<table>
<thead>
<tr>
<th><strong>Title</strong></th>
<th>Universal Taylor series for non-simply connected domains</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Authors(s)</strong></td>
<td>Gardiner, Stephen J.; Tsirivas, Nikolaos</td>
</tr>
<tr>
<td><strong>Publication date</strong></td>
<td>2010-05</td>
</tr>
<tr>
<td><strong>Publication information</strong></td>
<td>Comptes Rendus Mathématique, 348 (9-10): 521-524</td>
</tr>
<tr>
<td><strong>Publisher</strong></td>
<td>Elsevier</td>
</tr>
<tr>
<td><strong>Link to online version</strong></td>
<td><a href="http://dx.doi.org/10.1016/j.crma.2010.03.003">http://dx.doi.org/10.1016/j.crma.2010.03.003</a></td>
</tr>
<tr>
<td><strong>Item record/more information</strong></td>
<td><a href="http://hdl.handle.net/10197/2465">http://hdl.handle.net/10197/2465</a></td>
</tr>
<tr>
<td><strong>Publisher's version (DOI)</strong></td>
<td>10.1016/j.crma.2010.03.003</td>
</tr>
</tbody>
</table>
Universal Taylor series for non-simply connected domains

Séries universelles de Taylor pour les domaines non-simplement connexes

Stephen J. Gardiner and N. Tsirivas

Abstract

It is known that, for any simply connected proper subdomain \( \Omega \) of the complex plane and any point \( \zeta \) in \( \Omega \), there are holomorphic functions on \( \Omega \) that have “universal” Taylor series expansions about \( \zeta \); that is, partial sums of the Taylor series approximate arbitrary polynomials on arbitrary compacta in \( \mathbb{C} \setminus \Omega \) that have connected complement. This note shows that this phenomenon can break down for non-simply connected domains \( \Omega \), even when \( \mathbb{C} \setminus \Omega \) is compact. This answers a question of Melas and disproves a conjecture of Müller, Vlachou and Yavrian.

Résumé

Il est connu que, pour un sous-domaine propre simplement connexe \( \Omega \) du plan complexe et un point quelconque \( \zeta \) de \( \Omega \), il y a des fonctions holomorphes sur \( \Omega \) qui possèdent des séries de Taylor «universelles» autour de \( \zeta \); c’est-à-dire tout polynôme peut être approximé, sur tout compact de \( \mathbb{C} \setminus \Omega \) ayant un complémentaire connexe, par les sommes partielles de la série de Taylor. Cette note montre que ce résultat n’est plus vrai en général pour les domaines non-simplement connexes \( \Omega \), même lorsque \( \mathbb{C} \setminus \Omega \) est compact. Cela répond à une question de Melas et réfute une conjecture de Müller, Vlachou et Yavrian.

1 Introduction

Let \( \Omega \) be a proper subdomain of the complex plane \( \mathbb{C} \) and let \( \zeta \in \Omega \). A function \( f \) on \( \Omega \) is said to belong to the collection \( U(\Omega, \zeta) \), of holomorphic

\[ \text{Mathematics Subject Classification 30B30, 30E10.} \]

This research was supported by Science Foundation Ireland under Grant 09/RFP/MTH2149, and is also part of the programme of the ESF Network “Harmonic and Complex Analysis and Applications” (HCAA).
functions on $\Omega$ with universal Taylor series expansions about $\zeta$, if the partial sums

$$S_N(f, \zeta)(z) = \sum_{n=0}^{N} \frac{f^{(n)}(\zeta)}{n!}(z - \zeta)^n$$

of the Taylor series have the following property:

for every compact set $K \subset \mathbb{C}\setminus\Omega$ with connected complement and every function $g$ which is continuous on $K$ and holomorphic on $K^\circ$, there is a subsequence $(S_{N_k}(f, \zeta))$ that converges to $g$ uniformly on $K$.

Nestoridis [17], [18] has shown that $U(\Omega, \zeta) \neq \emptyset$ for any simply connected domain $\Omega$ and any $\zeta \in \Omega$. (The corresponding result, where $K$ is required to be disjoint from $\Omega$, had previously been established by Luh [12] and Chui and Parnes [4].) In fact, Nestoridis showed that possession of such universal Taylor series expansions is a generic property of holomorphic functions on simply connected domains $\Omega$, in the sense that $U(\Omega, \zeta)$ is a dense $G_\delta$ subset of the space of all holomorphic functions on $\Omega$ endowed with the topology of local uniform convergence (see also Melas and Nestoridis [14] and the survey of Kahane [11]).

