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SIMULTANEOUS REFORM OF TARIFFS AND QUOTAS*

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University College Dublin and CEPR
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Abstract

This paper presents a general result for simultaneous reform of tariffs and quotas in a small open economy, where some of the quota rents do not accrue to domestic residents. Absent highly perverse income effects, welfare must rise following a uniform proportionate reduction in tariffs and a uniform proportionate relaxation of quotas, weighted by their rent-retention parameters. Previous results are shown to be special cases of this one, and its implications for practical policy advice and its relationship with the policy of “tariffication” of quotas are noted.

JEL: F13

Keywords: International trade policy; piecemeal policy reform; tariffs and quotas; tariffication of quotas; trade liberalisation; voluntary export restraints.

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1 Introduction

The theory of piecemeal policy reform seeks rules of thumb for small policy changes which will guarantee an improvement in welfare, even when little detailed information on the structure of the economy is available. For changes in trade policy, the best-known result of this kind, associated with Foster and Sonnenschein (1970), Bruno (1972) and Hatta (1977a), is that welfare must rise if all tariffs are reduced by the same proportion.\(^1\) Falvey (1988) showed that this result also holds in the presence of “pure” quotas, where all the quota rents accrue to domestic residents.\(^2\) However, most real-world quantitative restrictions imply some loss of rents, typically mid-way between pure quotas and voluntary export restraints (VER’s) where all rents accrue to foreign exporters and so are lost to the domestic economy. The theory has been extended to take account of such mixed cases by Anderson and Neary (1992), but they did not present any results for simultaneous reform of tariffs and quotas.

This paper extends the theory of trade liberalization to derive a general result for simultaneous reform of all trade policies, when trade is distorted by quotas as well as tariffs and when quota-constrained imports differ in the share of rents retained by the importing country. Crucially, the result does not require any special assumptions about the structure of the economy. An alternative tradition derives results which hold under reasonable but nevertheless demanding restrictions on tastes and technology, for example, that some or all goods are general-equilibrium substitutes, as in Hatta (1977b) and Falvey (1988), or that tariff-constrained and quota-constrained goods are implicitly separable as in Anderson and Neary (1992). Though these results are of independent interest, it is clearly very desirable to find results which hold more generally.

Section 2 reviews the approach to modelling aggregate behaviour in the presence of tariffs and quotas developed in Anderson and Neary (1992). Section 3 presents the model of the economy and derives expressions for the marginal welfare effects of changes in tariffs and quotas which generalise those of Anderson and Neary (1992). These properties are then used in Section 4 to derive the main result of the paper. This section also explains the intuitive basis for the result, shows that it nests many previous reform rules in the literature, and relates it to the policy of “tarification” of quotas which was implemented in the Uruguay Round of trade liberalisation.

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\(^1\) Foster and Sonnenschein provided the first formal proof in the multi-commodity case, assuming that all goods are normal; Bruno showed that this assumption could be replaced by the much weaker assumption that the shadow price of foreign exchange is positive (see Section 3 below for details); and Hatta (1977a) provided a simple proof using the expenditure function.

\(^2\) Other papers which extend the theory of trade liberalization to take account of quotas (or of non-traded goods, which are formally equivalent to prohibitive quotas), include Hatta (1977b), Fukushima (1979) and Corden and Falvey (1985). The presentation here draws mostly on Anderson and Neary (1992) and Neary (1995). See also Lahtinen and Raimondos (1996).
2 Modelling Behaviour in the Presence of Tariffs and Quotas

Consider a competitive small open economy, in which some imports are subject to tariffs and others are subject to quotas. Imports of tariff-constrained goods are denoted by $m$, with domestic and world prices $\pi$ and $\pi^*$ respectively, which differ because of specific tariffs $t$, so $\pi = \pi^* + t$. Imports of quota-constrained goods are denoted by $q$, with domestic and world prices $p$ and $p^*$ respectively. It will be convenient to refer to the two groups of goods as the “$t$-goods” and the “$q$-goods”, respectively. Finally, exports and unconstrained imports can be grouped together as a composite numeraire good, with net imports $m_0$. The price of the numeraire, which is the same at home and abroad, is set equal to one (and omitted from the list of arguments of the behavioral functions for convenience).

