Sets of determination for the Nevanlinna class

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Abstract

This paper characterizes the subsets $E$ of the unit disc $D$ with the property that $\sup_{E} |f| = \sup_{D} |f|$ for all functions $f$ in the Nevanlinna class.

1 Introduction

Let $\mathcal{A}$ be a collection of holomorphic functions on the unit disc $D$, and let $\mathbb{T}$ denote the unit circle. A set $E \subset D$ is called a set of determination for $\mathcal{A}$ if $\sup_{E} |f| = \sup_{D} |f|$ for all $f \in \mathcal{A}$. Brown, Shields and Zeller [3] have shown that $E$ is a set of determination for $H^\infty$, the space of bounded holomorphic functions on $D$, if and only if almost every point of $\mathbb{T}$ can be approached nontangentially by a sequence of points in $E$. Massaneda and Thomas [6] have observed that the same characterization remains valid when $\mathcal{A}$ is the Smirnov class $\mathcal{N}^+$. However, the situation is more complicated for the Nevanlinna class $\mathcal{N}$, which consists of all holomorphic functions $f$ on $D$ that satisfy

$$\sup_{0 < r < 1} \int_{0}^{2\pi} \log^+ |f(re^{i\theta})| \, d\theta < \infty.$$ 

This is the main focus of [6], where a variety of conditions are shown to be either necessary or sufficient for $E$ to be a set of determination for $\mathcal{N}$, and some illustrative special cases are examined. (See also Stray [7], p.256.) The purpose of this paper is to give a complete characterization of such sets.

First we recall a related result of Hayman and Lyons [5] for the harmonic Hardy space $h^1$, which consists of those functions on $D$ that can be expressed as the difference of two positive harmonic functions. For $n \in \mathbb{N}$ and $0 \leq m < 2^{n+4}$ let

$$z_{m,n} = (1 - 2^{-n}) \exp(2\pi im/2^{n+4})$$

and

$$S_{m,n} = \left\{ re^{i\theta} : 2^{-n-1} \leq 1 - r \leq 2^{-n} \quad \text{and} \quad \frac{2\pi m}{2^{n+4}} \leq \theta \leq \frac{2\pi(m+1)}{2^{n+4}} \right\},$$

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and let $E_{m,n} = E \cap S_{m,n}$. The Poisson kernel for $\mathbb{D}$ is given by

$$P(z, w) = \frac{1 - |z|^2}{|z - w|^2} \quad (z \in \mathbb{D}, w \in \mathbb{T}).$$

**Theorem A** [5] Let $E \subset \mathbb{D}$. The following conditions are equivalent:

(a) $\sup_{E} h = \sup_{\mathbb{D}} h$ for all $h \in h^1$;

(b) $\sum_{m, n \neq 0} 2^{-n} P(z_{m,n}, w) = \infty$ for every $w \in \mathbb{T}$.

For any set $A$ which is contained in a disc of radius less than 1, and any $t \geq 0$, we define a capacity-related quantity $Q(A, t)$ as follows. We put $Q(A, t) = 0$ if either $t = 0$ or $A = \emptyset$; otherwise,

$$Q(A, t) = \min\{k \in \mathbb{N} : \exists \xi_1, \ldots, \xi_k \in \mathbb{C} \text{ such that } \sum_{j=1}^{k} \log \frac{1}{|z - \xi_j|} \geq t \quad (z \in A)\}.$$ 

Clearly $Q(\cdot, t)$ is translation-invariant and $Q(\{\zeta\}, \cdot) = \chi(0, \infty)$ for any $\zeta \in \mathbb{C}$. Also,

$$Q(\{\zeta_1, \zeta_2\}, t) = \begin{cases} 0 & \text{if } t = 0 \\ 1 & \text{if } |\zeta_1 - \zeta_2| \leq 2e^{-t} \text{ and } t > 0 \\ 2 & \text{otherwise} \end{cases}$$

and, if $A$ is a disc of radius of $r < 1$, then $Q(A, t)$ is the least integer $k$ satisfying $k \geq t/\log(1/r)$. We use $[t]$ to denote the integer part of a non-negative number $t$, and $tA$ to denote the set \{tz : z \in A\}. Our characterization of sets of determination for the Nevanlinna class is as follows.

**Theorem 1** Let $E \subset \mathbb{D}$. The following conditions are equivalent:

(a) $\sup_{E} |f| = \sup_{\mathbb{D}} |f|$ for all $f \in \mathcal{H}$;

(b) $\sum_{m, n} 2^{-n} Q(2^n E_{m,n}, [P(z_{m,n}, w)]) = \infty$ for every $w \in \mathbb{T}$.

