The Two-Child Paradox: Dichotomy and Ambiguity

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Abstract. Given that one of the children in a two-child family is a boy, what are the chances that the other is also a boy. The intuitive answer is 50 : 50. More careful investigation leads us to a 1-in-3 chance. We investigate circumstances under which these answers are correct. The imposition of further conditions yields some very surprising results.

To my sons Owen and Andrew, both born on Tuesday, one on Christmas Day.

1. Introduction

At the Ninth *Gathering 4 Gardner Conference* in March, 2010 [1], Gary Foshee presented a probabilistic puzzle, the solution of which was quite counter-intuitive. It has generated intensive discussion on the internet, with some intriguing contributions, and others that may charitably be described as misleading.

The problem raised by Foshee is simple to state. We consider only families having two children. We are told that one of the two children in a family is a boy born on a Tuesday, and are asked "what is the probability that there are two boys?" On first acquaintance, it seems that the information about the day of birth is irrelevant and cannot affect the result. As we shall see, things are not so straightforward. Foshee presented an answer that astounded his audience and that appeared to defy intuition [2].

We make the usual assumptions that boys and girls are born with equal probability, that the sex of each child is independent of that of the other, that each day of the week is equally probable, likewise each month, each star-sign, etc. These assumptions can be challenged, but we are not concerned here with genetic subtleties, chronological quirks or astrological aberrations.

The problem originally posed by Martin Gardner [3] was: "Given that there is at least one boy, what is the probability that there are

Problem	*Condition* (K)	*Probability* (P)
1	None	
2	The first-born child is a boy	
3	At least one of the children is a boy	
4	At least one is a boy born on a Tuesday	

TABLE 1. Probability of two boys in a two-child family.

two?" (No mention of Tuesdays here). Even this simpler problem led to extensive correspondence. In particular, cognitive psychologists have taken an interest in it from the point of view of human perception [4]. Interesting as this may be, it will not concern us here.

2. The Two-Child Paradox

We will confine attention to the following question: "Under stated conditions, what is the probability that, for a two-child family, there are two boys?" To motivate the discussion, why don't you start by completing Table 1. In each case, enter the value P that you think is the probability of two boys. Unless you have had previous exposure to Problem 4, is is unlikely that you will anticipate the correct answer.

We consider the simplest problem first: there are two children; what is the probability that there are two boys? There are no further conditions. There are four possible family configurations $\Omega = \{BB, BG, GB, GG\}$ where, for example, BG signifies that the first-born child is a boy and the second a girl. We arrange these in an array:

$$\begin{bmatrix} \underset{[1]}{BB} & \underset{[1]}{BG} \\ \\ \underset{[1]}{GB} & \underset{[1]}{GG} \end{bmatrix} \tag{1}$$

As each of the four possibilities is equally likely, we can assign equal relative frequencies or weights [in brackets] to all. In particular, two boys occur with probability $P = \frac{1}{4}$.

Now consider the second problem: the first-born child is a boy. The sample space is now $\Omega = \{BB, BG\}$, *i.e.*, we consider only the

first row of (1). As both events are equally likely, the probability of two boys is $P = \frac{1}{2}$.

For the third problem, where the condition is that at least one of the children is a boy, the sample space is $\Omega = \{BB, BG, GB\}$; we retain the first row *and* the first column of (1). As each event is equally likely, the probability of two boys is $P = \frac{1}{3}$. This is the problem that Martin Gardner popularized; it is often called the *Two-Child Paradox*. The intuitive answer is $P = \frac{1}{2}$, whereas mathematical reasoning leads us to the answer $P = \frac{1}{3}$.

3. Tuesday's Child

We now introduce a dichotomy: we assume that all children fall into one of two categories, denoted by subscripts 1 and 2, with relative frequencies L and M respectively, and write $N = L + M$. Furthermore, we assume that these frequencies are independent of the sex of the child. There are now sixteen possibilities for the configuration of a two-child family, which we can arrange, with obvious notation, in an array:

$$\begin{bmatrix} \begin{matrix} B_1B_1 \\ [L^2] \end{matrix} & \begin{matrix} B_1B_2 \\ [LM] \end{matrix} & \begin{matrix} B_1G_1 \\ [L^2] \end{matrix} & \begin{matrix} B_1G_2 \\ [LM] \end{matrix} \\ \\ \begin{matrix} B_2B_1 \\ [LM] \end{matrix} & \begin{matrix} B_2B_2 \\ [M^2] \end{matrix} & \begin{matrix} B_2G_1 \\ [LM] \end{matrix} & \begin{matrix} B_2G_2 \\ [M^2] \end{matrix} \\ \\ \begin{matrix} G_1B_1 \\ [L^2] \end{matrix} & \begin{matrix} G_1B_2 \\ [LM] \end{matrix} & \begin{matrix} G_1G_1 \\ [L^2] \end{matrix} & \begin{matrix} G_1G_2 \\ [LM] \end{matrix} \\ \\ \begin{matrix} G_2B_1 \\ [LM] \end{matrix} & \begin{matrix} G_2B_2 \\ [M^2] \end{matrix} & \begin{matrix} G_2G_1 \\ [LM] \end{matrix} & \begin{matrix} G_2G_2 \\ [M^2] \end{matrix} \end{bmatrix} \tag{2}$$

We have indicated [in brackets] the relative frequencies for each case. The total weight is $4N^2$, with the total for each of the four 2×2 blocks being N^2. Using this array to address Problem 3 (in which at least one child is a boy), we must consider the sample space comprising all cases except those in the bottom right-hand 2×2 block. We find immediately that $P = \frac{N^2}{3N^2} = \frac{1}{3}$, as before.