Consider first the benchmark case where the $q$-goods are not subject to binding quotas. The behavior of the economy is then most conveniently summarized by the trade expenditure function. With two categories of goods this is defined as:

$$E(\pi, p, u) \equiv e(\pi, p, u) - g(\pi, p),$$

where $e(\pi, p, u)$ and $g(\pi, p)$ are the household expenditure and GDP functions respectively. By Shepherd’s Lemma, the price derivatives of the trade expenditure function give the compensated net import demand functions for the $t$- and $q$-goods which apply in the absence of quotas:

$$E_{\pi}(\pi, p, u) = m^c(\pi, p, u).$$

$$E_p(\pi, p, u) = q^c(\pi, p, u)$$

Since the trade expenditure function is concave in $\{\pi, p\}$, each of these import demand functions is non-increasing in its own price. For convenience we will assume that they are strictly decreasing in their own price, and, more generally, that the matrices $E_{\pi\pi}$ and $E_{pp}$ are negative definite.\(^3\) The trade expenditure function may also be defined in an alternative way which will prove useful below:

$$E(\pi, p, u) \equiv \min_{m_0, m, q} \{m_0 + \pi' m + p' q : U(m_0, m, q) = u\},$$

where $U(m_0, m, q)$ is a Meade trade utility function defined over net imports $m_0$, $m$ and $q$ rather than final consumption.\(^4\)

\(^3\)This requires some substitutability in excess demand between the different groups of goods. I make this mild assumption throughout, without repeating the qualification.

\(^4\)See Chipman (1979) and Woodland (1980) for further details.
When the $q$-goods are restricted by binding quotas we need to adopt a different approach. Following Anderson and Neary (1992), we therefore introduce a new function, the distorted trade expenditure function. This equals net spending on the tariff-constrained goods conditional on the quota levels:

$$\tilde{E}(\pi, q, u) \equiv \min_{m_0, m} \{m_0 + \pi' m : U(m_0, m, q) = u\}. \quad (5)$$

Viewed as a function of $\pi$ and $u$ for given $q$, the distorted trade expenditure function behaves just like the standard trade expenditure function. The derivative of $\tilde{E}$ with respect to $u$ is the marginal cost of utility $e_u$, and, invoking Shepherd’s Lemma once again, its derivatives with respect to $\pi$ equal the compensated import demand functions for the $t$-goods conditional on the quotas:

$$\tilde{E}_\pi(\pi, q, u) = \tilde{m}^c(\pi, q, u). \quad (6)$$

These quota-constrained demand functions have properties with respect to $\pi$ similar to those of the unconstrained demand functions (2). In particular:

**Lemma 1**: $\tilde{E}$ is concave in $\pi$, and so the matrix of own-price derivatives of the quota-constrained demand functions for the $t$-goods, $\tilde{m}^c_\pi$, which equals $\tilde{E}_{\pi\pi}$, is negative definite.

Heuristically, the compensated net import demand functions for the $t$-goods slope downwards.

What about the properties of the distorted trade expenditure function as a function of the quota constraints? To derive these, note that the distorted and undistorted functions can be related provided the domestic prices of the quota-constrained goods are market-clearing. More precisely:

$$\tilde{E}(\pi, q, u) = E[\pi, p(\pi, q, u), u] - p(\pi, q, u)'q \quad (7)$$

where the market-clearing price vector $p$ is defined implicitly by:

$$q = E_p(\pi, p, u). \quad (8)$$

Differentiating (7) with respect to $q$ and using (8) to simplify gives an explicit expression for the prices $p$:

$$p(\pi, q, u) = -\tilde{E}_q(\pi, q, u). \quad (9)$$

Thus the derivatives of the distorted trade expenditure function with respect to the quota levels equal minus

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5This approach follows the analysis of household behaviour under rationing by Neary and Roberts (1980), who called $p$ the virtual prices of the rationed goods.
the inverse demand functions for the quota-constrained goods, expressing their market-clearing prices as functions of the exogenous variables. The key property of these inverse demand functions is given by the following:

*Lemma 2:* $\tilde{E}$ is convex in $q$, and so the matrix of derivatives of the inverse demand functions for the $q$-goods with respect to the quota levels, $p_q$, which equals $-\tilde{E}_{qq}$, is negative definite.