The situation when $\Omega$ is non-simply connected is much less well understood, despite much recent research: see, for example, [2], [3], [5], [6], [7], [9], [13], [15], [19], [22], [23], [24], [25]. Melas [13] (see also Costakis [5]) has shown that $U(\Omega, \zeta) \neq \emptyset$ for any $\zeta \in \Omega$ whenever $\mathbb{C}\setminus\Omega$ is compact and connected, and has asked if $U(\Omega, \zeta)$ can be empty when $\mathbb{C}\setminus\Omega$ is compact but disconnected. On the other hand, Müller, Vlachou and Yavrian [15] have shown, for non-simply connected domains $\Omega$, that thinness of the set $\mathbb{C}\setminus\Omega$ at infinity is necessary for $U(\Omega, \zeta)$ to be non-empty, and have conjectured that this condition is also sufficient. There is clearly a large gap between the results of [13] and [15]. Also there has been no known example of a domain $\Omega$ and points $\zeta_1, \zeta_2 \in \Omega$ such that $U(\Omega, \zeta_1) \neq \emptyset$ and $U(\Omega, \zeta_2) = \emptyset$.

The purpose of this note is to establish the following result. We denote by $D(a, r)$ the open disc of centre $a$ and radius $r$, and write $\overline{D} = D(0, 1)$. By a non-degenerate continuum we mean a connected compact set containing more than one element.

**Theorem 1** Let $\Omega$ be a domain of the form $\mathbb{C}\setminus(L \cup \{1\})$, where $L$ is a non-degenerate continuum in $\mathbb{C}\setminus\overline{D}$. Then $U(\Omega, 0) = \emptyset$.

The conjecture of Müller, Vlachou and Yavrian is thus disproved. Also, if we take $L$ to be $\overline{D}(-5/3, 1/3)$, then $U(\Omega, 0) = \emptyset$ by Theorem 1 and yet a result of the second author [22] tells us that $U(\Omega, -1/2) \neq \emptyset$ (see also Costakis and Vlachou [7]). Thus we now have an example of a domain where the existence of functions with universal Taylor series depends on the chosen centre for expansion. The result of Melas, that $U(\Omega, 0) \neq \emptyset$ if $\mathbb{C}\setminus\Omega$
is compact and connected, is now seen to be sharp in the sense that, by
Theorem 1, it can fail with the removal of one additional point from the
domain. Theorem 1 fails if $L$ is allowed to be a singleton [13].

2 Proof

Let $\Omega$ be as in the statement of Theorem 1, and suppose, for the sake of
contradiction, that there exists a function $f$ in $U(\Omega, 0)$. We can write $f = g + h$, where $g$ is the singular part of the Laurent expansion of $f$
associated with the singularity at 1, and $h$ is holomorphic on $\mathbb{C} \setminus L$. We denote the Taylor coefficients of $g$ and $h$ about 0 by $(a_n)$ and $(b_n)$, respectively. Since $(S_N(f, 0)(1))$ is dense in $\mathbb{C}$ and $(S_N(h, 0)(1))$ converges, we see that $g$ is non-zero.

Let $\rho = \inf\{|z| : z \in L\}$ and $0 < \delta < \varepsilon < \rho - 1$. The Taylor series for $g$ and $h$ about 0 converge absolutely in $D(0, \rho)$, respectively, so we can define the finite quantities

$$\alpha_\delta = \sum_{n=0}^{\infty} \frac{|a_n|}{(1 + \delta)^n} \quad \text{and} \quad \beta_\delta = \sum_{n=0}^{\infty} |b_n| \left(\frac{\rho}{1 + \delta}\right)^n.$$

Since $f \in U(\Omega, 0)$, we can choose a strictly increasing sequence $(N_k)$ of natural numbers such that

$$S_{N_k}(g, 0)(z) + S_{N_k}(h, 0)(z) \to 0 \quad \text{as} \quad k \to \infty, \text{uniformly on } L. \quad (1)$$

On $\overline{D}(0, \rho(1 + \varepsilon))$ we have

$$|S_{N_k}(h, 0)(z)| \leq \sum_{n=0}^{N_k} |b_n| \rho^n (1 + \varepsilon)^n \leq \{(1 + \varepsilon)(1 + \delta)\}^{N_k} \beta_\delta,$$

so by (1) we can choose $k_0$ such that

$$|S_{N_k}(g, 0)(z)| \leq \{(1 + \varepsilon)(1 + \delta)\}^{N_k} (\beta_\delta + 1) \quad (z \in L \cap \overline{D}(0, \rho(1+\varepsilon)); k \geq k_0).$$