Consider next the probability of two boys given that one child is a boy in Category 1, *i.e.*, that B_1 occurs. The sample space now comprises the first row and first column of the array (2). The total

weight is $4LN - L^2$. The weights for the three cases having two boys sum to $2LN - L^2$. Thus, the probability is

$$P = \frac{2LN - L^2}{4LN - L^2} = \frac{2-p}{4-p}\,, \tag{3}$$

where $p = L/N$ is the relative frequency of Category 1. We note the two limits

$$\lim_{p\to 0} P = \tfrac{1}{2} \qquad\qquad \lim_{p\to 1} P = \tfrac{1}{3}\,.$$

Thus, the sharper the condition (the smaller L compared to N) the higher the probability of a two-boy family given that condition.

Now let us consider Problem 4 in Table 1: at least one of the children is a boy born on a Tuesday. Then $L = 1$ and $N = 7$ so $p = 1/7$. Thus, by (3), the probability of two boys is

$$P = \frac{2 - \frac{1}{7}}{4 - \frac{1}{7}} = \frac{13}{27}\,.$$

The surprise here is not the particular numerical value, but the fact that the condition of being born on a Tuesday has *any influence whatsoever* on the result!

Let us consider another question: "Given that one child is a boy born on a Christmas Day that falls on a Tuesday, what is the probability of two boys?" (we ignore leap years). Then $p = 1/(7 \times 365) \approx 0.00039$ and

$$P = \frac{2-p}{4-p} \approx 0.49995\,.$$

For practical purposes, $P = \frac{1}{2}$. A sharper condition has increased P.

4. Paradox or Ambiguity?

It certainly seems at first amazing that the weekday of birth of one child can influence the probability of the sex of the other. The critical factor is that the information on one boy being born on a Tuesday is used at the outset to determine the sample space: we are really considering the question: "Among all two-child families for which at least one child is a boy born on a Tuesday, for what fraction of these families are there two boys?" To simplify matters, let us return to Gardner's problem: "Among all two-child families for which at least one child is a boy (born on *any* day of the week), for what fraction

of these families are there two boys?" We have found above that the answer is $P = \frac{1}{3}$.

Now consider this scenario: you are strolling on Dun Laoghaire pier and meet an old school-chum, Pat, whom you have not seen since your youth. He is accompanied by a boy, and introduces him thus: "This is Jack, one of my two children". What are the chances that his other child is a boy? The answer is $P = \frac{1}{2}$; Pat's family has not been pre-selected from those having at least one boy. Similarly, if Pat had said "This is one of my two children, Jack, who was born on a Tuesday", it would have changed nothing: the chance of his other child being a boy is still 50 : 50. We will demonstrate this now.

5. Bayes' Theorem

We examine the probability that there are two boys in a two-child family X. The possibilities are $X \in \{BB, BG, GB, GG\}$. We denote by H the hypothesis $X \in \{BB\}$. Now we introduce a further condition, K, that "at least one child is a boy". Bayes' Theorem [5] implies

$$P(H|K) = \frac{P(H)P(K|H)}{P(K)}\,. \tag{4}$$

The prior, or unconditional, probability of H is $P(H) = \frac{1}{4}$. Clearly $P(K|H) = 1$, as "two boys" implies "at least one boy". Everything now hangs on the value of $P(K)$.

We consider all two-child families with one or more boys. Since there are three equally likely outcomes out of four that at least one child is a boy, we have $P(K) = \frac{3}{4}$. Using this value in (4) we have

$$P(H|K) = \frac{\frac{1}{4}\cdot 1}{\frac{3}{4}} = \frac{1}{3}\,.$$

However, when you meet your old friend Pat on the pier with his son Jack, you must assume — in the absence of any other information — that he has randomly selected one of his two children to accompany him. The condition K now is that "Pat has brought his son (or one of them) along for a stroll". If Pat has two boys, he must choose one of them; if he has two girls, the chance of a boy is zero; if he has a boy and a girl, the chance of his choosing a boy is 50 : 50. Thus,

$$P(K|BB) = 1\,, \quad P(K|BG) = P(K|GB) = \tfrac{1}{2}\,, \quad P(K|GG) = 0\,.$$

The probability of the condition K may be partitioned as

$$\begin{aligned} P(K) &= P\big(K \cap (BB \cup BG \cup GB \cup GG)\big) \\ &= P(K \cap BB) + P(K \cap BG) + P(K \cap GB) + P(K \cap GG) \\ &= P(BB)P(K|BB) + P(BG)P(K|BG) + \\ &\qquad\qquad + P(GB)P(K|GB) + P(GG)P(K|GG) \end{aligned}$$

Substituting the numerical values in this gives

$$P(K) = \tfrac{1}{4} \cdot 1 + \tfrac{1}{4} \cdot \tfrac{1}{2} + \tfrac{1}{4} \cdot \tfrac{1}{2} + \tfrac{1}{4} \cdot 0 = \tfrac{1}{2}\,.$$

Finally, using this value in (4) we have

$$P(H|K) = \frac{\frac{1}{4} \cdot 1}{\frac{1}{2}} = \tfrac{1}{2}\,.$$

Pat's other child is equally likely to be a boy or girl! Moreover, you may ask Jack his birthday, whether he was born on a Tuesday, if he is a Gemini or likes bananas. It doesn't matter. None of this information has any influence on the probability of his sibling being a boy.

Of course, if you ask Jack does he have a brother . . . ?!!!

6. Conclusion

At the G4G Conference, Gary Foshee posed the question: "I have two children. One is a boy born on a Tuesday. What is the probability I have two boys?" He gave the answer $P = \frac{13}{27}$, and we have seen how this arises. But the question that Foshee actually answered was: "Of all two-child families with at least one child being a boy born on a Tuesday, what proportion of those families have two boys?" The correct answer to the question he actually posed is $P = \frac{1}{2}$.

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