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Universal Taylor series for non-simply connected domains

Séries universelles de Taylor pour les domaines non-simplement connexes

Stephen J. Gardiner and N. Tsirivas

Abstract

It is known that, for any simply connected proper subdomain $\Omega$ of the complex plane and any point $\zeta$ in $\Omega$, there are holomorphic functions on $\Omega$ that have “universal” Taylor series expansions about $\zeta$; that is, partial sums of the Taylor series approximate arbitrary polynomials on arbitrary compacta in $\mathbb{C}\setminus\Omega$ that have connected complement. This note shows that this phenomenon can break down for non-simply connected domains $\Omega$, even when $\mathbb{C}\setminus\Omega$ is compact. This answers a question of Melas and disproves a conjecture of Müller, Vlachou and Yavrian.

1 Introduction

Let $\Omega$ be a proper subdomain of the complex plane $\mathbb{C}$ and let $\zeta \in \Omega$. A function $f$ on $\Omega$ is said to belong to the collection $U(\Omega, \zeta)$, of holomorphic
functions on $\Omega$ with universal Taylor series expansions about $\zeta$, if the partial sums

$$S_N(f, \zeta)(z) = \sum_{n=0}^{N} \frac{f^{(n)}(\zeta)}{n!}(z - \zeta)^n$$

of the Taylor series have the following property:

For every compact set $K \subset \mathbb{C}\backslash\Omega$ with connected complement and every function $g$ which is continuous on $K$ and holomorphic on $K^\circ$, there is a subsequence $(S_{N_k}(f, \zeta))$ that converges to $g$ uniformly on $K$.

Nestoridis [17], [18] has shown that $U(\Omega, \zeta) \neq \emptyset$ for any simply connected domain $\Omega$ and any $\zeta \in \Omega$. (The corresponding result, where $K$ is required to be disjoint from $\Omega$, had previously been established by Luh [12] and Chui and Parnes [4].) In fact, Nestoridis showed that possession of such universal Taylor series expansions is a generic property of holomorphic functions on simply connected domains $\Omega$, in the sense that $U(\Omega, \zeta)$ is a dense $G_\delta$ subset of the space of all holomorphic functions on $\Omega$ endowed with the topology of local uniform convergence (see also Melas and Nestoridis [14] and the survey of Kahane [11]).

The situation when $\Omega$ is non-simply connected is much less well understood, despite much recent research: see, for example, [2], [3], [5], [6], [7], [9], [13], [15], [19], [22], [23], [24], [25]. Melas [13] (see also Costakis [5]) has shown that $U(\Omega, \zeta) \neq \emptyset$ for any $\zeta \in \Omega$ whenever $\mathbb{C}\backslash\Omega$ is compact and connected, and has asked if $U(\Omega, \zeta)$ can be empty when $\mathbb{C}\backslash\Omega$ is compact but disconnected. On the other hand, Müller, Vlachou and Yavrian [15] have shown, for non-simply connected domains $\Omega$, that thinness of the set $\mathbb{C}\backslash\Omega$ at infinity is necessary for $U(\Omega, \zeta)$ to be non-empty, and have conjectured that this condition is also sufficient. There is clearly a large gap between the results of [13] and [15]. Also there has been no known example of a domain $\Omega$ and points $\zeta_1, \zeta_2 \in \Omega$ such that $U(\Omega, \zeta_1) \neq \emptyset$ and $U(\Omega, \zeta_2) = \emptyset$.

The purpose of this note is to establish the following result. We denote by $D(a, r)$ the open disc of centre $a$ and radius $r$, and write $\mathbb{D} = D(0, 1)$. By a non-degenerate continuum we mean a connected compact set containing more than one element.

**Theorem 1** Let $\Omega$ be a domain of the form $\mathbb{C}\backslash(L \cup \{1\})$, where $L$ is a non-degenerate continuum in $\mathbb{C}\backslash\mathbb{D}$. Then $U(\Omega, 0) = \emptyset$.