(See Anderson and Neary (1992) for a formal proof.) Heuristically, the compensated inverse demand functions for the $q$-goods slope downwards.

### 3 The Welfare Effects of Changes in Trade Policy

The distorted trade expenditure function summarizes the behaviour of the private sector and it only remains to specify public sector behaviour, which is purely redistributive. It is standard to assume that all tariff revenue is redistributed in a lump-sum manner to the aggregate household. However, the same assumption is not plausible in the case of quota rents. Instead, we assume that a fraction $\omega_j$ of the quota rents on each good $j$ is lost to the domestic economy. As noted in the Introduction, $\omega_j$ is zero in the case of a pure quota and one in the case of a VER. Total quota rents retained at home and redistributed to households therefore equal $\sum_j (1 - \omega_j)(p_j - p_j^*)q_j$. In matrix form this can be written as $(p - p^*)(I - \omega)q$, where $I$ is the identity matrix and a bar under a vector denotes the corresponding diagonal matrix (so $\omega$ is a diagonal matrix with the rent-loss shares on the principal diagonal).

Armed with the properties of the distorted trade expenditure function and our assumptions about the disposition of quota rents, we are now ready to specify the general equilibrium of the economy. In equilibrium, net expenditure on the numeraire and on tariff-constrained goods $(\pi)$, plus net expenditure on quota-constrained goods $p'q$, must equal tariff revenue $t'm$ plus retained quota rents $(p - p^*)(I - \omega)q$:

$$\tilde{E}(\pi, q, u) + p'q = t'm + (p - p^*)(I - \omega)q.$$  \hfill (10)

The first step toward deriving the welfare effects of trade policy reform is to totally differentiate equation (10). (We simplify by using (6), (9) and the fact that $d\pi = dt$. We also assume for the present that the rent-loss parameters $\omega$ are constant, so we ignore terms in $d\omega$.) This yields:

$$e_u du = t'dm + (p - p^*)(I - \omega) dq - q'\omega dp.$$  \hfill (11)

This equation does not give the full effect of changes in trade policy, because the terms in $dm$ and $dp$ are
endogenous. Nevertheless, it is very helpful in providing intuition. Consider in turn the three terms on the right-hand side. The first shows that, as in standard models where tariffs are the only distortion, welfare rises if the tariff-weighted volume of tariff-constrained imports increases, or, equivalently, if tariff revenue rises at the initial tariffs. The second shows that, other things equal, welfare rises when quotas are relaxed (except for VER’s where $\omega_i = 1$, so all the rents are lost). Finally, the third term shows that welfare also rises when the domestic prices of quota-constrained goods fall, since (except for pure quotas where $\omega_i = 0$) this reduces total rents and hence reduces the amount transferred to foreigners.

The next step is to use the differentials of (6) and (9) to eliminate the endogenous terms $dm$ and $dp$ from (11). This yields the basic equation for the welfare effects of changes in trade policy in the presence of tariffs and quotas:

$$\mu^{-1}e_u du = \chi' dt + \rho' dq.$$  \hspace{1cm} (12)

As in Anderson and Neary (1992), the coefficient of the change in real income $e_u du$ can be interpreted as the inverse of the shadow price of foreign exchange, $\mu$, which measures the effect on welfare of a unit transfer of the numeraire good. Writing it in full:\footnote{To derive this we express the cross-derivatives of $\hat{E}$ with respect to prices and utility in terms of the derivatives of the distorted Marshallian import demand and virtual price functions with respect to income: $\hat{E}_{u u} = \delta_t e_u$ and $\hat{E}_{q u} = -p_I e_u$.}

$$\mu = \frac{1}{1 - \nu m_I + q' \omega p_I}.$$  \hspace{1cm} (13)

