \phi \right]
\]

\[
= \frac{1}{\pi} (-1)^p \text{Re} \left[ \sum_{n=0}^{N-1} \exp(-i\Delta \phi \Delta y np) (-1)^n cc f(\phi_n, \Delta t; \Theta, x) \Delta \phi \right]
\]

(32)

Provided \( \Delta \phi \) and \( \Delta y \) are restricted so that \( \Delta \phi \times \Delta y = 2\pi/N \) the FFT can be applied with

\[
f(y|x) = \frac{1}{\pi} (-1)^p \text{Re} [\text{FFT}(v(p))]
\]

(33)

where

\[
v(n) = (-1)^n cc f(\phi_n, \Delta t; \Theta, x) \Delta \phi
\]

(34)
and it can be applied with considerable computational speed if $N$ is a power of 2 because in this case the algorithm is of computational order $O(N \log_2 N)$. A trade-off exists here that is also present in other FFT option pricing methods. As $\Delta y$ becomes smaller $\Delta \phi$ must necessarily becomes larger and vice versa. So if the transformed stock price grid becomes finer the grid for computing the density function becomes coarser. After experimenting with different values of $N$ and $L$ it was found that $N = 512$ or $1024$ and $L = 10$ provided a good compromise between speed and accuracy. The option pricing method is denoted Q-FFT when the density function is calculated in this manner.

4 Numerical results

Q-FFT is superior in accuracy and speed relative to Q-FT so in this section we only consider Q-FFT as it is by far the dominant method. First Q-FFT is used to price European and American options under GBM and prices are compared to those from existing methods. Then Q-FFT is used to price European and Bermudan options under a JD process, a VG process and an NIG process and in each case the prices are benchmarked against existing pricing methods.

4.1 Geometric Brownian motion

Consider a call option when the underlying follows geometric Brownian motion. Let BS represent Black-Scholes prices and Q-FFT represent Q-FFT prices where the conditional density function is calculated using a FFT. The true American price is calculated with a binomial tree with 10,000 steps. Q-FFT R4, R10 and R20 are American options priced with a four point Richardson extrapolation scheme, and priced as $2P_{10} - P_5$ and $2P_{20} - P_{10}$ respectively, where $P_n$ is an $n$-times exercisable option. Numerical results are reported in Table 1 and Table 2.

The root mean square error (RMSE) is calculated w.r.t. the Black-Scholes (BS) price for Eu-
European options\textsuperscript{8}. The RMSE is calculated w.r.t. a binomial tree with 10,000 steps for American options. Q-FFT has an RMSE of 0.0004 when used to price European options and is extremely accurate. The RMSE is 0.0122 when American options are priced with the four point scheme R4 and is 0.0062 and 0.0046 when American options are priced with R10 and R20 respectively. The results for American options were compared with the results of Table 1 in Ju (1998). The RMSE’s from the American option pricing methods in Ju and this paper are recorded in Table 2. In this table BT800 is a binomial tree with 800 steps, GJ4 is the Geske-Johnson method (1984), MGJ2 is the Bunch and Johnson method (1992), HYS4 and HYS6 are four and six-point methods of Huang, Subrahmanyam, and Yu (1996), LUBA is the lower and upper bound method of Broadie and Detemple (1996), RAN4 and RAN6 are the four and six-point randomization methods of Carr (1998), and EXP3 is the three-point method of Ju (1998). The method performs well in comparison to other American option pricing models. Prices from Q-FFT R20 are not as accurate as BT800, LUBA, RAN6 or EXP3 however the method performs better than the other benchmark models. This performance does not take into account the computational time required to price the options using Q-FFT. The time required increases with the higher order approximations and although it only takes a few seconds to price each option, this is vastly slower than some of the competing methods such as RAN6 or EXP3. However the real gain from using Q-FFT comes from its ability to price exotic options under general exponential Lévy processes in a relatively simple manner. This is why we examine the methods ability to handle more complex stochastic processes in the next sections.

