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BOUNDS ON THE LEVELS OF COMPOSITION ALGEBRAS

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ABSTRACT

Certain families of quaternion and octonion algebras are conjectured to be of level and sublevel n . A proof of this conjecture is offered in the case where n is a power of two. Hoffmann's proof of the existence of infinitely many new values for the level of a quaternion algebra is generalised and adapted. Alternative constructions of quaternion and octonion algebras are introduced and justified in the case where n is a multiple of a two power.

1. Introduction

In [13], Lewis constructed quaternion algebras of level 2^k and $2^k + 1$, for all integers $k \geq 0$. Laghribi and Mammone recovered these values as the level of a quaternion algebra using function field techniques (see [10]). Pumplün employed this methodology in [18] to construct octonion algebras of level 2^k and $2^k + 1$. By constructing a quaternion algebra of sublevel 3, Krüskemper and Wadsworth produced the first example of a quaternion algebra whose sublevel was not a power of two (see [9]). The octonionic analogue of this result was established in [15]. The problems of determining which values are realisable as the level or sublevel of a quaternion or octonion algebra remain open.

In this article, we propose a natural construction of quaternion and octonion algebras to be of level and sublevel n for all n . We prove this conjecture for $n = 2^k$. In addition, we offer some generalisations of a recent argument of Hoffmann, proving the existence of infinitely many quaternion algebras of new sublevel values, and infinitely many octonion algebras of new level and sublevel values. In doing so, we produce the first examples of composition algebras whose known level is not of the form 2^k or $2^k + 1$ for some k . We conclude by introducing and studying a previously unconsidered family of composition algebras, which we suggest to be of level and sublevel $l \cdot 2^k$.

2. Preliminaries

All fields F are assumed to be of characteristic $\neq 2$, with F_0 denoting those fields that are formally real.

A *composition algebra* C is a unital F -algebra endowed with a non-degenerate quadratic form q that allows composition, that is $q(x \cdot y) = q(x) \cdot q(y)$ for all $x, y \in C$.

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Hurwitz proved that composition algebras are necessarily of dimension 1, 2, 4 or 8. The composition algebras of dimension 2 are the quadratic étale F -algebras, while those of dimension 4 are the (non-commutative) quaternion algebras, and those of dimension 8 are the (non-commutative and non-associative) octonion algebras. For $a, b \in F^\times$, the *quaternion algebra* $Q = \left(\frac{a, b}{F}\right)$ over F is a 4-dimensional F -vector space with basis $\{1, i, j, k\}$, where $i^2 = a, j^2 = b$ and $ij = -ji = k$. Analogously, for $a, b, c \in F^\times$, the *octonion algebra* $\left(\frac{a, b, c}{F}\right)$ over F is an 8-dimensional F -vector space with basis $\{1, i, j, k, e, ie, je, ke\}$, with multiplication, as described by the *Cayley-Dickson doubling process* (see [18]), satisfying $i^2 = a, j^2 = b$ and $e^2 = c$.

We define the *level* of a composition algebra C , denoted $s(C)$, as the least integer n such that -1 is a sum of n squares in C . If no such integer exists, we say that $s(C) = \infty$. The *sublevel* of C , denoted $\underline{s}(C)$, is the least positive integer n for which 0 is a sum of $n + 1$ squares of elements in C . If 0 is not expressible in this manner, we say that $\underline{s}(C) = \infty$. Note that $\underline{s}(C) \leq s(C)$.

In [17], Pfister showed that the level of a field, if finite, is a power of two, and moreover that any prescribed power of two may be realised as the level of a field. This classification extends to the case where C is a quadratic étale F -algebra. Furthermore, as composition algebras containing zero divisors are necessarily of level 1, we may restrict our attention to quaternion and octonion algebras that are division.

The quadratic form $T_C : C \rightarrow F$, given by $T_C(x) = \frac{1}{2}(x^2 + \overline{x^2})$, where $\overline{}$ denotes conjugation, is known as the *trace form*. For $C = \left(\frac{a, b}{F}\right)$, we have $T_C \simeq \langle 1, a, b, -ab \rangle$, whereas for $C = \left(\frac{a, b, c}{F}\right)$, $T_C \simeq \langle 1, a, b, -ab, c, -ac, -bc, abc \rangle$, where $\simeq$ denotes isometry. We define the *pure trace form* of a composition algebra C , denoted T_P , via the relation $T_C = \langle 1 \rangle \perp T_P$. As we will see, these trace forms are naturally related to squares in composition algebras.