Since

$$\log \frac{2^{-n}}{|z - z_{m,n}|} \geq -\frac{1}{2} \log \left(\frac{\pi}{8}\right)^2 + \left(\frac{1}{2}\right)^2 \geq \frac{1}{3} \quad (z \in S_{m,n}),$$

we have

$$3P(z_{m,n}, w) \log \frac{2^{-n}}{|z - z_{m,n}|} \geq P(z_{m,n}, w) \quad (z \in S_{m,n}, w \in \mathbb{T}).$$

By separate consideration of the cases $P(z_{m,n}, w) \geq 1$ and $P(z_{m,n}, w) < 1$, we see that

$$Q(2^n E_{m,n}, [P(z_{m,n}, w)]) \leq 4P(z_{m,n}, w).$$

(1)
Applying this inequality to terms where \( E_{m,n} \neq 0 \), it is now clear that condition (b) of Theorem 1 implies the corresponding condition of Theorem A. It is not difficult to check that condition (a) of Theorem 1 is equivalent to the assertion that, if \( \log |f| \leq h \) on \( E \), where \( f \in \mathcal{N} \) and \( h \in h^1 \), then \( \log |f| \leq h \) on all of \( \mathbb{D} \) (cf. [6]).

**Examples** Let \( U = \{ z : |z - \frac{1}{2}| < \frac{1}{3} \} \) and \( F = U \cap \{ z_{m,n} \} \).

(i) The set \( E = \mathbb{D} \setminus U \) is not a set of determination (for \( \mathcal{N} \)) because the series in condition (b) of Theorem A then converges when \( w = 1 \) (cf. Example 6.2 in [5]).

(ii) Further, even \( E \cup F \) is not a set of determination because each of the sets \( F_{m,n} \) contains at most 5 points and so

\[
\sum_{m,n} 2^{-n} Q(2^n F_{m,n}, [P(z_{m,n}, 1)]) \leq 5 \sum_{z_{m,n} \in F} 2^{-n} < \infty
\]

(cf. Example 1 in [6]).

(iii) On the other hand, \( E \cup [\frac{1}{2}, 1) \) is a set of determination since

\[
Q(2^n [1 - 2^{-n}, 1 - 2^{-n-1}], [P(z_{0,n}, 1)]) = Q\left([0, \frac{1}{2}], 2^n\right)
\]

and \( \inf_n 2^{-n} Q\left([0, \frac{1}{2}], 2^n\right) > 0 \) because \([0, \frac{1}{2}]\) is non-polar.

**2 Proof of Theorem 1**

Let \( G_U(\cdot, \cdot) \) denote the Green function of an open set \( U \), let

\[
D_\rho(z) = \{ \zeta : |\zeta - z| < \rho(1 - |z|) \} \quad (z \in \mathbb{D}, 0 \leq \rho < 1),
\]

and let \( A(g, z) \) denote the mean value of a function \( g \) over the disc \( D_{1/8}(z) \).

For potential theoretic background we refer to the book [2].

Suppose firstly that condition (b) of Theorem 1 holds and let \( f \in \mathcal{N} \). We will assume that \( \sup_E |f| < \infty \), for otherwise it is trivially true that \( \sup_E |f| = \sup_\mathbb{D} |f| \). Further, multiplication by a suitable constant enables us to arrange that \( \sup_E |f| \in [0, 1] \). Now let \( a \in (-\infty, 0] \) be such that \( a \geq \log \sup_E |f| \). We can write

\[
\log |f| = h_1 - h_2 - G_\mathbb{D} \mu,
\]

where \( h_1 \) and \( h_2 \) are positive harmonic functions and \( \mu \) is a sum of unit point masses on \( \mathbb{D} \) satisfying

\[
\int (1 - |z|) d\mu(z) < \infty.
\]
Further, by addition to both $h_1$ and $h_2$, we may assume that $h_1 \geq 1$. By the Riesz-Herglotz theorem there is a Borel measure $\nu_1$ on $T$ such that

$$h_1(z) = \int P(z, w) d\nu_1(w) \quad (z \in \mathbb{D}).$$

We know that

$$h_1 - a \leq h_2 + G_{\mathbb{D}} \mu \quad \text{on} \ E. \quad (2)$$

Also,

$$G_{\mathbb{D}}(z, \xi) - \mathcal{A}(G_{\mathbb{D}}(\cdot, \xi), z) \leq G_{D_{1/8}}(z, \xi) = \log \frac{(1 - |z|)/8}{|z - \xi|} \quad (\xi \in D_{1/8}(z)) \quad (3)$$

and $G_{\mathbb{D}}(z, \xi) - \mathcal{A}(G_{\mathbb{D}}(\cdot, \xi), z) = 0$ otherwise. Let $\varepsilon \in (0, 1)$ and

$$I_\varepsilon = \{(m, n) : G_{\mathbb{D}} \mu \geq \mathcal{A}(G_{\mathbb{D}} \mu, \cdot) + \varepsilon h_1 \text{ on } E_{m,n}\},$$