Any increase in real income has a multiplier effect which is greater than one to the extent that it raises demand for tariff-constrained imports. Offsetting this, when the rent-loss parameters $\omega_i$ are strictly positive, the multiplier effect is dampened to the extent that incipient increases in demand for quota-constrained goods push up their domestic prices and so increase the amount of rents lost. Because of the combined effect of these influences, $\mu$ may be either greater or less than unity. In any case, we assume throughout that it is positive.\footnote{This may be rationalised on stability grounds or by invoking a minimal degree of rationality of government policy. Alternatively, we can look for sufficient conditions to sign the individual terms. The term in $\mu$ which does not appear in the absence of quotas is $q' \omega p_I$. This can be shown to equal $-q' \omega p_{I q}$. Alternative sufficient conditions for this to be positive are: (a) from Hatta (1977a), that the $q$-goods are normal in demand and net substitutes; and (b) from Anderson and Neary (1992), that $\omega_i$ is the same for all goods and that the $q$-goods are homothetic in demand and have uniform import shares (so that $q_I = a q/I$, where $a$ is the common import share).}

The welfare effect of trade reform, or the marginal cost of protection, therefore depends on the coefficients of changes in the policy variables in (12), which we call the marginal costs of tariffs:

$$\chi' = \nu m_I^* - q' \omega p_I.$$  \hspace{1cm} (14)
and the shadow prices of quotas respectively:

\[
\rho' = t' \tilde{m}_q^c + (p - p^*)(I - \omega) - q' \omega p_q \tag{15}
\]

These formulae generalise the results of Anderson and Neary (1992) to allow for rent-loss parameters which differ across commodities. They are the central equations of the paper.

4 Simultaneous Trade Policy Reform

As explained in the introduction, we seek a rule for simultaneous changes in tariffs and quotas which guarantees a welfare improvement without the need to place restrictions on the structure of the economy. It transpires that such a rule can be devised by combining two results already in the literature. The first is the radial reduction in tariffs result, discussed in the introduction. The second is a result due to Anderson and Neary (1992), who showed that, in the absence of tariffs, welfare must rise following a uniform relaxation of quotas weighted by their rent-loss parameters.\(^8\) Combining these results, provided both sets of distortions are relaxed at the same rate, a welfare improvement is assured.

We first state the new result formally:

**Proposition 1:** Assume that the shadow price of foreign exchange is positive. Then a uniform proportionate reduction of tariffs combined with a uniform proportionate relaxation of quotas, weighted by the share of rents lost on each quota-constrained good, with both proportionate changes at the same rate, must raise welfare.

**Proof:** The policy rule implies that \(dt = -td\alpha\) and \(dq = \omega q d\alpha\), where \(d\alpha\) is a positive scalar. Substituting in (12), and using the expressions for \(\chi\) and \(\rho\) from (14) and (15), the change in welfare is:

\[
\mu^{-1} e_\alpha \left[ \frac{du}{d\alpha} \right] = -\chi' t + \rho' \omega q
\]

\[
= -t' \tilde{m}_c q + (p - p^*)(I - \omega) \omega q - q' \omega p_q \omega q \tag{16}
\]

All three terms on the right-hand side of this expression are positive scalars, the second because all its individual terms are positive, and the first and third because they are minus quadratic forms in negative definite matrices, from Lemmas 1 and 2 respectively. Hence a welfare gain is guaranteed.

\(^8\)See Theorem 2’ of Anderson and Neary (1992), p. 68.
While the proposition is not difficult to prove, providing intuition for it is more of a challenge. One approach to doing this is mathematical. Recall from Lemmas 1 and 2 that the function $\tilde{E}$ is concave in $\pi$ (and hence, for given world prices, in $t$) and convex in $q$. This implies that the second-derivative matrices $\tilde{E}_{\pi\pi}$ and $\tilde{E}_{qq}$ are negative definite and positive definite respectively. Hence the expressions $t^' \tilde{E}_{\pi\pi} dt$ and $q^' \tilde{E}_{qq} dq$ are both positive when $dt = -td\alpha$ and $dq = \omega qd\alpha$, since they are quadratic forms in the positive definite matrices $-\tilde{E}_{\pi\pi}$ and $\tilde{E}_{qq}$. Lemma 3 in the Appendix shows that this result can be extended to prove that for such a function the expression $x^' \tilde{E}_{xx} dx$ is also a positive quadratic form, where $x$ is a vector formed by stacking the two vectors $t$ and $q$, and $dx$ is such that $dt = -td\alpha$ and $dq = \omega qd\alpha$.