For $n > 1$, the *function field* of a regular n -dimensional F -quadratic form $\varphi \not\simeq \langle 1, -1 \rangle$, denoted $F(\varphi)$, is defined to be the quotient field of the integral domain $F[X_1, \dots, X_n]/(\varphi(X_1, \dots, X_n))$. The form φ is *isotropic* if there exists a non-zero vector x such that $\varphi(x) = 0$. By construction, every quadratic form is isotropic over its function field. An F -form ψ is a *subform* of φ if there exists an F -form χ such that $\varphi \simeq \psi \perp \chi$.

If an F -form τ is isotropic over $F(\varphi)$, then it is isotropic over $F(\psi)$, where ψ is a subform of φ . This observation, a direct consequence of [7, proposition 3.1] and [7, theorem 3.3], will be repeatedly employed throughout this paper.

The *Witt index* of φ , denoted $i_W(\varphi)$, is the dimension of a maximal totally isotropic subform of φ . The *first Witt index* of φ , denoted $i_1(\varphi)$, is the Witt index of φ over its function field. The *essential dimension* of φ is given by the following relation: $\dim_{es}(\varphi) = \dim(\varphi) - i_1(\varphi) + 1$. The *value set* of φ , denoted $D_F(\varphi)$, is the set of elements of $F^\times$ that are represented by φ . We use $m \times \varphi$ to denote the orthogonal sum $\varphi \perp \dots \perp \varphi$ of m copies of φ .

Karpenko and Merkurjev employed advanced algebro-geometric techniques involving Chow groups to prove the following powerful result, which we will invoke:

Theorem 2.1. [6, theorem 4.1] *Let φ and ψ be anisotropic over F and suppose that τ is isotropic over $F(\varphi)$. Then*

- (i) $\dim_{es}(\varphi) \leq \dim_{es}(\tau)$;
- (ii) *moreover, the equality $\dim_{es}(\varphi) = \dim_{es}(\tau)$ holds if and only if φ is isotropic over $F(\tau)$.*

For an overview of function fields of quadrics, or for further definitions and notation regarding quadratic forms, we refer the reader to [11].

3. Results

As motivated above, we will alternately consider quaternion and octonion algebras that are division, denoting them by C . To discriminate between these alternatives, we will use Q to denote a quaternion division algebra and O an octonion division algebra. We will denote the pure trace form of the algebra in question by T_P .

Since $D_F(n \times T_P) = \{\sum_{i=1}^n p_i^2 \mid p_i \in C, p_i \text{ pure for all } i\}$, the isotropy of $\langle 1 \rangle \perp n \times T_P$ implies that $s(C) \leq n$. Hence, it seems reasonable to suggest that if one were to construct composition algebras, subject to $\langle 1 \rangle \perp n \times T_P$ being isotropic, in a *suitably general fashion*, then their level and sublevel might actually equal n . Thus, we will consider

$$Q(n) := \left(\frac{x, y}{F} \right) \otimes_F F(\langle 1 \rangle \perp n \times T_P)$$

and

$$O(n) := \left(\frac{x, y, z}{F} \right) \otimes_F F(\langle 1 \rangle \perp n \times T_P),$$

where F denotes $F_0(x, y)$ and $F_0(x, y, z)$ respectively, and the F -quadratic form T_P denotes the pure trace form of the respective algebra, positing the following:

Conjecture 3.1. $s(Q(n)) = \underline{s}(Q(n)) = s(O(n)) = \underline{s}(O(n)) = n$ for all n .

As $\underline{s}(C) \leq s(C)$, it is clear that if the sublevel of $Q(n)$ or $O(n)$ equals n , then their level and sublevel coincide.

We can prove the conjecture, via elementary means, in the case where n is a power of two:

Theorem 3.2. $s(Q(n)) = \underline{s}(Q(n)) = s(O(n)) = \underline{s}(O(n)) = n$, for $n = 2^k$ where $k \geq 0$.

