Artificial Regression Based Mis-Specification Tests for Discrete Choice Models

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ARTIFICIAL REGRESSION BASED MIS-SPECIFICATION TESTS
FOR DISCRETE CHOICE MODELS

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Abstract: LM tests for omitted variables, neglected heteroscedasticity and other mis-specifications in general discrete choice models may be simply and conveniently calculated using an artificial regression. This artificial regression approach is likely to have better small sample properties than the more common outer product gradient (OPG) form of LM test.

Keywords: Discrete Choice, LM Mis-Specification Tests, Artificial Regressions

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Introduction

Davidson and MacKinnon (1984b) derive convenient Lagrange Multiplier (LM) tests for omitted variables and neglected heteroscedasticity in logit and probit models1. Their LM tests are convenient since they are based on artificial regressions. They are also likely to have good small sample properties since they do not use the outer product gradient (OPG) form of the LM test2. Instead the information matrix is calculated as the expectation of the outer product of the score and is not just approximated by the outer product of the score. Murphy (1994) shows that the Davidson and MacKinnon approach may be used to derive many other mis-specification tests in both binary choice models (eg. tests of normality in probit models and asymmetry in logit models) and some more general discrete choice models (eg. normality in censored bivariate probit models).

In this paper this approach is further extended and artificial regression based LM

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2 Engle (1984) also suggests using this approach.

tests for general discrete choice models are derived. The tests are likely to have
good small sample properties since the OPG form of the LM test is not used. The
proposed LM tests may be used to detect a range of mis-specifications such as
omitted variables, neglected heterogeneity, incorrect functional form and non-
normality/asymmetry in ordered probit/logit models.

General Discrete Choice Model

Consider a discrete choice model with a random sample of N individuals, denoted
by subscript i, and J + 1 alternative numbered from 0 to J. Let \( y_{ij} \) be an indicator
variable for individual i and alternative j. Thus, \( y_{ij} \) equals one if individual i selects
alternative j; otherwise \( y_{ij} \) equals zero. Let \( p_{ij} \) be the probability that i selects
alternative j. \( p_{ij} \) depends on the parameter vector \( \theta \). The true parameter is
assumed to be in the interior of the parameter space. For any individual both the
sum of the \( y_{ij} \)'s and the \( p_{ij} \)'s across the \( J + 1 \) alternatives equal one.

With a random sample the log likelihood is:

\[
I = \sum_{i=1}^{N} \sum_{j=0}^{J} y_{ij} \ln p_{ij}
\]

(1)

and the score equals:

\[
\frac{\delta I}{\delta \theta} = \sum_{i} \sum_{j} \frac{y_{ij}}{p_{ij}} \frac{\delta p_{ij}}{\delta \theta}
\]

(2)

which may be recast as:

\[
\frac{\delta I}{\delta \theta} = \sum_{i} \sum_{j} \left( \frac{y_{ij} - p_{ij}}{\sqrt{p_{ij}}} \right) \left( \frac{1}{\sqrt{p_{ij}}} \frac{\delta p_{ij}}{\delta \theta} \right)
\]

(3)

\[
= \sum_{i} \sum_{j} u_{ij} z_{ij}
\]

since \( \sum_{j} \frac{\delta p_{ij}}{\delta \theta} = 0 \). The \( u_{ij} \)'s may be thought of as standardised residuals. They
have zero means, variances equal to \( 1 - p_{ij} \) and covariances equal to \(- (p_{ij} p_{jk})^{1/2} \) when \( j \neq k \).

The information matrix is:

\[
I_{\theta \theta} = \lim_{N \to \infty} E \left\{ \frac{1}{N} \frac{\delta I}{\delta \theta} \frac{\delta I}{\delta \theta'} \right\}
\]

(4)

\[
= \lim_{N \to \infty} E \left\{ \frac{1}{N} \sum_{i} \sum_{j} \frac{1}{p_{ij}} \left( \frac{\delta p_{ij}}{\delta \theta} \right) \left( \frac{\delta p_{ij}}{\delta \theta'} \right) \right\}
\]

\[
= \lim_{N \to \infty} E \left\{ \frac{1}{N} \sum_{i} \sum_{j} u_{ij} z_{ij}' \right\}
\]
since the sample is random, $EY_{ij}^2 = EY_{ij} = p_{ij}$ and $EY_{ij}Y_{ik} = 0$ when $j \neq k$. The information matrix is assumed to be non-singular in the neighbourhood of the true parameter value.

**LM Test Statistic**

The LM test statistic of the null $\theta = \theta_0$ is:

$$LM = \frac{1}{N} \frac{\delta l}{\delta \hat{\theta}} \Gamma^{\theta_0^{-1}}_{\theta} \frac{\delta l}{\delta \hat{\theta}}$$  \hspace{1cm} (5)

where both the score and the observed information matrix $\Gamma^{\theta_0}_{\theta}$ are evaluated using the restricted parameter estimates $\hat{\theta}$. The observed information matrix is:

$$\Gamma^{\theta_0}_{\theta} = E \frac{1}{N} \sum_i \sum_j \frac{\delta l}{\delta \hat{\theta}} \frac{\delta l}{\delta \hat{\theta}'}$$  \hspace{1cm} (6)

$$= E \frac{1}{N} \sum_i \sum_j \frac{1}{\hat{p}_{ij}} \frac{\delta p_{ij}}{\delta \hat{\theta}} \frac{\delta p_{ij}}{\delta \hat{\theta}'}$$

Under standard weak regularity conditions, the LM test statistic has a chi-squared distribution with degrees of freedom equal to the number of restrictions under the null.