\textsuperscript{8}For European options the RMSE is calculated as:

\[ RMSE = \sqrt{\frac{\sum_{i=1}^{n} (V_{BSi} - V_{Q-FFTj})^2}{n}} \]
Table 1: Prices of European and American call options

\((E = \$100, t = 0, \Delta t = 0.5\) years, \(L = 10\) and \(N = 512\))

<table>
<thead>
<tr>
<th>((S, \sigma, r, q))</th>
<th>European prices</th>
<th>American prices</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>BS</td>
<td>Q-FFT</td>
</tr>
<tr>
<td>((80, 0.2, 0.03, 0.07))</td>
<td>0.2148</td>
<td>0.2149</td>
</tr>
<tr>
<td>((90, 0.2, 0.03, 0.07))</td>
<td>1.3451</td>
<td>1.3444</td>
</tr>
<tr>
<td>((100, 0.2, 0.03, 0.07))</td>
<td>4.5778</td>
<td>4.5778</td>
</tr>
<tr>
<td>((110, 0.2, 0.03, 0.07))</td>
<td>10.4208</td>
<td>10.4205</td>
</tr>
<tr>
<td>((120, 0.2, 0.03, 0.07))</td>
<td>18.3024</td>
<td>18.3017</td>
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<tr>
<td>((80, 0.4, 0.03, 0.07))</td>
<td>2.6506</td>
<td>2.6511</td>
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<tr>
<td>((90, 0.4, 0.03, 0.07))</td>
<td>5.6221</td>
<td>5.6228</td>
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<tr>
<td>((100, 0.4, 0.03, 0.07))</td>
<td>10.0211</td>
<td>10.0211</td>
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<tr>
<td>((110, 0.4, 0.03, 0.07))</td>
<td>15.7676</td>
<td>15.7678</td>
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<tr>
<td>((120, 0.4, 0.03, 0.07))</td>
<td>22.6502</td>
<td>22.6509</td>
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<tr>
<td>((80, 0.3, 0.00, 0.07))</td>
<td>1.0064</td>
<td>1.0058</td>
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<tr>
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<td>3.0040</td>
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<td>6.6943</td>
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<td>12.1666</td>
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<td>((120, 0.3, 0.00, 0.07))</td>
<td>19.1555</td>
<td>19.1555</td>
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<td>((80, 0.3, 0.07, 0.03))</td>
<td>1.6644</td>
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<td>4.4945</td>
</tr>
<tr>
<td>((100, 0.3, 0.07, 0.03))</td>
<td>9.2506</td>
<td>9.2506</td>
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<td>15.7980</td>
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<tr>
<td>((120, 0.3, 0.07, 0.03))</td>
<td>23.7062</td>
<td>23.7062</td>
</tr>
</tbody>
</table>

| RMSE   | 0.0004 | 0.0122 | 0.0062 | 0.0046 |
Table 1: Columns 2 - 6 represent Black-Scholes European prices, Q-FFT European prices, binomial tree American prices with 10,000 time steps, Q-FFT American prices using 4 point Richardson extrapolation and extrapolation schemes given by $2P_{10}^5-P_5$ and $2P_{20}^{20}-P_{10}$.

Table 2: Comparison of accuracy of various numerical methods

<table>
<thead>
<tr>
<th>Method</th>
<th>RMSE</th>
<th>Method</th>
<th>RMSE</th>
</tr>
</thead>
<tbody>
<tr>
<td>BT800</td>
<td>0.0012</td>
<td>RAN4</td>
<td>0.0104</td>
</tr>
<tr>
<td>GJ4</td>
<td>0.0090</td>
<td>RAN6</td>
<td>0.0035</td>
</tr>
<tr>
<td>MGJ2</td>
<td>0.0805</td>
<td>EXP3</td>
<td>0.0013</td>
</tr>
<tr>
<td>HSY4</td>
<td>0.0231</td>
<td>Q-FFT R4</td>
<td>0.0122</td>
</tr>
<tr>
<td>HSY6</td>
<td>0.0089</td>
<td>Q-FFT R10</td>
<td>0.0062</td>
</tr>
<tr>
<td>LUBA</td>
<td>0.0012</td>
<td>Q-FFT R20</td>
<td>0.0046</td>
</tr>
</tbody>
</table>

Table 2: RMSE from different American option pricing methods is calculated w.r.t. a binomial tree with 10,000 steps.

4.2 Finite activity Lévy processes

Finite activity Lévy processes are commonly referred to as jump-diffusion processes and have been very popular in option pricing literature because markets frequently undergo discontinuous jumps and because jumps can explain the implied volatility smile across strike prices especially well at short time horizons. For more on jump-diffusion processes see, for example, Bates (1996) and Duffie, Pan and Singleton (2000).