PROOF. Suppose $\underline{s}(Q(2^k)) \leq 2^k - 1$. For $k \geq 2$, $\langle 1 \rangle \perp (2^k - 1) \times T_P$ is isotropic over $F(\varphi)$ by [15, theorem 3.5], where $\varphi := \langle 1 \rangle \perp 2^k \times T_P$. Since $\psi := \langle 1 \rangle \perp 2^k \times \langle x \rangle$ is a subform of φ , $\langle 1 \rangle \perp (2^k - 1) \times T_P$ is isotropic over $F(\psi)$. Springer's theorem implies that either $\langle 1 \rangle \perp (2^k - 1) \times \langle x \rangle$ or $(2^k - 1) \times \langle 1, -x \rangle$ is isotropic over $F_0(x)(\psi)$. However, $\langle 1 \rangle \perp (2^k - 1) \times \langle x \rangle$ is anisotropic over $F_0(x)(\psi)$ by [3, theorem 1]. Hence, $(2^k - 1) \times \langle 1, -x \rangle$ must be isotropic over $F_0(x)(\psi)$, implying that $2^k \times \langle 1, -x \rangle$ is

hyperbolic over $F_0(x)(\psi)$. The Cassels-Pfister Subform Theorem implies that ψ is a subform of $2^k \times \langle 1, -x \rangle$, giving the desired contradiction.

In the case where $k = 1$, we must additionally ensure that $2 \times T_P$ is anisotropic over $F(\langle 1 \rangle \perp 2 \times T_P)$. Since $\langle 1 \rangle \perp 2 \times T_P$ is not a subform of $2^k \times (\langle 1 \rangle \perp -T_P)$, an application of the Cassels-Pfister Subform Theorem confirms the anisotropy.

A similar method of argument resolves the octonionic case. ■

As remarked above, the isotropy of $\langle 1 \rangle \perp n \times T_P$ encodes n as an upper bound for the level of a composition algebra. Indeed, for $n = 2^k - 1$, we have that $s(C) \leq n$ if and only if $\langle 1 \rangle \perp n \times T_P$ is isotropic. This does not hold in general however, as is evidenced by Koprowski's [8] construction of a quaternion algebra of level 2^k such that $\langle 1 \rangle \perp 2^k \times T_P$ is anisotropic for all k , and Pumplün's analogous octonion algebra (see [18]). For $n = 2^k$, we do have the following equivalence however:

Theorem 3.3. $s(C) \leq 2^k \iff (2^k + 1) \times \langle 1 \rangle \perp (2^k - 1) \times T_P$ is isotropic, where $k \geq 2$ if C is a quaternion algebra and $k \geq 3$ if C is an octonion algebra.

PROOF. By generalising [14, lemma 3] and [12, theorem 2.2] to the class of composition algebras, we yield that $s(C) \leq 2^k \iff (2^k + 1) \times \langle 1 \rangle \perp (2^k - 1) \times T_P$ or $\langle 1 \rangle \perp 2^k \times T_P$ is isotropic, where $k \geq 0$. Hence, it suffices to prove that the isotropy of $\langle 1 \rangle \perp 2^k \times T_P$ implies that of $(2^k + 1) \times \langle 1 \rangle \perp (2^k - 1) \times T_P$, for k as above. Suppose $\langle 1 \rangle \perp 2^k \times T_P$ is isotropic, implying that $2^k \times T_C \simeq (2^k \times \langle 1 \rangle) \otimes T_C$ is isotropic. We know that $i_W(2^k \times T_C) \geq 2^k$ by [1, proposition 1.4]. For k as above, $2^k \times \langle 1 \rangle \perp (2^k - 1) \times T_P$ is a subform of $2^k \times T_C$ of codimension $< 2^k$, and thus is also isotropic. ■

Since $Q(n)$ and $O(n)$ coincide with constructions of Laghribi and Mammone [10] and Pumplün [18] when $n = 2^k + 1$ for $k \geq 1$, the level component of Conjecture 3.1 has been established for such values of n (see [10, theorem 2.1] and [18, theorem 3.1]). Indeed, both results extend to the case where $k = 0$ by Theorem 3.2. Invoking Theorem 3.3, we offer succinct proofs of these results:

Theorem 3.4. $s(Q(n)) = s(O(n)) = n$, for $n = 2^k + 1$.

