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Holographic interferometry and the fractional Fourier transformation

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The fractional Fourier transform (FRT) is shown to be of potential use in analyzing the motion of a surface by use of holographic interferometry. The extra degree of freedom made available by the use of the FRT allows information regarding both translational and tilting motion to be obtained in an efficient manner. © 2000 Optical Society of America

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The fractional Fourier transform (FRT) has received a great deal of attention in the optics literature.[1–3] Holographic interferometry is extremely popular because of its simplicity of implementation and because it can be applied to detect both in-plane and out-of-plane motions.[4,5] In this Letter we combine the holographic interferometric principle with the FRT to measure combined tilting and in-plane motion of surfaces.

In holographic interferometry the holographically reproduced field of an object interferes with the field produced by the same object some time later. Any interim deformations of the object lead to phase changes in the resultant field, producing an interference pattern. One can then analyze this pattern to extract the phase and so deformation information. We propose the superposition of the optically produced FRT's of the original and a shifted field. The resultant intensity information can then be processed to give accurate information regarding the displacement and tilting of the object.

For the sake of simplicity and brevity we prove our results for the one-dimensional case, using normalized notation. System implementation and limitations are not addressed here.

The initial field produced by an object is written as $u(x)$. Following some slight tilting of the object, the field becomes $u(x)\exp(j\kappa x)$. We superimpose these two fields and capture the resultant intensity interference pattern:

$$|u(x) + u(x)\exp(j\kappa x)|^2 = |u(x)|^2[2 + \exp(-j\kappa x) + \exp(+j\kappa x)] = 2I(x)[1 + \cos(\kappa x)], \quad (1)$$

where $I(x) = |u(x)|^2$. The magnitude of the change in spatial frequency, k, can be found from the period of the cosinusoidal intensity variation.

When the object no longer rotates but rather translates in the plane, moving by a distance ξ, $u(x)$ becomes $u(x-\xi)$. Using an optical implementation of the Fourier transform (FT), we now record a hologram of the FT of the object field, $\tilde{U}(k) = \mathrm{FT}[u(x)]$. One can then interferometrically compare this result with the FT of the shifted field. In this case the intensity distribution in the spatial frequency plane is given by

$$|\tilde{U}(k) + \tilde{U}(k)\exp(jk\xi)|^2 = |\tilde{U}(k)|^2[2 + \exp(-jk\xi) + \exp(+jk\xi)] = 2I(k)[1 + \cos(k\xi)], \quad (2)$$

where $I(k) = |\tilde{U}(k)|^2$. The magnitude of the shift in x can be found from the period of the cosinusoidal variation.

Let us now assume that our surface undergoes a tilting motion and a translation motion simultaneously. In this case the field for comparison is given by $u(x-\xi)\exp(j\kappa x)$. Using an optical implementation of the FRT, we assume that we can record a hologram of the original reflected field and of the field from the moved object. Optical implementations of the FRT have been described in the literature, and they result in an output plane that contains a fixed ratio of mixed spatial frequency and positional domain information.

The FRT was previously defined in the literature by use of normalized parameters[6] to be

$$F_\theta[u(x)] = U(x,\theta) = \frac{1}{\sqrt{2\pi|\sin\theta|}} \times \exp\left\{-j\frac{\pi}{2}\left[\frac{1}{2} + J\left(\frac{\theta}{\pi}\right)\right] + \frac{j}{2}x^2\cot\theta\right\} \times \int_{-\infty}^{+\infty} u(x')\exp\left(+\frac{j}{2}x'^2\cot\theta - jxx'\csc\theta\right)\mathrm{d}x', \quad (3)$$

where θ is of the order of the FRT and $J(Z)$ denotes the largest integer smaller than Z. The field following

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optical application of the FRT to the moved object field is given by

$$F_\theta[u(x-\xi)\exp(j\kappa x) = \frac{1}{\sqrt{2\pi|\sin\theta|}} \times \exp\left\{-j\frac{\pi}{2}\left[\frac{1}{2}+J\left(\frac{\theta}{\pi}\right)\right]+\frac{j}{2}x^2\cot\theta\right\} \times \int_{-\infty}^{+\infty} u(x'-\xi)\exp(j\kappa x') \times \exp\left(+\frac{j}{2}x'^2\cot\theta - jxx'\csc\theta\right)\mathrm{d}x'. \quad (4)$$

We wish to identify the value of θ that will ensure that the variation in x and k produces a simple positional shift in the FRT plane. To this end we manipulate Eq. (4) until the shifting can be clearly identified. We introduce a dummy variable $y = x' - \xi$ into Eq. (4) and expand the exponential arguments, giving

$$\frac{1}{\sqrt{2\pi|\sin\theta|}}\exp\left\{-j\frac{\pi}{2}\left[\frac{1}{2}+J\left(\frac{\theta}{\pi}\right)\right]+\frac{j}{2}x^2\cot\theta + \frac{j}{2}\xi^2\cot\theta - jx\xi\csc\theta + j\kappa\xi\right\}\int_{-\infty}^{+\infty}u(y)\exp(j\kappa y) \times \exp\left[+\frac{j}{2}(y^2+2y\xi)\cot\theta - jxy\csc\theta\right]\mathrm{d}y. \quad (5)$$

