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Title	Reevaluation of the direct method of calculating Fresnel and other linear canonical transforms
Authors(s)	Healy, John J., Sheridan, John T.
Publication date	2010-04-01
Publication information	Healy, John J., and John T. Sheridan. "Reevaluation of the Direct Method of Calculating Fresnel and Other Linear Canonical Transforms" 35, no. 7 (April 1, 2010).
Publisher	Optical Society of America
Item record/more information	http://hdl.handle.net/10197/3381
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Publisher's version (DOI)	10.1364/OL.35.000947

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Reevaluation of the direct method of calculating Fresnel and other linear canonical transforms

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Received December 21, 2009; revised February 5, 2010; accepted February 18, 2010;
posted February 26, 2010 (Doc. ID 121706); published March 19, 2010

The linear canonical transform may be used to simulate the effect of paraxial optical systems on wave fields. Using a recent definition of the discrete linear canonical transform, phase space diagram analyses of the sampling requirements of the direct method of calculating the Fresnel and other linear canonical transforms are more favorable than previously thought. Thus the direct method of calculating these transforms may be used with fewer samples than previously reported simply by making use of an appropriate reconstruction filter on the samples output by the algorithm. © 2010 Optical Society of America

OCIS codes: 000.4430, 070.2025, 070.2590, 090.1995.

The linear canonical transforms (LCTs), in particular the Fresnel transform, are used as propagation models in many applications, e.g., digital holography [1,2]. As such, issues relating to their numerical approximation have been explored in depth by the optics community, in terms of sampling the signals [3–9], defining discrete approximations to the transforms [10–15], and developing fast algorithms to evaluate these discrete LCTs [4,8,11,12,16]. Most of the proposed fast algorithms decompose the optical system into an equivalent concatenation of subsystems for which established algorithms are known. In [4], an analysis of the sampling requirements of many of these algorithms was presented, which consolidated and improved on many previous sampling analyses in this area. The analysis assumed an initial width and bandwidth of the signal, and it used phase space diagrams (PSDs) to track how these characteristics evolved as the wave field was propagated through each subsystem. In [9], this analysis was modified to account for the effects of sampling. However, Stern has defined a discrete LCT [13], which we have previously argued [16] possesses desirable properties for numerical simulation, with consequent benefits in terms of the number of samples required. Using this discrete LCT requires the analysis of sampling requirements to be reviewed and updated. In this Letter, we will update the analysis of [4] for the direct method (DM) of calculating the LCT. We will show that the sampling methodology of [13] results in a reduced sampling rate over that of [4].

A theorem has been proposed, defining a sampling rate proportional to the width of a wave field's LCT [3,5–7]. Sampling at or above this rate permits perfect recovery of the wave field from its samples. For the Fourier transform, another special case of the LCT, this theorem reduces to the Shannon sampling theorem [17]. Stern proposed a discrete LCT [13] using this theorem in the discretization of the input and output domains of the continuous transform. The

resulting signal's PSD is a parallelogram of the form of Fig. 1(a) [15].

Two sides of the parallelogram are parallel to the frequency axis. These indicate the width, W , of the wave field at the input of the system. The other two sides are separated by a distance $\hat{W} = W_T/B$, where W_T is the width of the wave field at the output of the LCT system, T is the $ABCD$ matrix characterizing that system, and B the upper right element of that matrix. These sides have slope $-A/B$. Note that if the system is a Fourier transform ($A=D=0$, $B=-C=1$), this slope is 0; i.e., the initial PSD is assumed to be a rectangle, the height of which indicates the bandwidth of the signal. We note that Fig. 1 constitutes different initial assumptions to most sampling discussions, with the width of the output replacing the traditional role of bandwidth.

We now consider the effect each stage of the DM algorithm has on the PSD of Fig. 1(a). We show that we avoid the interpolation and decimation stages of that algorithm, and furthermore reduce the sampling rates required for calculating the algorithm. For this particular algorithm, the effects of sampling the input and the output domains are decoupled; thus we do not need to guard against the overlaps of the PSD

Fig. 1. Tracing the PSD of a signal through a general LCT system decomposed into a chirp, scaled Fourier transform, and a second chirp. (a) Initial PSD, assuming width W and output width W_T . The black dot in the upper right corner is to aid the reader in following the transformations. (b) After the first chirp multiplication. (c) After the Fourier transform. (d) After the second chirp multiplication.