PROOF. Suppose $s(Q(2^k + 1)) \leq 2^k$. Hence $(2^k + 1) \times \langle 1 \rangle \perp (2^k - 1) \times T_P$ is isotropic over $F(\langle 1 \rangle \perp (2^k + 1) \times T_P)$ by Theorem 3.3 (we can assume that $k \geq 2$ by Theorem 3.8). Since $y(2^k + 1) \times \langle 1, -x \rangle$ is a subform of $\langle 1 \rangle \perp (2^k + 1) \times T_P$, $(2^k + 1) \times \langle 1 \rangle \perp (2^k - 1) \times T_P$ is isotropic over $F(y(2^k + 1) \times \langle 1, -x \rangle)$, and hence over $F((2^k + 1) \times \langle 1, -x \rangle)$, since $F(y(2^k + 1) \times \langle 1, -x \rangle) \cong F((2^k + 1) \times \langle 1, -x \rangle)$. Springer's theorem implies that either $(2^k + 1) \times \langle 1 \rangle \perp (2^k - 1) \times \langle x \rangle$ or $(2^k - 1) \times \langle 1, -x \rangle$ is isotropic over $F_0(x)((2^k + 1) \times \langle 1, -x \rangle)$. However, both forms are anisotropic by [3, theorem 1]. Hence $s(Q(2^k + 1)) \not\leq 2^k$.

The same method resolves the octonionic case. ■

It is possible to say more regarding the levels and sublevels of $Q(n)$ and $O(n)$

by employing the powerful machinery of Theorem 2.1. At a seminar in University College Dublin [4], Hoffmann kindly communicated his method of showing the existence of infinitely many quaternion algebras whose level is neither 2^k nor $2^k + 1$ for some k . The family of quaternion algebras that he considers is $\{Q(n)\}$. The following result represents a key step in his argument:

Lemma 3.5. [5, lemma 4.1 (iii)] $i_1(\langle 1 \rangle \perp n \times T_P) = 1$ for all n , where T_P represents the pure trace form of $Q(n)$.

We have the corresponding result for $O(n)$:

Lemma 3.6. $i_1(\langle 1 \rangle \perp n \times T_P) = 1$ for all n , where T_P represents the pure trace form of $O(n)$.

PROOF. For P an ordering of F_0 , let P' represent an extension to F such that x, y and z are negative. Hence, $\langle 1 \rangle \perp n \times T_P$ is indefinite with respect to P' , implying that P' extends to $F(\langle 1 \rangle \perp n \times T_P)$ by [2, theorem 3.5 and remark 3.6]. Now, since $\langle 1 \rangle \perp n \times T_P$ has only one positive coefficient with respect to P' , we can conclude that $i_1(\langle 1 \rangle \perp n \times T_P) = 1$. ■

Hoffmann invokes lemma 3.5 to prove the following result:

Theorem 3.7. [5, corollary 4.3] For $n = m + 1 + \lfloor \frac{m}{3} \rfloor$, $s(Q(n)) \in [m + 1, n]$.

In the above, $m + 1 + \lfloor \frac{m}{3} \rfloor$ is the least value of n which ensures that the essential dimension of $\langle 1 \rangle \perp n \times T_P$ is as great as that of $\langle 1 \rangle \perp m \times T_{Q(n)}$, thereby ensuring that $s(Q(n)) \not\leq m$ via Theorem 2.1. For the sake of consistency, we offer the following reformulation of the above result: $s(Q(n)) \in [n - \lfloor \frac{n}{4} \rfloor, n]$ for all n .

We next invoke Lemma 3.6 to similar effect, obtaining analogous bounds for $s(O(n))$. In addition, we apply Hoffmann's methodology to considerations of sub-levels:

Theorem 3.8. (i) $s(O(n)) \in [n - \lfloor \frac{n}{8} \rfloor, n]$ for all n .

(ii) $s(Q(n)) \in [n - \lfloor \frac{n+3}{4} \rfloor, n]$ for all n .

(iii) $s(O(n)) \in [n - \lfloor \frac{n+7}{8} \rfloor, n]$ for all n .