To appreciate the economics underlying the proposition, consider the individual terms on the right-hand side of (16). From equation (11), the second term reflects the effects of the quota relaxation at given import volumes $m$ and domestic prices $p$. Fixing $m$ and $p$ in this way rules out second-best complications, so any quota reform must raise welfare since it reduces the amount of rents lost. A quota reform of the type $dq = \omega qd\alpha$ must strictly raise welfare provided that not all quotas have either zero ($\omega_i = 0$) or full ($\omega_i = 1$) rent loss.

As for the first and third terms on the right-hand side of (16), these reflect the direct effects of the tariff and quota reforms. The first term, $-t^' \tilde{m}_c^' t$, reflects the welfare gain arising from the increase in imports of the $t$-goods following a uniform proportionate reduction in tariffs. The third term, $-q^' \tilde{p}_q^' q$, reflects the welfare gain arising from the reduction in domestic prices $p$ of the $q$-goods (with a consequent fall in rents lost) following a uniform proportionate relaxation of $\omega$-weighted quotas. The fact that these direct effects on welfare are unambiguously positive is well-established in the literature.\(^9\)

The trade reforms also have indirect effects, which might be expected to render their net impact on welfare ambiguous. These indirect effects are captured by two additional terms (not shown in equation (16)) which appear in the full expression for $du$, and which are indeterminate in sign : $t^' \tilde{m}_c^' q$ and $q^' \tilde{p}_q^' t$. However, these two scalars cancel, because $\tilde{m}_c^' q$ (which equals $\tilde{E}_{\pi q}$) is the transpose of $-p_{\pi}$ (which equals $\tilde{E}_{qq}$). In words, the effect of the uniform quota relaxation on tariff revenue is exactly equal to the effect of the uniform tariff reduction on lost quota rents. Crucial for this result is the assumption that both types of trade distortion are relaxed at the same rate. As a result the indirect effects play no role and the net effect of the trade reform on welfare is unambiguously positive.

\(^9\)Hatta (1977b) and Fukushima (1979) showed that a uniform proportionate reduction in tariffs must raise welfare in the presence of non-traded goods, provided all goods are net substitutes. Falvey (1988) extended this result to tariff reductions in the presence of quotas with full retention, and Neary (1995) showed that the qualification of net substitutability is unnecessary. As for a uniform proportionate relaxation of $\omega$-weighted quotas, Anderson and Neary (1992) showed that this must raise welfare in the absence of tariffs.
It is clear that Proposition 1 encompasses as special cases all the results already in the literature for uniform proportionate relaxations of trade distortions in a small open economy.\textsuperscript{10} This is true of the results of Hatta (1977a), Falvey (1988) and Neary (1995) that a uniform proportionate reduction in tariffs raises welfare either when quotas are absent or when all quota rents are retained. It is also true of the result of Anderson and Neary (1992) that, in the absence of tariffs, a uniform proportionate relaxation of quotas raises welfare with partial rent retention.\textsuperscript{11} All these results are corollaries of Proposition 1 since they apply only in special cases when one set of trade policy instruments is either absent (no tariffs in the case of quota relaxations only) or benign (full rent retention in the case of tariff reductions only).

Finally, Proposition 1 highlights the importance in trade policy reform of taking account of the rents lost to the domestic economy. While some authors have argued that this consideration also applies to tariffs (see in particular the discussion of “revenue seeking” by Bhagwati and Srinivasan (1980)), it seems most serious in the case of quotas. This suggests that the model should have implications for the issue of “tariffication”: abolishing quantitative restrictions and replacing them by their equivalent quotas. This policy switch was applied to agricultural trade in the Uruguay Round for example. In the present model, it is equivalent to a combination of two policies: first, a switch from the economy described by equation (10) where behaviour is summarised in terms of the distorted trade expenditure function to an otherwise identical economy expressed in terms of the undistorted trade expenditure function (1); and second, a reduction in the rent-loss parameters $\omega$. The first change is neutral in itself. To see the effects of the second, we can first return to equation (10) and totally differentiate it with respect to $\omega$, which yields

$$
\mu^{-1}e_u du = -(p-p^*)'q\omega
$$

The right-hand side is positive provided $d\omega$ is negative. Reducing the rent-loss parameters in any way (not necessarily proportionally) thus unambiguously lowers the amount of rents lost and raises welfare. After the tariffication process is carried out, so it is $q$ rather than $p$ which adjusts, the welfare effect of changes in $\omega$ is also given by (17), except that the shadow price of foreign exchange takes a slightly different form.\textsuperscript{12}

Thus tariffication of quotas, to the extent that it reduces infra-marginal rent loss, is unambiguously welfare-improving.