Suppose the risk neutral dynamics of the asset price are given by the following process:

$$dS_t/S_t = (r - q - \lambda \mu) dt + \sigma dW_t + (e^J - 1) dN_t(\lambda) \quad (35)$$

where $dW_t$ is a Brownian motion under the risk neutral volatility measure, $N_t$ is a pure jump Poisson process with instantaneous intensity $\lambda$ and $\mu$ is the risk neutral mean relative jump size with $\mu = E_t^Q [e^J - 1]$. $J$ is the random percentage jump size with distribution $\nu$ conditional on a jump occurring. For tractability
purposes the risk neutral distribution of the jump size $J$ is taken to be normal with $\nu(J) \sim N(\mu_J, \sigma_J)$, hence $\mu = E_t^Q[e^{J} - 1] = e^{\mu_J + \frac{1}{2}\sigma_J^2} - 1$. The term $\lambda \mu$ appears in the risk neutral returns process to compensate for the discontinuous jump term and ensures that the discounted stock price process is a martingale. This model was originally proposed by Merton (1976), however in this setting we can use any jump size distribution that has a tractable Fourier transform such as lognormal and exponential jumps. Letting $x_t = \log(S_t/E)$, this model has a closed form CCF given by

$$\text{ccf}(\phi, \Delta t; \Theta, x_t) = \exp \left[ i\phi x_t + i\phi \left( r - q - \frac{1}{2} \sigma^2 + w \right) \Delta t - \Psi(\phi, \Delta t; \Theta) \Delta t \right]$$

(36)

where

$$\Psi(\phi, \Delta t; \Theta) = \frac{1}{2} \sigma^2 \phi^2 - \lambda \left( e^{i\phi \mu_J - \frac{1}{2} \phi^2 \sigma_J^2} - 1 \right)$$

(37)

and

$$w = -\lambda \mu = -\lambda \left( e^{\mu_J + \frac{1}{2}\sigma_J^2} - 1 \right)$$

(38)

$\Psi$ is the characteristic exponent and $w$ is risk neutral compensator\(^9\) that ensures that the discounted stock price follows a martingale. Note that the parameter vector is given by $\Theta = \{\sigma, \lambda, \mu_J, \sigma_J\}'$ and that the CCF is conditioned only on the observable transformed stock price $x_t$.

Let us consider a put option in the jump-diffusion case. Let FT represent European put prices computed using standard Fourier transform techniques, pioneered by Heston (1993) and extended to the case of jump-diffusions by Bates (1996). Let FD represent European put prices computed using a mixed implicit-explicit finite difference method based on Das (1999) where the number of time steps $N_t = 1000$ and the number of asset price steps $N_s = 300$. Let Q-FFT represent European put prices computed using Q-FFT. The benchmark Bermudan put prices are calculated for ten evenly spaced exercise dates using the finite difference method based on Das (1999). Q-FFT10 are Bermudan put prices (with ten exercise dates)

\(^9\)To solve for $w$ set

$$\text{ccf}(\phi = -i) = \exp(x_t + (r - q) \Delta t) \text{ i.e. } E_t^Q[S_{t+\Delta t}] = S_t \exp((r - q) \Delta t)$$

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computed using Q-FFT. Numerical results are reported in Table 3.

