Research Repository UCD

Provided by the author(s) and University College Dublin Library in accordance with publisher policies., Please cite the published version when available.

Title	Sums of squares in certain quaternion and octonion algebras
Authors(s)	O'Shea, James
Publication date	2011-03
Publication information	Comptes Rendus Mathematique, 349 (5-6): 239-242
Publisher	Elsevier Masson SAS
Link to online version	http://dx.doi.org/10.1016/j.crma.2011.01.022
Item record/more information	http://hdl.handle.net/10197/2869
Publisher's statement	All rights reserved.
Publisher's version (DOI)	10.1016/j.crma.2011.01.022

Downloaded 2019-03-25T00:14:03Z

The UCD community has made this article openly available. Please share how this access benefits you. Your story matters! (@ucd_oa)

Some rights reserved. For more information, please see the item record link above.

SUMS OF SQUARES IN CERTAIN QUATERNION AND OCTONION ALGEBRAS

JAMES O'SHEA

Abstract. Formulae for the levels and sublevels of certain quaternion and octonion algebras are established. Corollaries concerning the equality of levels and sublevels of quaternion algebras with those of associated octonion algebras are presented.

Let R be a not necessarily associative ring with unity. The *level* and *sublevel of* R, respectively denoted by $s(R)$ and $\underline{s}(R)$, are defined as follows:

$$s(R) = \inf\{n \in \mathbb{N} \mid \text{there exist } \; x_1, \ldots, x_n \in R \; \text{ such that } \textstyle\sum_{t=1}^{n} {x_t}^2 = -1\},$$

$$\underline{s}(R) = \inf\{n \in \mathbb{N} \mid \text{there exist } \; x_1, \ldots, x_{n+1} \in R \setminus \{0\} \; \text{ such that } \; \textstyle\sum_{t=1}^{n+1} {x_t}^2 = 0\}.$$

Let F be a field of characteristic different from 2. For $a, b \in F^\times$, the *quaternion algebra* $(a,b)_F$ is a 4-dimensional F-vector space with basis $\{1, i, j, k\}$ satisfying $i^2 = a, j^2 = b$ and $ij = -ji = k$. For $a, b, c \in F^\times$, the *octonion algebra* $(a,b,c)_F$ is isomorphic to $(a,b)_F \oplus (a,b)_F e$, where $e^2 = c$, with its multiplication being determined by $(u_1, v_1)(u_2, v_2) = (u_1u_2 + c\overline{v_2}v_1, v_2u_1 + v_1\overline{u_2})$, where $u_1, u_2, v_1, v_2 \in (a,b)_F$ (here, $^-$ denotes conjugation).

The related problems of determining the numbers attainable as the levels and sublevels of quaternion and octonion algebras remain open, and motivate our investigations. Given $a, b \in F^\times$, we study whether the level (respectively, the sublevel) of $(a,b)_F$ equals that of $(a,b,x)_{F((x))}$.

We will provide a partial answer to these questions, by showing that the respective equalities hold whenever the level or sublevel of $(a,b)_F$ belongs to an associated family of intervals. Moreover, we will show that these equalities always hold for a particular class of quaternion algebras, conjectured to contain members of level and sublevel n for all $n \in \mathbb{N}$ (see [4]).

Throughout, we will employ standard concepts and notation regarding quadratic forms. Our notation coincides with that employed in [3], aside from our usage of $n \times \varphi$ to denote the orthogonal sum of $n \in \mathbb{N}$ copies of a quadratic form φ. Moreover, for $a, b \in F^\times$, we will let $k(a)$ (respectively $k(a,b)$) denote the least $n \in \mathbb{N}$ such that $n \times \langle 1, -a \rangle$ (respectively $n \times \langle 1, -a, -b, ab \rangle$) is isotropic (over F, unless stated otherwise). If such an n exists, then $n = 2^k + 1$ for some $k \in \mathbb{Z}$. Otherwise, the quantity is said to be infinite.

In order to obtain the aforementioned results, we will establish characterisations of the levels and sublevels of certain quaternion and octonion algebras, namely those with "transcendental parameters". These characterisations provide analogues of a theorem of Tignol and Vast (see [5]), the statement of which is included in the following result.