The conjecture of Müller, Vlachou and Yavrian is thus disproved. Also, if we take $L$ to be $\overline{D}(-5/3, 1/3)$, then $U(\Omega, 0) = \emptyset$ by Theorem 1 and yet a result of the second author [22] tells us that $U(\Omega, -1/2) \neq \emptyset$ (see also Costakis and Vlachou [7]). Thus we now have an example of a domain where the existence of functions with universal Taylor series depends on the chosen centre for expansion. The result of Melas, that $U(\Omega, 0) \neq \emptyset$ if $\mathbb{C}\backslash\Omega$
is compact and connected, is now seen to be sharp in the sense that, by Theorem 1, it can fail with the removal of one additional point from the domain. Theorem 1 fails if \( L \) is allowed to be a singleton [13].

2 Proof

Let \( \Omega \) be as in the statement of Theorem 1, and suppose, for the sake of contradiction, that there exists a function \( f \) in \( U(\Omega, 0) \). We can write \( f = g + h \), where \( g \) is the singular part of the Laurent expansion of \( f \) associated with the singularity at 1, and \( h \) is holomorphic on \( \mathbb{C} \setminus L \). We denote the Taylor coefficients of \( g \) and \( h \) about 0 by \( (a_n) \) and \( (b_n) \), respectively. Since \( (S_N(f, 0)(1)) \) is dense in \( \mathbb{C} \) and \( (S_N(h, 0)(1)) \) converges, we see that \( g \) is non-zero.

Let \( \rho = \inf \{|z| : z \in L\} \) and \( 0 < \delta < \varepsilon < \rho - 1 \). The Taylor series for \( g \) and \( h \) about 0 converge absolutely in \( \mathbb{D} \) and \( D(0, \rho) \), respectively, so we can define the finite quantities

\[
\alpha_\delta = \sum_{n=0}^{\infty} \frac{|a_n|}{(1 + \delta)^n} \quad \text{and} \quad \beta_\delta = \sum_{n=0}^{\infty} |b_n| \left( \frac{\rho}{1 + \delta} \right)^n.
\]

Since \( f \in U(\Omega, 0) \), we can choose a strictly increasing sequence \( (N_k) \) of natural numbers such that

\[
S_{N_k}(g, 0)(z) + S_{N_k}(h, 0)(z) \to 0 \quad \text{as} \quad k \to \infty, \text{ uniformly on} \ L. \quad (1)
\]

On \( \overline{D}(0, \rho(1 + \varepsilon)) \) we have

\[
|S_{N_k}(h, 0)(z)| \leq \sum_{n=0}^{N_k} |b_n| \rho^n (1 + \varepsilon)^n \leq \{(1 + \varepsilon)(1 + \delta)\}^{N_k} \beta_\delta,
\]

so by (1) we can choose \( k_0 \) such that

\[
|S_{N_k}(g, 0)(z)| \leq \{(1 + \varepsilon)(1 + \delta)\}^{N_k} (\beta_\delta + 1) \quad (z \in L \cap \overline{D}(0, \rho(1 + \varepsilon)); \ k \geq k_0).
\]

We also have

\[
|S_{N_k}(g, 0)(z)| \leq \sum_{n=0}^{N_k} |a_n| (1 + \varepsilon)^n \leq \{(1 + \varepsilon)(1 + \delta)\}^{N_k} \alpha_\delta \quad (z \in \overline{D}(0, 1 + \varepsilon)),
\]

so

\[
|S_{N_k}(g, 0)(z)| \leq \{(1 + \varepsilon)(1 + \delta)\}^{N_k} \gamma_\delta \quad (z \in A_\varepsilon; \ k \geq k_0), \quad (2)
\]

where \( \gamma_\delta = \max\{\alpha_\delta, \beta_\delta + 1\} \) and

\[
A_\varepsilon = \overline{D}(0, 1 + \varepsilon) \cup [L \cap \overline{D}(0, \rho(1 + \varepsilon))].
\]
Let $G_\varepsilon$ denote the Green function for the domain $D_\varepsilon = (\mathbb{C} \cup \{\infty\}) \setminus A_\varepsilon$ with pole at infinity. Then