The RMSE are calculated w.r.t. the standard FT method for European options and w.r.t. the finite difference method for Bermudan options. The RMSE of Q-FFT is 0.0001 which is extremely accurate, while the RMSE of FD is 0.0039 which is reasonably accurate. The RMSE of Q-FFT10 is 0.0069 and one can see that the difference between the finite difference Bermudan prices and the Q-FFT Bermudan prices is still very small and certainly lies within a reasonable accuracy range.
<table>
<thead>
<tr>
<th>$(S, \sigma_J, \mu_J)$</th>
<th>European</th>
<th>Bermudan</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>HES</td>
<td>FD</td>
</tr>
<tr>
<td>$(80, 0.02, 0.0)$</td>
<td>16.1006</td>
<td>16.1023</td>
</tr>
<tr>
<td>$(90, 0.02, 0.0)$</td>
<td>6.8791</td>
<td>6.8836</td>
</tr>
<tr>
<td>$(100, 0.02, 0.0)$</td>
<td>1.4603</td>
<td>1.4593</td>
</tr>
<tr>
<td>$(110, 0.02, 0.0)$</td>
<td>0.1314</td>
<td>0.1314</td>
</tr>
<tr>
<td>$(120, 0.02, 0.0)$</td>
<td>0.0055</td>
<td>0.0056</td>
</tr>
<tr>
<td>$(80, 0.02, -0.02)$</td>
<td>16.1053</td>
<td>16.1085</td>
</tr>
<tr>
<td>$(90, 0.02, -0.02)$</td>
<td>6.9964</td>
<td>7.0081</td>
</tr>
<tr>
<td>$(100, 0.02, -0.02)$</td>
<td>1.6937</td>
<td>1.6998</td>
</tr>
<tr>
<td>$(110, 0.02, -0.02)$</td>
<td>0.2256</td>
<td>0.2276</td>
</tr>
<tr>
<td>$(120, 0.02, -0.02)$</td>
<td>0.0194</td>
<td>0.0198</td>
</tr>
<tr>
<td>$(80, 0.02, 0.02)$</td>
<td>16.1336</td>
<td>16.1335</td>
</tr>
<tr>
<td>$(90, 0.02, 0.02)$</td>
<td>7.0891</td>
<td>7.0839</td>
</tr>
<tr>
<td>$(100, 0.02, 0.02)$</td>
<td>1.6477</td>
<td>1.6408</td>
</tr>
<tr>
<td>$(110, 0.02, 0.02)$</td>
<td>0.1638</td>
<td>0.1633</td>
</tr>
<tr>
<td>$(120, 0.02, 0.02)$</td>
<td>0.0068</td>
<td>0.0070</td>
</tr>
<tr>
<td>$(80, 0.04, 0.0)$</td>
<td>16.1787</td>
<td>16.1791</td>
</tr>
<tr>
<td>$(90, 0.04, 0.0)$</td>
<td>7.3525</td>
<td>7.3521</td>
</tr>
<tr>
<td>$(100, 0.04, 0.0)$</td>
<td>2.0443</td>
<td>2.0457</td>
</tr>
<tr>
<td>$(110, 0.04, 0.0)$</td>
<td>0.3475</td>
<td>0.3506</td>
</tr>
<tr>
<td>$(120, 0.04, 0.0)$</td>
<td>0.0427</td>
<td>0.0439</td>
</tr>
<tr>
<td><strong>RMSE</strong></td>
<td>0.0039</td>
<td>0.0001</td>
</tr>
</tbody>
</table>
Table 3: Columns 2 - 6 represent FT, FD and Q-FFT European prices, and FD and Q-FFT Bermudan prices.

4.3 Infinite activity Lévy processes

Infinite activity Lévy processes are those Lévy processes with an infinite jump arrival rate, that is an infinite number of jumps can occur in a finite time horizon. They have become increasingly popular in finance because of their ability to fit asset return processes and option prices across strike prices in a parsimonious manner. The fact that on very small time scales market dynamics undergo a large number of very small discrete jumps is another theoretically appealing property of infinite activity Lévy processes.

4.3.1 Variance gamma process

The variance gamma process for asset returns was first proposed by Madan and Seneta (1990), and extended by Madan and Milne (1991) and Madan, Carr and Chang (1998). The process is an arithmetic Brownian motion with drift evaluated at a random time change that follows a gamma process thus the VG process is a pure jump process. European options under VG can be priced in terms of special functions. Methods for pricing path dependent options under VG processes include the finite difference method of Hirsa and Madan (2004), the lattice methods of Kellizi and Webber (2003) and the multinomial tree method Maller, Solomon and Szimayer (2004).

Denote an arithmetic Brownian motion with drift $\theta$ and volatility $\sigma$ as follows:

$$b(t; \theta, \sigma) = \theta t + \sigma W(t)$$

(39)

Then a VG process has a log stock price process given by

$$X(t; \sigma, \nu, \theta) = b(T^\nu; \theta, \sigma)$$

(40)

where $T^\nu = \gamma(t; 1, \nu)$ is a gamma process with unit mean and variance $\nu$. Letting $x_t = \ln(S_t/E)$ this model
has a simple closed form CCF given by

\[
ccf(\phi, \Delta t; \Theta, x_t) = \exp \left[ i\phi x_t + i\phi (r - q + w) \Delta t - \Psi(\phi, \Delta t; \Theta) \Delta t \right]
\]  