and let $I'_{\varepsilon}$ denote the complementary set of pairs $(m, n)$. (We note that $(m, n) \in I_\varepsilon$ whenever $E_{m,n} = \emptyset$.) If $(m, n) \in I_\varepsilon$, then we see from (3) that

$$\varepsilon h_1(z) \leq G_{\mathbb{D}} \mu(z) - \mathcal{A}(G_{\mathbb{D}} \mu, z) = \int_{D_{1/8}(z)} (G_{\mathbb{D}}(z, \xi) - \mathcal{A}(G_{\mathbb{D}}(\cdot, \xi), z)) d\mu(\xi) \leq \int_{A_{m,n}} \log \frac{2^{-n}}{|z - \xi|} d\mu(\xi) \quad (z \in E_{m,n}),$$

where

$$A_{m,n} = \{\xi : \text{dist}(\xi, S_{m,n}) < 2^{-n-3}\}.$$

(Here we have used the fact that the diameter of $2^n A_{m,n}$ is less than 1.) By Harnack’s inequalities there is an absolute constant $c_1 > 1$ such that $h(\xi_1) \leq c_1 h(\xi_2)$ for any positive harmonic function $h$ on $\mathbb{D}$, any points $\xi_1, \xi_2 \in S_{m,n}$, and any choice of $(m, n)$. For any $w \in T$ we thus have

$$P(z_{m,n}, w) \leq \frac{c_1}{\varepsilon h_1(z_{m,n})} P(z_{m,n}, w) \int_{A_{m,n}} \log \frac{2^{-n}}{|z - \xi|} d\mu(\xi) \quad (z \in E_{m,n}),$$

and so

$$Q(2^n E_{m,n}, [P(z_{m,n}, w)]) \leq \left(\frac{c_1}{\varepsilon h_1(z_{m,n})} P(z_{m,n}, w) + 1\right) \mu(A_{m,n}).$$

Integration of the above inequality with respect to $d\nu_1(w)$ yields

$$\int Q(2^n E_{m,n}, [P(z_{m,n}, w)]) d\nu_1(w) \leq \left(\frac{c_1}{\varepsilon} + h_1(0)\right) \mu(A_{m,n}).$$
Since no point of \( \mathbb{D} \) can lie in more than 4 of the sets \( A_{m,n} \), and \( 1 - |z| > 2^{-n-2} \) when \( z \in A_{m,n} \), we see that

\[
\int \sum_{(m,n) \in I_e} 2^{-n} Q(2^n E_{m,n}, [P(z_{m,n}, w)])\,d\nu_1(w) \\
\leq 2^4 \left( \frac{c_1}{\varepsilon} + h_1(0) \right) \int (1 - |z|)d\mu(z) < \infty,
\]

so

\[
\sum_{(m,n) \in I_e} 2^{-n} Q(2^n E_{m,n}, [P(z_{m,n}, w)]) < \infty \quad \text{for } \nu_1\text{-almost every } w \in \mathbb{T},
\]

and hence, by hypothesis,

\[
\sum_{(m,n) \in I'_e} 2^{-n} Q(2^n E_{m,n}, [P(z_{m,n}, w)]) = \infty \quad \text{for } \nu_1\text{-almost every } w \in \mathbb{T}.
\]

In view of (1) we now see that

\[
\sum_{(m,n) \in I'_e} 2^{-2n} |w - z_{m,n}|^{-2} = \infty \quad \text{for } \nu_1\text{-almost every } w \in \mathbb{T}.
\] (4)

For each \((m,n) \in I'_e\) we can find \( \zeta_{m,n} \in E_{m,n} \) such that

\[ G_{\mathbb{D} \mu}(\zeta_{m,n}) < A(G_{\mathbb{D} \mu}, \zeta_{m,n}) + \varepsilon h_1(\zeta_{m,n}). \]

Let \( F = \{ \zeta_{m,n} : (m,n) \in I'_e \} \). Then

\[ (1 - \varepsilon)h_1 - a \leq h_2 + A(G_{\mathbb{D} \mu}, \cdot) \quad \text{on } F, \] (5)

in view of (2). Also, by (4),

\[
\int_{F_\rho} |w - z|^{-2} d\lambda(z) = \infty \quad (0 < \rho < 1)
\] (6)

for \( \nu_1\)-almost every \( w \in \mathbb{T} \), where \( F_\rho = \bigcup_{\zeta \in F} D_\rho(\zeta) \) and \( \lambda \) denotes area measure. At this point we could invoke Theorem 2 of [4], but for the sake of completeness we will extract the relevant reasoning in the next paragraph.