**Artificial Regression Based LM Test Statistic**

The LM test statistic may be calculated as:

$$LM = (\sum_i \sum_j \tilde{u}_{ij} \tilde{z}_{ij})' (\sum_i \sum_j \tilde{z}_{ij} \tilde{z}_{ij})^{-1} (\sum_i \sum_j \tilde{u}_{ij} \tilde{z}_{ij})$$  \hspace{1cm} (7)

where $\tilde{u}_{ij}$ and $\tilde{z}_{ij}$ are the estimates of $u_{ij}$ and $z_{ij}$ at the restricted parameter estimates:

$$\tilde{u}_{ij} = \frac{y_{ij} - \hat{p}_{ij}}{\sqrt{\hat{p}_{ij}}}$$

$$\tilde{z}_{ij} = \frac{1}{\sqrt{\hat{p}_{ij}}} \frac{\delta p_{ij}}{\delta \hat{\theta}_{ij}}$$

In (7) the LM test statistic is simply the explained sum of squares from the
unccentred auxiliary regression of the \( \hat{u}_{ij} \)'s on the \( \hat{z}_{ij} \)'s across all \( J + 1 \) alternatives and \( N \) individuals.

The LM test statistic is also asymptotically equal to \( NJ \) times the \( R^2 \) from this auxiliary regression. The proof is the same as in McFadden (1987). Note that:

\[
NJ R^2 = \frac{LM}{\frac{1}{NJ} \sum_{i} \sum_{j} \hat{u}_{ij}}
\]

and:

\[
\frac{1}{NJ} \sum_{i} \sum_{j} \hat{u}_{ij} \hat{u}_{ij} = \frac{1}{NJ} \sum_{i} \sum_{j} (\hat{u}_{ij}^2 - u_{ij}^2) + \frac{1}{NJ} \sum_{i} \sum_{j} u_{ij}^2
\]

Use the mean value theorem to expand the first term as the product of a stochastically bounded expression and \( \hat{\theta} - \theta \) which has a plim of zero. Thus the first term has a plim of zero. Finally, note that the expectation of the second term is one, since \( E u_{ij}^2 = 1 - p_{ij} \) and \( \sum_i \sum_j E u_{ij}^2/NJ = 1 \), and the variance tends to zero as \( N \) tends to infinity. Thus the plim of \( NJ \) times the \( R^2 \) from the auxiliary regression equals the LM test statistic.

**Binary Choice Model**

In the special case of two alternatives 0 and 1, the log likelihood equals:

\[
\ell = \sum_{i=1}^{N} ((1-y_i)\ln(1-p_i) + y_i\ln p_i)
\]

(1')

and the score equals:

\[
\frac{\delta \ell}{\delta \theta} = \sum_{i} \left[ \frac{y_i - p_i}{1-p_i} + \frac{y_i}{p_i} \right] \frac{\delta p_i}{\delta \theta}
\]

\[
= \sum_{i} \left( \frac{y_i - p_i}{\sqrt{p_i (1-p_i)}} \right) \left( \frac{1}{\sqrt{p_i (1-p_i)}} \right) \frac{\delta p_i}{\delta \theta}
\]

\[
= \sum_{i} r_i x_i
\]

(3')

where \( y_i \) is an indicator variable (ie. \( y_i \) is one if alternative one is chosen and zero otherwise) and \( p_i \) is the probability that alternative one is chose. The \( r_i \) are just the scaled residuals - they have a zero mean and a unit variance. Note that, using the notation of the previous section:
\[ y_{10} = 1 - y_1, \ y_{11} = y_1, \ p_{10} = 1 - p_1, \ p_{11} = p_1, \ r_i = \sum_{j=0}^{1} u_{ij}, \ x_i = \sum_{j=0}^{1} z_{ij} \]

The information matrix is:

\[ I_{g0} = \lim_{N \to \infty} E \frac{1}{N} \sum_i x_i x_i' \]  \hspace{1cm} (5')

and the LM test statistic is:

\[ LM = \left( \sum_i \hat{r}_i \hat{x}_i' \right) \left( \sum_i \hat{x}_i \hat{x}_i' \right)^{-1} \left( \sum_i \hat{r}_i \hat{x}_i \right) \]  \hspace{1cm} (7')

where the observed scaled residuals and regressors are:

\[ \hat{r}_i = \frac{y_i - \hat{P}_i}{\sqrt{\hat{P}_i(1 - \hat{P}_i)}} \]

\[ \hat{x}_i = \frac{1}{\sqrt{\hat{P}_i(1 - \hat{P}_i)}} \delta \hat{P}_i \]

As Davidson and MacKinnon (1984) point out, the LM test statistic is just the explained sum of squares from the uncentred auxiliary regression of \( \hat{r}_i \) on \( \hat{x}_i \). Asymptotically \( N \) times the \( R^2 \) from this regression equals the LM test statistic.

Conclusion

LM test statistics for omitted variables, neglected heteroscedasticity and other misspecifications in general discrete choice models may be readily calculated using an artificial regression, the same as in binary choice models. However the form of the artificial regression is different in the general case. This artificial regression approach is both convenient and likely to have better small sample properties than the more common outer product gradient form of LM statistic.

References