(41)

where

\[
\Psi(\phi, \Delta t; \Theta) = \frac{1}{v} \ln \left[ 1 - i\theta v \phi + \frac{1}{2} \sigma^2 v \phi^2 \right]
\]  

(42)

and

\[
w = \frac{1}{v} \ln \left[ 1 - \theta v - \frac{1}{2} \sigma^2 v \right]
\]  

(43)

As before \( \Psi \) is the characteristic exponent and \( w \) is the risk neutral compensator. The parameter vector is given by \( \Theta = \{\sigma, \theta, v\} \)' and the CCF is conditioned only on the observable transformed stock price \( x_t \).

Option prices from Q-FFT are compared to those from Kellezi and Webber (2003), denoted as KW, to benchmark Q-FFT with results available in the literature. Table 4 contains results for European call options and Bermudan put options with 10 evenly spaced exercise dates. The parameter vector is given by \( \Theta = \{0.12, -0.14, 0.2\} \)', the current stock price is \( S_t = 100 \) and the exercise price takes on values \( E = \{90, 95, \ldots, 110\} \).'
Table 4: European and Bermudan options under VG

\( (t = 0, \Delta t = 1 \text{ year}, r = 0.10, q = 0, L = 10 \text{ and } N = 1024) \)

<table>
<thead>
<tr>
<th>( E )</th>
<th>European call prices</th>
<th>European and Bermudan put prices</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Analytical</td>
<td>KW</td>
</tr>
<tr>
<td>90</td>
<td>19.09935</td>
<td>19.09936</td>
</tr>
<tr>
<td>95</td>
<td>15.07047</td>
<td>15.07048</td>
</tr>
<tr>
<td>100</td>
<td>11.37002</td>
<td>11.37002</td>
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<tr>
<td>105</td>
<td>8.11978</td>
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<td>110</td>
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<td>5.42960</td>
</tr>
<tr>
<td>120</td>
<td>1.92110</td>
<td>1.92110</td>
</tr>
</tbody>
</table>

RMSE \( 6.5 \times 10^{-6} \) \( 1.5 \times 10^{-4} \) \( 0.01603 \)

Table 4: Columns 2 - 7 represent analytical, KW and Q-FFT European call prices, Q-FFT European put prices, KW and Q-FFT Bermudan put prices.

The RMSE of KW and Q-FFT are calculated w.r.t. the analytical prices for the European call prices. Both methods are very accurate with KW more accurate than Q-FFT. The RMSE of Q-FFT for the Bermudan put prices is calculated w.r.t. the KW prices and is reasonably accurate. The Bermudan prices from both methods are relatively close as can be seen from the table.

4.3.2 Normal inverse Gaussian process

Generalized hyperbolic distributions contain as a subset the normal inverse Gaussian (NIG) distribution which has been used as a model of asset returns by Eberlein and Keller (1995) and Barndorff-Nielsen (1998) among others. European options can be priced using numerical integration or FFTs. Methods for pricing path dependent options under NIG processes include the lattice methods of Kellizi and Webber (2003) and the multinomial tree method Maller, Solomon and Szimayer (2004).
Let $T_v$ be the first time that a Brownian motion with drift $v$ reaches the positive level $t$. The density of $T_v$ is inverse Gaussian. An NIG process is an arithmetic Brownian motion with drift $\theta$ and volatility $\sigma$ evaluated at this Gaussian time and thus, like the VG process, the NIG process is a pure jump process. Denoting an arithmetic Brownian motion with drift as in Eq. 39 an NIG process has a log stock price process given by

$$X(t; \sigma, v, \theta) = b(T_v; \theta, \sigma)$$

(44)

Letting $x_t = \ln(S_t/E)$ this model has a closed form CCF given by

$$ccf(\phi, \Delta t; \Theta, x_t) = \exp \left[ i\phi x_t + i\phi (r - q + w) \Delta t - \Psi(\phi, \Delta t; \Theta) \Delta t \right]$$

(45)

where

$$\Psi(\phi, \Delta t; \Theta) = \delta \left( \sqrt{\alpha^2 - (\beta + i\phi)^2} - \sqrt{\alpha^2 - \beta^2} \right)$$

(46)

and

$$w = \delta \left( \sqrt{\alpha^2 - (\beta + 1)^2} - \sqrt{\alpha^2 - \beta^2} \right)$$

(47)

As before $\Psi$ is the characteristic exponent and $w$ is the risk neutral compensator. The parameter vector is written as $\Theta = \{\alpha, \beta, \delta\}'$ to conform with conventional notation, where the relationship with the parameters $\{\sigma, v, \theta\}'$ is given as follows

$$\alpha^2 = \frac{v^2}{\sigma^2} + \frac{\theta^2}{\sigma^4}, \quad \beta = \frac{\theta}{\sigma^2}, \quad \delta = \sigma$$

(48)

The CCF is conditioned only on the observable transformed stock price $x_t$.