Theorem 1. *Let* $a, b \in F^\times$, $Q = (a,x)_{F((x))}$ *and* $O = (a,b,x)_{F((x))}$. *Then*

(a) $s(Q) = \min\{s(F(\sqrt{a})), k(a)\}$ *and* $\underline{s}(Q) = \min\{s(F(\sqrt{a})), k(a) - 1\}$,

(b) $s(O) = \min\{s((a,b)_F), k(a,b)\}$ *and* $\underline{s}(O) = \min\{\underline{s}((a,b)_F), k(a,b) - 1\}$.

Proof. (a) The level equality is the aforementioned result from [5]. We will prove the sublevel equality.

As $F(\sqrt{a}) \subseteq Q$, we have that $\underline{s}(Q) \leqslant \underline{s}(F(\sqrt{a}))$, which equals $s(F(\sqrt{a}))$. Letting $k(a) = n$, there exist $\gamma_1, \ldots, \gamma_n, \delta_1, \ldots, \delta_n \in F$, not all zero, such that $\sum_{t=1}^{n} {\gamma_t}^2 - a \sum_{t=1}^{n} {\delta_t}^2 = 0$. Hence $\sum_{t=1}^{n} (\gamma_t j + \delta_t k)^2 = 0$ in Q, whereby $\underline{s}(Q) \leqslant n-1$. Hence, we have that $\underline{s}(Q) \leqslant \min\{s(F(\sqrt{a})), k(a)-1\}$.

Suppose that $\underline{s}(Q) = n$. Hence, there exist $\alpha_t, \beta_t, \gamma_t, \delta_t \in F(\!(x)\!)$, not all zero, such that

$$\sum_{t=1}^{n+1} \alpha_t^2 + a\sum_{t=1}^{n+1} \beta_t^2 + x\left(\sum_{t=1}^{n+1} \gamma_t^2 - a\sum_{t=1}^{n+1} \delta_t^2\right) = 0$$

and

$$\sum_{t=1}^{n+1} \alpha_t\beta_t = \sum_{t=1}^{n+1} \alpha_t\gamma_t = \sum_{t=1}^{n+1} \alpha_t\delta_t = 0.$$

Multiplying across these equations by x^{2d} for a suitable choice of $d \in \mathbb{Z}$, we have that

$$\sum_{t=1}^{n+1} (x^d\alpha_t)^2 + a\sum_{t=1}^{n+1} (x^d\beta_t)^2 + x\left(\sum_{t=1}^{n+1} (x^d\gamma_t)^2 - a\sum_{t=1}^{n+1} (x^d\delta_t)^2\right) = 0$$

and

$$\sum_{t=1}^{n+1} (x^d\alpha_t)(x^d\beta_t) = \sum_{t=1}^{n+1} (x^d\alpha_t)(x^d\gamma_t) = \sum_{t=1}^{n+1} (x^d\alpha_t)(x^d\delta_t) = 0,$$

where at least one of $x^d\alpha_t, x^d\beta_t, x^d\gamma_t, x^d\delta_t \in F[\![x]\!]$ is not divisible by x for some t.

If $x^d\alpha_t$ or $x^d\beta_i$ is not divisible by x for some t, then taking residues modulo x gives that

$$\sum_{t=1}^{n+1} \left(\overline{x^d\alpha_t}\right)^2 + a\sum_{t=1}^{n+1} \left(\overline{x^d\beta_t}\right)^2 = 0 \quad \text{and} \quad \sum_{t=1}^{n+1} \left(\overline{x^d\alpha_t}\right)\left(\overline{x^d\beta_t}\right) = 0,$$

whereby $\underline{s}(F(\sqrt{a})) = s(F(\sqrt{a})) \leqslant n$.

If $x^d\alpha_t$ and $x^d\beta_t$ are divisible by x for all t, then dividing by x and taking residues modulo x gives that

$$\sum_{t=1}^{n+1} \left(\overline{x^d\gamma_t}\right)^2 - a\sum_{t=1}^{n+1} \left(\overline{x^d\delta_t}\right)^2 = 0,$$

whereby $k(a) - 1 \leqslant n$, completing the proof of (a).

(b) The sublevel equality can be proven by arguing as above. We will prove the level equality.