PROOF. (i) Since $i_1(\langle 1 \rangle \perp n \times T_P) = 1$ for all n , if we have that $\dim(\langle 1 \rangle \perp n \times T_P) > \dim(\langle 1 \rangle \perp m \times T_{O(n)})$, then Theorem 2.1 implies that $\langle 1 \rangle \perp m \times T_{O(n)}$ remains anisotropic over $F(\langle 1 \rangle \perp n \times T_P)$, bounding $s(O(n))$ in $[m + 1, n]$. Comparing dimensions, we see that $n - \lfloor \frac{n}{8} \rfloor - 1$ is the greatest value of m such that the above inequality holds. Hence, $s(O(n)) \in [n - \lfloor \frac{n}{8} \rfloor, n]$.

Arguing as above, one may show the anisotropy of $(m + 1) \times T_{Q(n)}$ and $(m + 1) \times T_{O(n)}$ for the relevant values of m , thereby proving (ii) and (iii). ■

Part (i) of the above result implies that $s(O(6)) = 6$ and $s(O(7)) = 7$, yielding

the first examples of composition algebras whose known level is not of the form 2^k or $2^k + 1$ for some k . This resolves a further two cases of the level component of Conjecture 3.1.

Without placing any restrictions on the value that n may take, the above results represent the sharpest bounds on the level and sublevel of $Q(n)$ and $O(n)$ currently available. For a large class of values however, we may further reduce the intervals in which these quantities are known to lie, by exploiting Theorem 2.1 more fully:

Theorem 3.9. *Let k denote the 2-adic order of m .*

- (i) $s(Q(n))$ and $\underline{s}(Q(n)) \in [m, n]$ for $n \geq m + \left\lfloor \frac{m-2^k}{3} \right\rfloor + 1$.
- (ii) $s(O(n))$ and $\underline{s}(O(n)) \in [m, n]$ for $n \geq m + \left\lfloor \frac{m-2^k}{7} \right\rfloor + 1$.

PROOF. (i) [1, theorem 1.4] implies that $i_1(m \times T_{Q(n)}) \geq 2^k$. Comparing the essential dimensions of $\langle 1 \rangle \perp n \times T_P$ and $m \times T_{Q(n)}$, we see that $m + \left\lfloor \frac{m-2^k}{3} \right\rfloor + 1$ is the least value of n such that $\dim(\langle 1 \rangle \perp n \times T_P) - 1 > \dim(m \times T_{Q(n)}) - 2^k$. Hence, for $n \geq m + \left\lfloor \frac{m-2^k}{3} \right\rfloor + 1$, $m \times T_{Q(n)}$ is anisotropic over $F(\langle 1 \rangle \perp n \times T_P)$ by Theorem 2.1, implying that $\underline{s}(Q(n)) \in [m, n]$.

(ii) Invoke the analogous argument to that presented above. ■

Of course, if $n = 2^k + h$ for $h \geq 0$ sufficiently small, we may invoke our earlier arguments to yet again increase the lower bounds on the levels and sublevels of $Q(n)$ and $O(n)$ relative to those derived from the above results:

Theorem 3.10. (i) $s(Q(n)), \underline{s}(Q(n)), s(O(n))$ and $\underline{s}(O(n)) \in [2^k, n]$, where 2^k is the largest 2-power $\leq n$.

- (ii) $s(Q(n))$ and $s(O(n)) \in [2^k + 1, n]$, where 2^k is the largest 2-power $< n$.

PROOF. (i) In the proof of Theorem 3.2, we showed that $\langle 1 \rangle \perp (2^k - 1) \times T_P$ is anisotropic over $F(\langle 1 \rangle \perp 2^k \times T_P)$. For $n \geq 2^k$, $\langle 1 \rangle \perp 2^k \times T_P$ is a subform of $\langle 1 \rangle \perp n \times T_P$. Hence, $\langle 1 \rangle \perp (2^k - 1) \times T_P$ is anisotropic over $F(\langle 1 \rangle \perp n \times T_P)$, implying the result.