Let \( 0 < \rho < 1/8 \). If \( z' \in D_\rho(z) \), then by the mean value inequality

\[
G_{\mathbb{D} \mu}(z') \geq \frac{1}{\pi(\rho + 1/8)^2(1 - |z|)^2} \int_{\{\zeta : |\zeta - z'| < (\rho + 1/8)(1 - |z|)} G_{\mathbb{D} \mu}(\zeta)\,d\lambda(\zeta)
\]

\[
\geq \frac{1}{(\rho + 1/8)^2} A(G_{\mathbb{D} \mu}, z),
\]

and by Harnack’s inequalities

\[
\frac{1 - \rho}{1 + \rho} h_j(z) \leq h_j(z') \leq \frac{1 + \rho}{1 - \rho} h_j(z) \quad (j = 1, 2),
\]
so (5) yields

\[(1 - \varepsilon) \frac{1 - \rho}{1 + \rho} h_1 - a \leq \frac{1 + \rho}{1 - \rho} h_2 + (8\rho + 1)^2 G_{D} \mu \text{ on } F_{\rho}. \] (7)

Condition (6) is known to ensure that the reduced function $R_{P_{\gamma}(\cdot, w)}^{F_{\rho}}$, where

\[R_{\alpha}^{F_{\rho}} = \inf \{ v : v \text{ is positive and superharmonic on } D \text{ and } v \geq u \text{ on } F_{\rho} \}, \]

coincides with $P(\cdot, w)$ (see Corollary 7.4.6 in [1]). Since this condition holds \(\nu_1\)-almost everywhere on \(T\), we have

\[R_{h_1}^{F_{\rho}} = \int R_{P_{\gamma}(\cdot, w)}^{F_{\rho}} d\nu_1(w) = \int P(\cdot, w) d\nu_1(w) = h_1. \]

Also, \(h_1 \geq 1\), so \(\nu_1\) majorizes normalized arclength measure on \(T\), and we similarly have \(R_{1}^{F_{\rho}} \equiv 1\). Hence, on taking reductions over \(F_{\rho}\), we see that the inequality in (7) extends to all of \(D\). (Recall that \(a \leq 0\).) We can now let \(\rho \to 0^+\) and \(\varepsilon \to 0^+\) to see that \(\log |f| \leq a\) on \(D\). It is now clear that (b) implies (a).

Next suppose that condition (b) of Theorem 1 fails. Then there exists \(w_0 \in T\) such that

\[\sum_{m,n} 2^{-n} q_{m,n} < \infty, \text{ where } q_{m,n} = \mathcal{Q}(2^n E_{m,n}, [P(z_{m,n}, w_0)]). \] (8)

For each \(m, n\) we can choose points \(\xi_{k,m,n} (k = 1, \ldots, q_{m,n})\) such that

\[\sum_{k=1}^{q_{m,n}} \log \frac{2^{-n}}{|z - \xi_{k,m,n}|} \geq P(z_{m,n}, w_0) - 1 \quad (z \in E_{m,n}), \] (9)

and without loss of generality we can assume that \(\xi_{k,m,n}\) lies in the convex hull \(\text{conv}(S_{m,n})\) of \(S_{m,n}\). In view of (8), the Blaschke product

\[B(z) = \prod_{k,m,n} \frac{\xi_{k,m,n} - z}{|z - \xi_{k,m,n}|} \]

converges on \(D\). There is an absolute constant \(c_2 > 0\) such that

\[G_{D}(z, \xi) \geq c_2 \log \frac{2^{-n}}{|\xi - z|} \quad (z, \xi \in \text{conv}(S_{m,n})) \]

for any pair \((m, n)\). For a given pair \((m_0, n_0)\) we thus have

\[- \log |B(z)| = \sum_{k,m,n} G_{D}(z, \xi_{k,m,n}) \geq \sum_{k=1}^{q_{m_0,n_0}} G_{D}(z, \xi_{k,m_0,n_0}) \]

\[\geq c_2 \sum_{k=1}^{q_{m_0,n_0}} \log \frac{2^{-n_0}}{|\xi_{k,m_0,n_0} - z|} \quad (z \in S_{m_0,n_0}) \]
so, by (9),

\[ c_2 - \log |B(z)| \geq c_2 P(z_{m_0,n_0}, w_0) \geq \frac{c_2}{c_1} P(z, w_0) \quad (z \in E_{m_0,n_0}). \quad (10) \]

Let

\[ f(z) = B(z) \exp \left( \frac{c_2}{c_1} \left( \frac{w_0 + z}{w_0 - z} \right) \right) \quad (z \in \mathbb{D}). \]

Then \( |f(z)| \leq (c_2/c_1) P(z, w_0) \), so \( f \in \mathcal{N} \), and certainly \( f \) is unbounded on \( \mathbb{D} \). However, \( |f| \leq e^{c_2} \) on \( E \), by (10). Hence condition (a) of Theorem 1 also fails.

**References**


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