As $(a,b)_F \subset O$, we clearly have that $s(O) \leqslant s\left((a,b)_F\right)$. For $k(a,b) = n$, we have that $n \times \langle 1, -a, -b, ab\rangle$ is isotropic over F, whereby it is isotropic over $F(\!(x)\!)$, and thus represents $\frac{-1}{x}$. Hence, there exist $\varepsilon_t, \zeta_t, \eta_t, \vartheta_t \in F(\!(x)\!)$, not all zero, such that $\sum_{t=1}^{n} {\varepsilon_t}^2 - a\sum_{t=1}^{n} {\zeta_t}^2 - b\sum_{t=1}^{n} {\eta_t}^2 + ab\sum_{t=1}^{n} {\vartheta_t}^2 = \frac{-1}{x}$. Thus, we have that $\sum_{t=1}^{n} (\varepsilon_t e + \zeta_t ie + \eta_t je + \vartheta_t ke)^2 = -1$ in O, whereby $s(O) \leqslant n$. Hence $s(O) \leqslant \min\{s\left((a,b)_F\right), k(a,b)\}$.

Suppose that $s(O) = n$. Hence, there exist $\alpha_t, \beta_t, \gamma_t, \delta_t, \varepsilon_t, \zeta_t, \eta_t, \vartheta_t \in F(\!(x)\!)$, not all zero, such that

$$\sum_{t=1}^{n} \alpha_t^2 + a\sum_{t=1}^{n} \beta_t^2 + b\sum_{t=1}^{n} \gamma_t^2 - ab\sum_{t=1}^{n} \delta_t^2 + x\left(\sum_{t=1}^{n} \varepsilon_t^2 - a\sum_{t=1}^{n} \zeta_t^2 - b\sum_{t=1}^{n} \eta_t^2 + ab\sum_{t=1}^{n} \vartheta_t^2\right) = -1$$

and

$$\sum_{t=1}^{n}\alpha_t\beta_t=\sum_{t=1}^{n}\alpha_t\gamma_t=\sum_{t=1}^{n}\alpha_t\delta_t=\sum_{t=1}^{n}\alpha_t\varepsilon_t=\sum_{t=1}^{n}\alpha_t\zeta_t=\sum_{t=1}^{n}\alpha_t\eta_t=\sum_{t=1}^{n}\alpha_t\vartheta_t=0.$$

If $\alpha_t,\dots,\vartheta_t\in F[\![x]\!]$ for all t, then, taking residues modulo x, we obtain that $s\left((a,b)_F\right)\leqslant n$.

Alternatively, multiplying across these equations by x^{2d} for a suitable choice of $d\in\mathbb{N}$, we have that

$$\sum_{t=1}^{n}(x^d\alpha_t)^2+\ \dots\ -ab\sum_{t=1}^{n}(x^d\delta_t)^2+x\left(\sum_{t=1}^{n}(x^d\varepsilon_t)^2-\ \dots\ +ab\sum_{t=1}^{n}(x^d\vartheta_t)^2\right)=-x^{2d},$$

and

$$\sum_{t=1}^{n}(x^d\alpha_t)(x^d\beta_t)=\dots=\sum_{t=1}^{n}(x^d\alpha_t)(x^d\vartheta_t)=0,$$

where at least one of $x^d\alpha_t,x^d\beta_t,x^d\gamma_t,x^d\delta_t,x^d\varepsilon_t,x^d\zeta_t,x^d\eta_t,x^d\vartheta_t\in F[\![x]\!]$ is not divisible by x for some t.

If $x^d\alpha_t,x^d\beta_t,x^d\gamma_t$ or $x^d\delta_t$ is not divisible by x for some t, then taking residues modulo x gives that

$$\sum_{t=1}^{n}\left(\overline{x^d\alpha_t}\right)^2+\ \dots\ -ab\sum_{t=1}^{n}\left(\overline{x^d\delta_t}\right)^2=0$$

and

$$\sum_{t=1}^{n}\left(\overline{x^d\alpha_t}\right)\left(\overline{x^d\beta_t}\right)=\ \dots\ =\sum_{t=1}^{n}\left(\overline{x^d\alpha_t}\right)\left(\overline{x^d\delta_t}\right)=0.$$

Hence $\underline{s}\left((a,b)_F\right)\leqslant n-1$, whereby $s\left((a,b)_F\right)\leqslant n$ by [2, Theorem].