(ii) Similarly, this result represents a re-interpretation of Theorem 3.4. ■

For certain values of n , the lower bounds obtained for $s(Q(n)), \underline{s}(Q(n)), s(O(n))$ and $\underline{s}(O(n))$ in the above results are actually optimal with respect to the standard isotropy tests. For example, $s(Q(15))$ and $\underline{s}(Q(15)) \in [12, 15]$ as a consequence of Theorem 3.9. However, $\langle 1 \rangle \perp 12 \times T_{Q(15)}$ is isotropic over $F(\langle 1 \rangle \perp 15 \times T_P)$, since it is a subform of $16 \times T_{Q(15)}$ of codimension 15, where $i_W(16 \times T_{Q(15)}) \geq 16$ by [1, theorem 1.4]. Hence, in order to prove Conjecture 3.1 via trace form arguments, stronger consequences of $\underline{s}(Q(n))$ and $\underline{s}(O(n))$ having $n - 1$ as an upper bound are required.

3.1. Constructions for the case where $n = l \cdot 2^k$

If we seek to construct quaternion and octonion algebras of level and sublevel n for all values of n , the natural candidates to consider are $Q(n)$ and $O(n)$. However, if we restrict our attention to certain values of n , other constructions appear to be equally, if not more, appropriate.

We offer the following natural extension of [12, theorem 2.2]:

Theorem 3.11. *If $(2^h + 1) \times \langle 1 \rangle \perp (l \cdot 2^k - 1) \times T_P$ is isotropic, where h denotes the 2-adic order of $l \cdot 2^k$, then $s(C) \leq l \cdot 2^k$.*

PROOF. Consider the case where C is a quaternion algebra. If $(2^h + 1) \times \langle 1 \rangle \perp (l \cdot 2^k - 1) \times T_P$ is isotropic, then there exists $-A \in D_F((\langle 1 \rangle \perp (l \cdot 2^k - 1) \times T_P))$ for some nonzero $A \in D_F(2^h \times \langle 1 \rangle)$. Hence, for some $\alpha \in F$ and $B, C, D \in D_F((l \cdot 2^k - 1) \times \langle 1 \rangle) \cup \{0\}$, we have $-A = \alpha^2 + aB + bC - abD$, whereby $-1 = \frac{1}{A^2}(\alpha^2 A + aAB + bAC - abAD)$. Let $A = \sum_{\lambda=1}^{2^h} x_\lambda^2$. We show that there exist $y_\lambda \in F$ such that $\sum_{\lambda=1}^{l \cdot 2^k} y_\lambda^2 = AB$ and $\vec{x} \cdot \vec{y} = 0$, where $\vec{x} \cdot \vec{y}$ denotes the scalar product of the two vectors (taking $x_{2^h+1}, \dots, x_{l \cdot 2^k} = 0$ if $2^h < l \cdot 2^k$). If $B = 0$, let each $y_\lambda = 0$. If $B \neq 0$, then $\langle A, AB \rangle \cong A \cdot \langle 1, B \rangle$ is a subform of $A \cdot (l \cdot 2^k) \times \langle 1 \rangle \cong (l \cdot 2^k) \times \langle 1 \rangle$, since $B \in D_F((l \cdot 2^k - 1) \times \langle 1 \rangle)$ and $A \cdot 2^h \times \langle 1 \rangle \cong 2^h \times \langle 1 \rangle$. Therefore such a $\vec{y}$ exists. Similarly $\vec{z}, \vec{w}$ exist such that $\sum_{\lambda=1}^{l \cdot 2^k} z_\lambda^2 = AC$, $\sum_{\lambda=1}^{l \cdot 2^k} w_\lambda^2 = AD$ and $\vec{x} \cdot \vec{z} = \vec{x} \cdot \vec{w} = 0$. It follows that $\sum_{\lambda=1}^{l \cdot 2^k} \left(\frac{\alpha x_\lambda}{A} + \frac{y_\lambda}{A} i + \frac{z_\lambda}{A} j + \frac{w_\lambda}{A} k \right)^2 = \frac{1}{A^2}(\alpha^2 A + aAB + bAC - abAD) = -1$. Therefore $s(C) \leq l \cdot 2^k$.

The same method of proof applies in the octonionic case. ■

We will consider

$$Q' := \left(\frac{x, y}{F} \right) \otimes_F F((2^h + 1) \times \langle 1 \rangle \perp (l \cdot 2^k - 1) \times T_P)$$

and

$$O' := \left(\frac{x, y, z}{F} \right) \otimes_F F((2^h + 1) \times \langle 1 \rangle \perp (l \cdot 2^k - 1) \times T_P),$$

where h is the 2-adic order of $l \cdot 2^k$, F denotes $F_0(x, y)$ and $F_0(x, y, z)$ respectively, and the F -quadratic form T_P denotes the pure trace form of the respective algebra. As before, we posit the natural conjecture:

Conjecture 3.12. $s(Q') = \underline{s}(Q') = s(O') = \underline{s}(O') = l \cdot 2^k$ for all l and k .

