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Holography: an interpretation from the phase-space point of view

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Phase-retrieval techniques (PRTs) have found important applications in various fields such as in astronomy, radiography, crystallography, and security. The determination of the phase of a scattered wave field is of interest because it carries important information about the object surface or inner structure. Generally, PRTs fall mainly into two main categories: interferometric (InF) and noninterferometric (NInF) approaches. The latter category can be further classified into methods based on iterative technique, transport-of-intensity equation (TIE), phase-space tomography, and moments of phase-space distributions. All of these NInF approaches essentially use intensity measurements of the scattered field at various domains to retrieve the object phase. At each of these domains, the phase-space distribution, e.g., the ambiguity function (AF) associated with the wavefront has the same value, but the coordinates undergo an affine transform specified by the ABCD matrix. Thus the Fourier transform of each measured intensity corresponds to a plane through the ambiguity space with a slope with respect to the propagation distance. The intensity measurements required by the NInF approaches mentioned above can then be regarded as specific samplings in the ambiguity space, and these techniques have a unified interpretation based on phase-space tomography.

InF approaches, including digital holography (DH), have received much attention in the last decades and have been successfully applied in many areas. Usually the principle of holography has been primarily interpreted using communication theory. It is not until recently that attempts have been made to interpret holography using the Wigner distribution function (WDF). However, as we will show, the WDF representation does not depict holography most conveniently, while the AF representation provides a much more insightful interpretation. This may be instructive in developing a sampling strategy in the reconstruction of DH. It may also provide a unified framework for the common formulation of InF and NInF PRTs.

For simplicity, we consider a one-dimensional (1D) signal $s(x)$, since the generalization to two-dimensions is straightforward. The WDF of $s(x)$ is defined as

$$W_s(x, ν) = \int s \left(x + \frac{\bar{x}}{2}\right) s^* \left(x - \frac{\bar{x}}{2}\right) \exp[-j2πν\bar{x}]d\bar{x}. \quad (1)$$

One important property of the WDF is its bilinearity, which is useful in analyzing the coherent superposition of two signals $y(x) = s(x) + r(x)$. The WDF of $y(x)$ is

$$W_y(x, ν) = W_s(x, ν) + W_r(x, ν) + W_{sr}(x, ν) + W_{rr}(x, ν), \quad (2)$$

where

$$W_{sr}(x, ν) = \int s \left(x + \frac{\bar{x}}{2}\right) r^* \left(x - \frac{\bar{x}}{2}\right) \exp[-j2πν\bar{x}]d\bar{x}, \quad (3)$$

and straightforward for the definition of $W_{rr}(x, ν)$. Equation (2) shows that the cross terms arise because of the bilinearity. Now let us consider the case when a plane wave of special frequency $ν_0$, i.e., $r(x) = \exp[j2πν_0x]$ is incident at the interference plane whose longitudinal coordinate is $z = 0$. The WDF of $r(x)$ is $δ(ν - ν_0)$, and according to Eq. (3) $W_{sr}(x, ν) = 2 \exp[j4π(ν - ν_0)x]S(2ν - ν_0)$, $W_{rr}(x, ν) = -2S(2ν - ν_0)\ldots S^*(2ν - ν_0)$, where $S$ and $S^*$ are the Fourier transforms of $s$ and $s^*$. As schematically shown in Fig. 1, both cross terms occupy the same spatial frequency region centered at the coordinate $(0, ν_0/2)$, while the WDFs of the signal and the reference are centered at origin and $(0, ν_0)$, respectively. This is clearly not the familiar picture of off-axis holograms as interpreted using the carrier wave theory.

Now we consider the AF representation. The AF was initially proposed and is commonly used in radar and sonar to track moving targets. The AF of the signal $s(x)$ is defined as

$$AF(x, ν) = \int s \left(x + \frac{\bar{x}}{2}\right) s^* \left(x - \frac{\bar{x}}{2}\right) \exp[-j2πν\bar{x}]d\bar{x}. \quad (4)$$
importance in our case because they result in the ho-
were made to eliminate them. However, they are of

Referring to the definition of the AF, Eq.(6) is the
wave
distribution
which corresponds to the case of in-line holography.
To cancel out the DC and conjugate terms, one ap-
approach is to introduce π/2-stepwise phase shifting
into the reference beam [14]. The signs of the last
three terms in Eq. (8) then change. The reconstruc-
tion is achieved by algebraic manipulations of these
AFs.

Alternatively, one can shift the CCD position and
record several holograms longitudinally at the planes
zi, (i=1,2,...), parallel to the first plane (denoted by
z0=0). The Fourier transforms of these holograms
can then be expressed as

where

Referring to the definition of the AF, Eq. (6) is the
Fourier spectrum of the recorded hologram h(x) =|y(x)|2. We illustrate this in Fig. 2. It can be clearly
seen that the Fourier spectra of the object wavefront
S(\nu) and of its conjugate S*(\nu) are located on either

side of the origin, with shifts in spatial frequency of
−\nu0 and +\nu0, respectively, while the spectra of the DC
terms A0(\nu) and 0 occupy the lower frequencies
around the origin. This is consistent with the picture
of the hologram spectrum provided by the carrier
wave theory [15].

We can see from Eq. (7) that the line x=0 contains
the full complex amplitude and not only its intensity
as in the NInF cases. Thus appropriate filtering al-

If the spatial frequency of the reference \nu0 is insuf-
1, all terms in Eq. (5) overlap; the AF of y(x) then
terms

\nu=x/2.

\nu0=0) is sufficiently large, are well separated, as sche-
matically shown in Fig. 2.

It is interesting to examine the line x=0 in the am-

If the signal s(x) propagates along the z axis,
A(x,\nu,\bar{x}) then lies at the origin in the 2D ambiguity
space, as does A0(\nu,\bar{x}). The cross terms, A0(\nu,\bar{x}) and
A00(\nu,\bar{x}) are centering at (−\nu0,0) and (+\nu0,0), and if
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matically shown in Fig. 2.
In conclusion, we have shown that the formation of holograms can be interpreted as the result of the bilinearity of the AF. Compared with the WDF representation [16], the present approach provides a picture consistent with the carrier wave theory. One important prediction of this interpretation is that the reconstruction of a hologram may be possible by intensity measurements along the z axis. This may result in a new reconstruction algorithm by solving Eq. (9). Compared with the NInF techniques, such as the phase-space tomography of the autoterm of the AF, the holographic approaches can be regarded as the phase-space tomography or filtering of the cross term of the AF in terms of wavefront reconstruction. This provides a unified picture for the formulation of both these two categories of PRTs. However, as shown in detail in Table 1, each of these techniques has its own applicability. Generally speaking, InF techniques have simple reconstruction algorithms, while NInF techniques have simpler measurement setups.

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