This is equal to

$$\frac{1}{\sqrt{2\pi|\sin\theta|}}\exp\left\{-j\frac{\pi}{2}\left[\frac{1}{2}+J\left(\frac{\theta}{\pi}\right)\right] + \frac{j}{2}\cot\theta\left[x^2+\xi^2-2\frac{x\xi}{\cos\theta}+2\kappa\xi\tan\theta\right]\right\} \times \int_{-\infty}^{+\infty}u(y)\exp\left[+\frac{j}{2}y^2\cot\theta - j \times (x-\kappa\sin\theta-\xi\cos\theta)y\csc\theta\right]\mathrm{d}y. \quad (6)$$

Comparing expression (6) with Eq. (4), we see that inside the integral the x parameter has been shifted by an amount $\kappa\sin\theta + \xi\cos\theta$. We want this shift to be a maximum as a function of θ. Taking the derivative of the shift with respect to θ and putting it equal to zero, we find that this maximum occurs when $\theta = \tan^{-1}(\kappa/\xi)$. This defines a line in phase space given by the equation $k = \kappa x/\xi$, which fixes the mixing relationship between position and spatial frequency in the FRT plane. When θ takes this value the shift has the magnitude of the hypotenuse of the triangle formed by the two orthogonal shifts, i.e., ξ and κ.

Before proceeding we must also examine the exponential term that appears outside the integral. Substituting the formula $\theta = \tan^{-1}(\kappa/\xi)$ gives

$$\frac{1}{\sqrt{2\pi|\sin\theta|}}\exp\left\{-j\frac{\pi}{2}\left[\frac{1}{2}+J\left(\frac{\theta}{\pi}\right)\right] + \frac{j}{2}\cot\theta\left[\left(x-\sqrt{\xi^2+\kappa^2}\right)^2\right]+\frac{j}{2}\kappa\xi\right\} \times \int_{-\infty}^{+\infty}u(y)\exp\left[+\frac{j}{2}y^2\cot\theta - j\left(x-\sqrt{\xi^2+\kappa^2}\right)y\csc\theta\right]\mathrm{d}y. \quad (7)$$

Therefore we can write that, when $\theta = \tan^{-1}(\kappa/\xi)$,

$$F_\theta[u(x-\xi)\exp(j\kappa x)] = F_\theta\left[u\left(x-\sqrt{\xi^2+\kappa^2}\right)\right]\exp(j\kappa\xi/2). \quad (8)$$

Taking the FT of Eq. (8) and recalling, first, that the FT is the special case of a FRT of order $\theta = \pi/2$ and, second, the semigroup property of the FRT, i.e., $F_{\theta 1}F_{\theta 2} = F_{\theta 2}F_{\theta 1} = F_{\theta 1+\theta 2}$, we get

$$\mathrm{FT}\{F_\theta[u(x-\xi)\exp(j\kappa x)]\} = F_\theta\left\{F_{\pi/2}\left[u\left(x-\sqrt{\xi^2+\kappa^2}\right)\right]\exp(j\kappa\xi/2)\right\} = F_\theta[\tilde{U}(k)]\exp(j\kappa\xi/2)\exp\left(jk\sqrt{\xi^2+\kappa^2}\right). \quad (9)$$

Calculating the resultant interference pattern gives

$$\left|F_\theta[\tilde{U}(k)] + F_\theta[\tilde{U}(k)]\exp(j\kappa\xi/2)\exp\left(jk\sqrt{\xi^2+\kappa^2}\right)\right|^2 = |F_\theta[\tilde{U}(k)]|^2\left\{2+\exp\left[j\left(\kappa\xi/2+k\sqrt{\xi^2+\kappa^2}\right)\right] + \exp\left[-j\left(\kappa\xi/2+k\sqrt{\xi^2+\kappa^2}\right)\right]\right\} = 2|F_\theta[\tilde{U}(k)]|^2\left[1+\cos\left(k\sqrt{\xi^2+\kappa^2}+\frac{\kappa\xi}{2}\right)\right]. \quad (10)$$

Once again we have a cosinusoidal interference pattern the period of which is related to the total plane shift in the system. The extra phase term is a geometric phase, which shifts the interference pattern but does not vary its period.

In conclusion, it has been shown that, by optically generating FRT planes of suitable order within a holographic interferometric system, one can extract unambiguous translation and tilt information for a surface. Such a system is potentially useful when a fixed linear relationship exists between the variations of spatial frequency k and position x. We note that similarly a generalization of the speckle photographic technique can also be made with the optical FRT.[7]

Although only the one-dimensional case is discussed here, the generalization to two dimensions appears simple. The ability to optically implement dynamic FRT systems of different orders in different directions means that different linear tilt–translation relationships in orthogonal directions can be accommodated.[8] The combination of such a dynamic optical FRT system with a dynamic holographic recording system could be used to permit the independent determination of surface tilt and translation motions by use of a single optical channel. Finally, we note that generalized linear optical transformations[9] exist that may be useful in examining more-complex types of motion.

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