We also have

$$|S_{N_k}(g, 0)(z)| \leq \sum_{n=0}^{N_k} |a_n| (1 + \varepsilon)^n \leq \{(1 + \varepsilon)(1 + \delta)\}^{N_k} \alpha_\delta \quad (z \in \overline{D}(0, 1 + \varepsilon)), $$

so

$$|S_{N_k}(g, 0)(z)| \leq \{(1 + \varepsilon)(1 + \delta)\}^{N_k} \gamma_\delta \quad (z \in A_\varepsilon; k \geq k_0), \quad (2)$$

where $\gamma_\delta = \max\{\alpha_\delta, \beta_\delta + 1\}$ and

$$A_\varepsilon = \overline{D}(0, 1 + \varepsilon) \cup [L \cap \overline{D}(0, \rho(1 + \varepsilon))].$$
Let $G_\varepsilon$ denote the Green function for the domain $D_\varepsilon = (\mathbb{C} \cup \{\infty\}) \setminus A_\varepsilon$ with pole at infinity. Then

$$G_\varepsilon(z) - \log |z| \to - \log \mathcal{C}(A_\varepsilon) \quad (|z| \to \infty),$$

where $\mathcal{C}(A)$ denotes the logarithmic capacity of a set $A$ (see Section 5.8 of [1], or Section 5.2 of [21]). Thus we can choose $r_{\delta, \varepsilon} > \max \{|z| : z \in L\}$ such that

$$G_\varepsilon(z) \leq \log |z| - \log \mathcal{C}(A_\varepsilon) + \delta \quad (|z| \geq r_{\delta, \varepsilon}). \quad (3)$$

Bernstein’s lemma (Theorem 5.5.7 in [21]) tells us that any polynomial $q$ of degree $n \geq 1$ satisfies

$$\left( \frac{|q(z)|}{\max_{A_\varepsilon} |q|} \right)^{1/n} \leq e^{G_\varepsilon(z)} \quad (z \in D_\varepsilon \setminus \{\infty\}).$$

Applying this inequality to the polynomial $S_{N_k}(g, 0)$, and using (2) and then (3), we obtain

$$|S_{N_k}(g, 0)(z)| \leq \{(1 + \varepsilon)(1 + \delta)\}^{N_k} \gamma_\delta e^{N_k G_\varepsilon(z)}$$

$$\leq \left\{ \frac{(1 + \varepsilon)(1 + \delta)e^\delta |z|}{\mathcal{C}(A_\varepsilon)} \right\}^{N_k} \gamma_\delta \quad (|z| \geq r_{\delta, \varepsilon}; k \geq k_0).$$

We next adapt an argument from pp.498,499 of Gehlen [8]. Let $\nu \in (0, 1)$. Since

$$|a_n|^{1/n} = \left| \frac{1}{2\pi i} \int_{|z|=r_{\delta, \varepsilon}} \frac{S_{N_k}(g, 0)(z)}{z^{n+1}} \, dz \right|^{1/n}$$

$$\leq \left\{ \frac{(1 + \varepsilon)(1 + \delta)e^\delta}{\mathcal{C}(A_\varepsilon)} \right\}^{N_k/n} \gamma_\delta \frac{1/n}{r_{\delta, \varepsilon}^{N_k/n-1}} \quad (n \leq N_k; k \geq k_0),$$

we obtain

$$\limsup_{k \to \infty} \max_{\nu N_k \leq n \leq N_k} |a_n|^{1/n} \leq \left( \frac{(1 + \varepsilon)(1 + \delta)e^\delta}{\mathcal{C}(A_\varepsilon)} \right)^{1/\nu} \gamma_\delta^{1/\nu-1} = \lambda, \quad \text{say.} \quad (4)$$

Since $L$ is a non-degenerate continuum that intersects $\{|z| = \rho\}$, we have

$$\mathcal{C}(L \cap \overline{D}(0, \rho(1 + \varepsilon))) > 0$$

and so

$$\mathcal{C}(A_\varepsilon) > \mathcal{C}(\overline{D}(0, 1 + \varepsilon)) = 1 + \varepsilon.$$
Theorem (Theorem 11.2.2 in Hille [10]) tells us that there is an entire function $F$ of exponential type $0$ such that $F(n) = a_n$ for all $n \geq 0$. However, Theorem V of Pólya [20] says that, for any $\mu > 0$, however small, such a function $F$ has the property that the sequence \{\(n \in \mathbb{N} : |F(n)| > e^{-\mu n}\)\} is of density 1. This contradicts (4) with $\lambda < 1$. Thus our original assumption, that there exists $f$ in $U(\Omega, 0)$, must be false, and the proof of the theorem is complete.

Remarks. 1) The assumption that $L$ is a continuum can be relaxed. It is enough to suppose that $L$ is a compact subset of $\mathbb{C} \setminus \mathbb{W}$ such that $\mathcal{C}(D(0, \rho^2) \cap L) > 0$ where $\rho = \inf\{|z| : z \in L\}$.
2) The proof actually shows that there is no holomorphic function $f$ on $U$ such that $(S_N(f, 0))$ is divergent at $z = 1$ and has a subsequence that is uniformly bounded on $L$.

References


School of Mathematical Sciences,
University College Dublin, Belfield, Dublin 4, Ireland.

stephen.gardiner@ucd.ie
nikolaos.tsirivas@ucd.ie