In the case where l is a 2-power, Q' represents a construction of Laghribi and Mammone (see [10]), with O' coinciding with one of Pumplün (see [18]), allowing us to conclude that $s(Q') = s(O') = 2^k$. Moreover, $\underline{s}(Q') = \underline{s}(O') = 2^k$ for $k \geq 2$, by [12, theorem 2.5] and [15, theorem 3.11], with an ad-hoc argument proving the case where $k = 1$. Thus, the conjecture holds for l a power of two.

As was the case with $Q(n)$ and $O(n)$, the ‘bounding forms’ for these constructions are of minimal first Witt index:

Theorem 3.13. $i_1((2^h + 1) \times \langle 1 \rangle \perp (l \cdot 2^k - 1) \times T_P) = 1$, for T_P the pure trace form of Q' or O' .

PROOF. We begin by considering the quaternion algebra case. Let ψ denote $(2^h + 1) \times \langle 1 \rangle \perp (l \cdot 2^k - 1) \times \langle x \rangle$, with ϑ denoting $(2^h + 1) \times \langle 1 \rangle \perp (l \cdot 2^k - 1) \times T_P$.

In the case where l is a 2-power, we recall [10, proposition 3.4], which states that $2^k \times \langle 1 \rangle \perp (2^k - 1) \times T_P$ is anisotropic over $F((2^k + 1) \times \langle 1 \rangle \perp (2^k - 1) \times T_P)$. Since $2^k \times \langle 1 \rangle \perp (2^k - 1) \times T_P$ is a 1-codimensional subform of $(2^k + 1) \times \langle 1 \rangle \perp (2^k - 1) \times T_P$, Theorem 2.1 allows us to conclude that $i_1((2^k + 1) \times \langle 1 \rangle \perp (2^k - 1) \times T_P) = 1$, for all k .

Thus, we may assume that l is odd, whereby k becomes the 2-adic order of $l \cdot 2^k$.

Suppose there exist k and l such that $i_1(\vartheta) > 1$. Hence $2^k \times \langle 1 \rangle \perp (l \cdot 2^k - 1) \times T_P$ is isotropic over $F(\vartheta)$, since it is a 1-codimensional subform of ϑ . Since ψ is a subform of ϑ , $2^k \times \langle 1 \rangle \perp (l \cdot 2^k - 1) \times T_P$ is isotropic over $F(\psi)$. Springer's Theorem implies that either $2^k \times \langle 1 \rangle \perp (l \cdot 2^k - 1) \times \langle x \rangle$ or $(l \cdot 2^k - 1) \times \langle 1, -x \rangle$ is isotropic over $F_0(x)(\psi)$.

Suppose $(l \cdot 2^k - 1) \times \langle 1, -x \rangle$ becomes isotropic over $F_0(x)(\psi)$. Hence $2^n \times \langle 1, -x \rangle$ becomes hyperbolic over $F_0(x)(\psi)$, where $2^n > l \cdot 2^k - 1$. The Cassels-Pfister Subform Theorem implies that ψ is a subform of $2^n \times \langle 1, -x \rangle$, which is clearly false.

Thus $2^k \times \langle 1 \rangle \perp (l \cdot 2^k - 1) \times \langle x \rangle$ must be isotropic over $F_0(x)(\psi)$. Hence, $\dim_{es}(\psi) = \dim_{es}(2^k \times \langle 1 \rangle \perp (l \cdot 2^k - 1) \times \langle x \rangle)$ by Theorem 2.1.