$$G_\varepsilon(z) - \log |z| \to -\log \mathcal{C}(A_\varepsilon) \quad (|z| \to \infty),$$

where $\mathcal{C}(A)$ denotes the logarithmic capacity of a set $A$ (see Section 5.8 of [1], or Section 5.2 of [21]). Thus we can choose $r_{\delta,\varepsilon} > \max\{|z| : z \in L\}$ such that

$$G_\varepsilon(z) \leq \log |z| - \log \mathcal{C}(A_\varepsilon) + \delta \quad (|z| \geq r_{\delta,\varepsilon}).$$

(3)

Bernstein’s lemma (Theorem 5.5.7 in [21]) tells us that any polynomial $q$ of degree $n \geq 1$ satisfies

$$\left( \frac{|q(z)|}{\max_{A_\varepsilon} |q|} \right)^{1/n} \leq e^{G_\varepsilon(z)} \quad (z \in D_\varepsilon \setminus \{\infty\}).$$

Applying this inequality to the polynomial $S_{N_k}(g, 0)$, and using (2) and then (3), we obtain

$$|S_{N_k}(g, 0)(z)| \leq \left\{ \frac{(1 + \varepsilon)(1 + \delta)e^\delta}{\mathcal{C}(A_\varepsilon)} \right\}^{N_k} \gamma_\delta \quad (|z| \geq r_{\delta,\varepsilon}; k \geq k_0).$$

We next adapt an argument from pp.498,499 of Gehlen [8]. Let $\nu \in (0, 1)$. Since

$$|a_n|^{1/n} = \left| \frac{1}{2\pi i} \int_{|z|=r_{\delta,\varepsilon}} \frac{S_{N_k}(g, 0)(z)}{z^{n+1}} \, dz \right|^{1/n},$$

we obtain

$$\limsup_{k \to \infty} \max_{\nu N_k \leq n \leq N_k} |a_n|^{1/n} \leq \left\{ \frac{(1 + \varepsilon)(1 + \delta)e^\delta}{\mathcal{C}(A_\varepsilon)} \right\}^{1/\nu} \gamma_\delta^{1/\nu} r_{\delta,\varepsilon}^{1/\nu-1} = \lambda, \text{ say.}$$

(4)

Since $L$ is a non-degenerate continuum that intersects $\{|z| = \rho\}$, we have

$$\mathcal{C}(L \cap \overline{D}(0, \rho(1 + \varepsilon))) > 0$$

and so

$$\mathcal{C}(A_\varepsilon) > \mathcal{C}(\overline{D}(0, 1 + \varepsilon)) = 1 + \varepsilon.$$
theorem (Theorem 11.2.2 in Hille [10]) tells us that there is an entire function \( F \) of exponential type \( 0 \) such that \( F(n) = a_n \) for all \( n \geq 0 \). However, Theorem V of Pólya [20] says that, for any \( \mu > 0 \), however small, such a function \( F \) has the property that the sequence \( \{ n \in \mathbb{N} : |F(n)| > e^{-\mu n} \} \) is of density 1. This contradicts (4) with \( \lambda < 1 \). Thus our original assumption, that there exists \( f \) in \( U(\Omega, 0) \), must be false, and the proof of the theorem is complete.

**Remarks.** 1) The assumption that \( L \) is a continuum can be relaxed. It is enough to suppose that \( L \) is a compact subset of \( \mathbb{C} \setminus \mathbb{D} \) such that \( \mathbb{C}(D(0, \rho^2) \cap L) > 0 \) where \( \rho = \inf \{|z| : z \in L\} \).
2) The proof actually shows that there is no holomorphic function \( f \) on \( \Omega \) such that \( (S_N(f, 0)) \) is divergent at \( z = 1 \) and has a subsequence that is uniformly bounded on \( L \).

**References**


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