To benchmark results option prices from Q-FFT are compared to those from Kellezi and Webber (2003). Table 5 contains results for European call options and Bermudan put options with 10 evenly spaced exercise dates. The parameter vector is given by $\Theta = \{28.42141, -15.08623, 0.31694\}'$, the current stock price is $S_t = 100$ and the exercise price takes on values $E = \{90, 95, \ldots, 110\}'$.

As with the VG case the European options from Q-FFT are extremely accurate and the Bermudan prices are very close to the prices from Kellezi and Webber.

---

10 These parameters were chosen by Kellezi and Webber so that the density function of the NIG process has the same first four moments as the density function from the VG process.
Table 5: European and Bermudan options under NIG

\[(t = 0, \Delta t = 1 \text{ year}, r = 0.10, q = 0, L = 10 \text{ and } N = 1024)\]

<table>
<thead>
<tr>
<th>E</th>
<th>Reference</th>
<th>KW</th>
<th>Q-FFT</th>
<th>European call prices</th>
<th>European and Bermudan put prices</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>Q-FFT (Euro)</td>
</tr>
<tr>
<td>90</td>
<td>19.09330</td>
<td>19.09330</td>
<td>19.09321</td>
<td>0.52858</td>
<td>0.74482</td>
</tr>
<tr>
<td>95</td>
<td>15.06077</td>
<td>15.06077</td>
<td>15.06061</td>
<td>1.02017</td>
<td>1.49554</td>
</tr>
<tr>
<td>100</td>
<td>11.35994</td>
<td>11.35993</td>
<td>11.35969</td>
<td>1.84343</td>
<td>2.84445</td>
</tr>
<tr>
<td>105</td>
<td>8.11561</td>
<td>8.11561</td>
<td>8.11537</td>
<td>3.12330</td>
<td>5.17297</td>
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<tr>
<td>110</td>
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<td>5.43724</td>
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</tr>
<tr>
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<td>1.94359</td>
<td>1.94296</td>
<td>10.52345</td>
<td>18.80693</td>
</tr>
</tbody>
</table>

RMSE

Table 5: Columns 2 - 7 represent analytical, KW and Q-FFT European call prices, Q-FFT European put prices, KW and Q-FFT Bermudan put prices

5 Conclusion

A numerical method is developed that can be used to price weakly path dependent exotic options on Lévy processes with closed form conditional characteristic functions. The numerical method combines two existing techniques: the Fourier transform and quadrature option pricing methods, retaining nice features of both. The numerical method is very general in terms of the underlying processes, with the only constraint being that the log stock price process has to have a closed form conditional characteristic function. The method can easily handle infinite activity Lévy processes since a characteristic function based approach is used thus avoiding any problems other lattice based approaches encounter when using the Lévy measure in a neighbourhood of zero. The method is not specific to the stochastic process under consideration and to move from one stochastic process to another one simply needs to replace the characteristic function.
There are many possible avenues of future research. The accuracy and speed of the method can be improved by using more refined quadrature routines and by constructing trees in a more efficient manner. The method can also be extended to incorporate “piecewise Lévy processes” where, for example, the density function has increments that are stationary on each time interval \([t, t_1], [t_1, t_2]\) and \([t_2, t + \Delta t]\), but the density function can change from one interval to the next, thereby increasing calibration performance with respect to option prices over both strike price and maturity. However such a method is a rather ad-hoc solution to the calibration problem. Another possibly more theoretically appealing solution to the calibration problem is the application of the method to Lévy processes that incorporate stochastic volatility as a continuous time stochastic process or in a regime switching model. In this way the underlying process would capture the implied volatility smile across strike prices and the term structure of implied volatility with a combination of the stochastic volatility and Lévy processes.

References