Consider the form $2^k \times \langle 1 \rangle \perp l \cdot 2^k \times \langle x \rangle \simeq 2^k \times \langle 1 \rangle \otimes (\langle 1 \rangle \perp l \times \langle x \rangle)$. Now $i_1(2^k \times \langle 1 \rangle \perp l \cdot 2^k \times \langle x \rangle) \geq 2^k$ by [1, theorem 1.4]. In addition, choosing an ordering P of $F_0(x)$ such that x is negative implies that $i_1(2^k \times \langle 1 \rangle \perp l \cdot 2^k \times \langle x \rangle) \leq 2^k$ by [2, theorem 3.5 and remark 3.6]. Thus $i_1(2^k \times \langle 1 \rangle \perp l \cdot 2^k \times \langle x \rangle) = 2^k$, implying that $2^k \times \langle 1 \rangle \perp (l \cdot 2^k - 1) \times \langle x \rangle$ is isotropic over $F_0(2^k \times \langle 1 \rangle \perp l \cdot 2^k \times \langle x \rangle)$ and hence that $\dim_{es}(2^k \times \langle 1 \rangle \perp l \cdot 2^k \times \langle x \rangle) = \dim_{es}(2^k \times \langle 1 \rangle \perp (l \cdot 2^k - 1) \times \langle x \rangle)$ by Theorem 2.1.

We may therefore conclude that $i_1(\psi) = 2^k$. Consequently $(2^k + 1) \times \langle 1 \rangle \perp ((l - 1) \cdot 2^k) \times \langle x \rangle$ is isotropic over $F_0(\psi)$, since it is a subform of ψ of codimension $2^k - 1$. Invoking Theorem 2.1 yields that $i_1((2^k + 1) \times \langle 1 \rangle \perp ((l - 1) \cdot 2^k) \times \langle x \rangle) = 1$.

Since l is odd, 2^{k+1} divides $(l - 1) \cdot 2^k$. Thus $2^{k+1} \times \langle 1 \rangle \perp ((l - 1) \cdot 2^k) \times \langle x \rangle \simeq 2^{k+1} \times \langle 1 \rangle \otimes (\langle 1 \rangle \perp \frac{l-1}{2} \times \langle x \rangle)$. Now $i_1(2^{k+1} \times \langle 1 \rangle \perp ((l - 1) \cdot 2^k) \times \langle x \rangle) = 2^{k+1}$ by [1, theorem 1.4] and [2, theorem 3.5 and remark 3.6]. Hence $(2^k + 1) \times \langle 1 \rangle \perp ((l - 1) \cdot 2^k) \times \langle x \rangle$ becomes isotropic over $F_0(2^{k+1} \times \langle 1 \rangle \perp ((l - 1) \cdot 2^k) \times \langle x \rangle)$, whereby $\dim_{es}(2^{k+1} \times \langle 1 \rangle \perp ((l - 1) \cdot 2^k) \times \langle x \rangle) = \dim_{es}((2^k + 1) \times \langle 1 \rangle \perp ((l - 1) \cdot 2^k) \times \langle x \rangle)$ by Theorem 2.1. Thus we conclude that $i_1((2^k + 1) \times \langle 1 \rangle \perp ((l - 1) \cdot 2^k) \times \langle x \rangle) = 2^k + 1$, contradicting the supposition that $2^k \times \langle 1 \rangle \perp (l \cdot 2^k - 1) \times \langle x \rangle$ is isotropic over $F_0(x)(\psi)$.

Hence $2^k \times \langle 1 \rangle \perp (l \cdot 2^k - 1) \times T_P$ is in fact anisotropic over $F(\vartheta)$, contradicting the supposition that there exist k and l such that $i_1(\vartheta) > 1$.

A similar method of argument resolves the octonion algebra case. ■

In light of the above result, we note that

$$\dim_{es}((2^h + 1) \times \langle 1 \rangle \perp (l \cdot 2^k - 1) \times T_P) > \dim_{es}(\langle 1 \rangle \perp (l \cdot 2^k) \times T_P),$$

provided that $k \geq 1$ (resp. $k \geq 2$) in the quaternionic (resp. octonionic) case.

Thus, Theorem 2.1 allows us to place lower bounds on the level and sublevel of Q' and O' that are greater than those we can currently impose on the level and sublevel of $Q(l \cdot 2^k)$ and $O(l \cdot 2^k)$. For example, while we currently cannot make any stronger statement regarding $Q(48)$ than $\underline{s}(Q(48))$ and $s(Q(48)) \in [36, 48]$, we have that $\underline{s}(Q')$ and $s(Q') \in [40, 48]$ for $l = 3$ and $k = 4$.

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