If $x^d\alpha_t,x^d\beta_t,x^d\gamma_t$ and $x^d\delta_t$ are divisible by x for all t, then dividing by x and taking residues modulo x gives that

$$\sum_{t=1}^{n}\left(\overline{x^d\varepsilon_t}\right)^2-a\sum_{t=1}^{n}\left(\overline{x^d\zeta_t}\right)^2-b\sum_{t=1}^{n}\left(\overline{x^d\eta_t}\right)^2+ab\sum_{t=1}^{n}\left(\overline{x^d\vartheta_t}\right)^2=0,$$

whereby $k(a,b)\leqslant n$. □

Corollary 1. *(a) Let $a\in F^\times$. For $O=(a,x,y)_{F(\!(x)\!)(\!(y)\!)}$, we have that*

$$s(O)=\min\left\{s(F(\sqrt{a})),k(a)\right\}\quad\text{and}\quad \underline{s}(O)=\min\left\{s(F(\sqrt{a})),k(a)-1\right\}.$$

(b) For $Q=(x,y)_{F(\!(x)\!)(\!(y)\!)}$ and $O=(x,y,z)_{F(\!(x)\!)(\!(y)\!)(\!(z)\!)}$, we have that

$$s(O)=\underline{s}(O)=s(Q)=\underline{s}(Q)=s(F).$$

Proof. (a) Since $k(a,x)$ over $F(\!(x)\!)$ equals $k(a)$ over F, by [3, Ch.VI, Theorem 1.4], an application of Theorem 1(b), followed by one of Theorem 1(a), establishes these statements.

(b) Invoking [3, Ch.VI, Theorem 1.4], we see that $s(F(\!(x)\!)(\sqrt{x}))=s(F)$ and that $k(x)$ over $F(\!(x)\!)$ equals $s(F)+1$. Hence, applications of Corollary 1(a) and Theorem 1(a) establish the result. □

For F a formally real field and $a\in F^\times$ a sum of squares, one sees that $k(a)$ is finite, whereas $s(F(\sqrt{a}))$ is infinite (see [3, Ch.VIII, Lemma 1.4]). In contrast, the finiteness of $k(a,b)$ encodes an upper bound on $s\left((a,b)_F\right)$, allowing us to establish the following corollary.

Corollary 2. *Let $k \geqslant 0$ be an integer and $a, b \in F^{\times}$.*

(a) If $1 + \lfloor \frac{2}{3} \cdot 2^k \rfloor < s\left((a,b)_F\right) \leqslant 2^k + 1$, then $s\left((a,b,x)_{F((x))}\right) = s\left((a,b)_F\right)$.

(b) If $\lfloor \frac{2}{3} \cdot 2^k \rfloor < \underline{s}\left((a,b)_F\right) \leqslant 2^k$, then $\underline{s}\left((a,b,x)_{F((x))}\right) = \underline{s}\left((a,b)_F\right)$.

Proof. If $k(a,b) < 2^k + 1$, then the Pfister form $2^k \times \langle 1, -a, -b, ab\rangle$ is hyperbolic, whereby its neighbour $\left(1 + \lfloor \frac{2}{3} \cdot 2^k \rfloor\right) \times \langle a, b, -ab\rangle$ is isotropic, implying that $\underline{s}\left((a,b)_F\right) \leqslant \lfloor \frac{2}{3} \cdot 2^k \rfloor$ and $s\left((a,b)_F\right) \leqslant 1 + \lfloor \frac{2}{3} \cdot 2^k \rfloor$. Hence, we must have that $k(a,b) \geqslant 2^k + 1$, whereby Theorem 1(b) gives the result. □

The existence of quaternion algebras whose levels and sublevels lie outside of the above intervals was established in [1] and [4], through the consideration of algebras of the form $(x,y)_{F_0(x,y)(\varphi)}$, where F_0 is formally real and the quadratic form φ over $F_0(x,y)$ is such that $\langle 1, x, y, -xy\rangle \subset \varphi \subset n \times \langle 1, x, y, -xy\rangle$ for some $n \in \mathbb{N}$. Without placing any restrictions on the quaternion algebra $(a,b)_F$, we cannot say whether Corollary 2 holds when $s((a,b)_F)$ or $\underline{s}((a,b)_F)$ take such values.