\textsuperscript{10}Of course, it does not deal with the case of unilateral reform of tariffs and quotas in a large economy, as in Neary (1995), nor with that of multilateral reforms of tariffs and quotas by a group of countries as in Woodland and Turunen-Red (2000).

\textsuperscript{11}Strictly speaking, Proposition 1 does not nest the result of Corden and Falvey (1985), whereby welfare is raised by any quota reduction provided all rents are retained and there are no tariffs. In such a case the rule $dq = \omega q du$ is clearly degenerate, since $\omega$ is zero for all $t$. As Corden and Falvey showed in this case, for arbitrary positive $dq$ the change in welfare is proportional to $(p-p^*)'dq$ and so is positive.

\textsuperscript{12}Differentiating (10) with the left-hand side equal to the undistorted trade expenditure function (1) and with $m$ and $q$ determined by (2) and (3) respectively, yields: $(\mu')^{-1}e_u du = -(p-p^*)'q\omega$, where $\mu' = [1 - t'm_I - (p-p^*)'(I-\omega)q_I]^{-1}$. 

\vspace{3cm} 8
5 Conclusion

This paper has presented a new result on simultaneous reform of tariffs and quotas in a distorted small open economy. The policy rule whose efficacy is established in Proposition 1 involves a uniform proportionate relaxation of all distortions. It is convenient for practical advice, if somewhat surprising, that both tariffs and quotas should be relaxed at the same rate, even though they are measured in different units. It is also intuitively plausible that the quotas which should be relaxed fastest are those which lose the most rent for the domestic economy. Combining these policy reform rules ensures that second-best problems are avoided and a welfare gain is assured.

As far as the practical applicability of the trade reform rule in Proposition 1 is concerned, it clearly requires that all trade policy instruments can be altered at once. On the other hand, it has minimal informational requirements: no parameters of the home economy need be known, and the only assumption which must be made is to rule out interactions between initial distortions and income effects which are sufficiently perverse that the shadow price of foreign exchange is negative. Though unlikely to be directly applicable in any particular application, the result hopefully provides a benchmark with which actual liberalisation plans (whether in a unilateral or multilateral context) can be compared.
Appendix

As noted in the text, the formal underpinnings of Proposition 1 can be expressed as follows.

**Lemma 3**: Consider a function $F(x)$ where the vector $x$ is partitioned in two and the vector $y$ is a simple transformation of $x$:

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \text{and} \quad y = \begin{bmatrix} -x_1 \\ x_2 \end{bmatrix}$$

(18)

Suppose that $F$ is strictly concave in $x_1$ and strictly convex in $x_2$. Then, the expression $x'F_{xx}y$ is strictly positive.

**Proof**: The proof is immediate:

$$x'F_{xx}y = x_1'F_{11}y_1 + x_1'F_{12}y_2 + x_2'F_{21}y_1 + x_2'F_{22}y_2$$

$$= -x_1'F_{11}x_1 + x_1'F_{12}x_2 - x_2'F_{21}x_1 + x_2'F_{22}x_2$$

$$= -x_1'F_{11}x_1 + x_2'F_{22}x_2 > 0$$

(19)

In the last line, $x_1'F_{11}x_1$ is negative because $F$ is strictly concave in $x_1$ and $x_2'F_{22}x_2$ is positive because $F$ is strictly convex in $x_2$. Hence the whole expression is strictly positive.

Q.E.D.

Note that Lemma 3 does not yield Proposition 1 immediately. It can be checked that much of the expression for $du$ in (16) can be written as $x'F_{xx}dx$, where $dx_1 = -x_1d\alpha$ and $dx_2 = x_2d\alpha$, representing a uniform proportionate decrease in $x_1$ and a uniform proportionate increase in $x_2$ at the same rate. Lemma 3 then applies directly. However, there is an additional term in the expression for $du$ in (16), $(p-p^*)(I-\omega)\omega q$, which is not covered by the Lemma. Fortunately, since all the elements of this term are individually positive, it does not affect the result.
References