At present, the existence of quaternion algebras of level 6 (respectively, sublevel 5) remains unknown, prompting us to ask the following question.

Question 1. Let F_0 be a formally real field and $\psi = 8 \times \langle 1, -x, -y, xy\rangle$ be a quadratic form over $F_0(x,y)$. Is $(x,y)_{F_0(x,y)(\psi)}$ of level 6 (and thus sublevel 5)?

Since $6 \times \langle x, y, -xy\rangle$ is a Pfister neighbour of ψ, it is isotropic over $F_0(x,y)(\psi)$, whereby the sublevel and level of $(x,y)_{F_0(x,y)(\psi)}$ are at most 5 and 6 respectively. It seems reasonable to suggest that these upper bounds are attained. Should this be the case, Theorem 1(b) would imply that the associated octonion algebra $(x,y,z)_{F_0(x,y)(\psi)((z))}$ has strictly smaller level and sublevel, since $k(x,y) = 5$ over $F_0(x,y)(\psi)$. Thus, we suspect that Corollary 2 does not hold for all possible level and sublevel values of $(a,b)_F$.

However, restricting ourselves to the aforementioned class of quaternion algebras, conjectured to contain members of level and sublevel n for all $n \in \mathbb{N}$, we can prove the level and sublevel equalities.

Corollary 3. *Let F_0 be a formally real field and φ a quadratic form over $F_0(x,y)$ such that $\langle 1, x, y, -xy\rangle \subset \varphi \subset n \times \langle 1, x, y, -xy\rangle$ for some $n \in \mathbb{N}$. Let $Q = (x,y)_{F_0(x,y)(\varphi)}$ and $O = (x,y,z)_{F_0(x,y)(\varphi)((z))}$. Then $s(O) = s(Q)$ and $\underline{s}(O) = \underline{s}(Q)$.*

Proof. For S an ordering of F_0, let T denote an extension of S to $F_0(x,y)$ such that x and y are negative with respect to T. By [3, Ch.XIII, Theorem 3.1], T extends to an ordering of $F_0(x,y)(\varphi)$. However, $\langle 1, -x, -y, xy\rangle$ is positive definite with respect to T, whereby $k(x,y) = \infty$ over $F_0(x,y)(\varphi)$. Hence, the result follows from invoking Theorem 1(b). □

Remark All of the above results hold if the respective Laurent series fields are replaced by rational function fields.

Acknowledgements

I would like to thank Jean-Pierre Tignol, Thomas Unger and Karim Johannes Becher for their guidance and help. I am beholden to David Grimm and the referee for some helpful comments. I gratefully acknowledge the support I received through an IRCSET-Marie Curie International Mobility Fellowship from the Irish Research Council for Science, Engineering and Technology, co-funded by Marie Curie Actions under FP7. I greatly appreciate the hospitality and support of the Zukunftskolleg.

References

[1] D. W. Hoffmann, Levels of quaternion algebras, *Archiv der Mathematik* 90, no. 5 (2008) 401-411.

[2] D. W. Hoffmann, Levels and sublevels of quaternion algebras, *Mathematical Proceedings of the Royal Irish Academy* 110A, no. 1 (2010) 95-98.

[3] T. Y. Lam, Introduction to Quadratic Forms over Fields, *American Mathematical Society* (2005).

[4] J. O'Shea, Bounds on the levels of composition algebras, *Mathematical Proceedings of the Royal Irish Academy* 110A, no. 1 (2010) 21-30.

[5] J.-P. Tignol, N. Vast, Représentation de -1 comme somme de carrés dans certaines algèbres de quaternions, *C. R. Acad. Sci. Paris Sér. I Math.* 305, no. 13 (1987) 583-586.

James O'Shea
Zukunftskolleg, Universität Konstanz, 78457 Konstanz, Germany.
School of Mathematical Sciences, Science Centre, University College Dublin, Belfield, Dublin 4, Ireland.
E-mail: james.oshea@ucd.ie