replicas discussed in [9]. Consequently, we may discuss discrete systems using PSDs, which do not include the replicas created by sampling. The DM decomposes the LCT into a chirp, magnification, a fast Fourier transform (FFT), and a second chirp [4]. It is unusual in that its parameters are observable in the kernel of the LCT without any manipulation, and so is the obvious decomposition of the LCT. However, this decomposition has been dismissed in the LCT algorithms literature as naïve, and it is claimed that it will in general lead to very high sampling rates [8]. In the Fresnel case ($A=D=1$, $B=\lambda z$, $C=0$), several analyses have concluded that for small values of z , the algorithm is numerically intensive [4] (and references therein). Often, separate constraints are used for different ranges of propagation distance, e.g., [18]. However, underpinning these arguments is Shannon's sampling theorem, which may not be the most appropriate choice.

We will now justify the claim that at every intermediate stage of the algorithm [labeled (1) above], the PSD retains the general form of Fig. 1(a). This means that we can continue to represent the signal with N samples, with no interpolation or oversampling. Figure 1(b) shows the effect of the first chirp [16]. The PSD is skewed in the k direction to transform the PSD into a rectangle. We may now consider the wave field to be Shannon sampled. Next, magnification stretches the PSD in x and squeezes it in k (or vice versa for demagnification). We omit this step in Fig. 1 as the PSD remains rectangular. The third step is an FFT, rotating the PSD by 90° , as shown in Fig. 1(c). The samples may still be interpreted as a Shannon sampled signal. Finally, the samples are multiplied by a chirp, skewing them in k . This produces a PSD of the form of Fig. 1(d). Throughout the stages of the calculation, the PSD has retained the general form of Fig. 1(a). This is significant because the number of samples required to represent a signal using the LCT sampling theorem is given by the area of the enclosing parallelogram of this form. Thus, as the LCT is area preserving in phase space, the number of samples required to represent the signal remains constant throughout the calculation. This is in stark contrast to the analysis presented in [4] and references therein, which indicate that the DM can require enormous oversampling, depending on the specific transform parameters. The analog output of the optical system may be well approximated using a reconstruction filter as discussed in [3] or [5], with the only source of error being aliasing due to the approximation of the width of the signal as finite in two domains.

Table 1. Comparison of Sampling Rates from [4] with Those of this Letter for an Example Input and Various Propagation Distances, z

z [m]	0.02	0.2	0.81	2
Method of [4]	41,000	5000	2013	3500
Proposed method	41,000	5000	1988	1400

Table 1 compares the proposed DM calculation with a DM that uses the sampling methodology of [4]. The signal is assumed to have width 0.02 m, bandwidth 50000 m^{-1} (hence, for Shannon sampling of the input signal, we need $0.02 \times 50000 = 1000$ samples), and $\lambda = 0.5 \times 10^{-6}$. For practical purposes, $\hat{W}$ was established using the methodology of [4], which is a conservative (i.e., high) estimate, particularly for small z . We could also have chosen to assume some $\hat{W}$ and determined the input bandwidth, but the method chosen is a sterner test of our proposed method. The resulting sampling rates are identical for small values of z because (i) of how $\hat{W}$ is determined and (ii) because the requirements established by [4] are dominated by the first chirp multiplication for small z , for which the determination of sampling rates is coincidental to that of the proposed method. We note that for small z (or B), the proposed method also requires a high sampling rate, though with a less conservative estimate of $\hat{W}$, further savings are possible over [4]. These results are dependent on the signal parameters. In particular, if the bandwidth is much less than the width, it is possible for our method to result in unnecessarily high sampling rates. However, these cases are rare in optical propagation problems. We may conclude that this algorithm is superior to that of [4] in typical situations. The algorithm is valid for all $z > 0$, but may require high sampling rates for small z .

We now justify the claim that the effects of sampling the input and the output domains are decoupled, and consequently the preceding analysis may be applied to numerical simulations. Figure 2(a) establishes the initial PSD identically to Fig. 1(a). The signal is then sampled, resulting in a periodic replication in the k direction, as shown in Fig. 2(b) [9,16].

Fig. 2. Tracing the PSD of a signal through a numerical LCT using the DM. (a) Initial PSD, assuming width W and output width $\hat{W}$. (b) First sampling operation. The ellipses indicate infinite, periodic replication. (c) First chirp multiplication. (d) Discrete-space Fourier transform. (e) Second sampling operation. Note that (d) and (e) are achieved in one step using an FFT. (f) Second chirp multiplication.

As before, the signal is chirped, as in Fig. 2(c), and then a Fourier transform is performed, resulting in a periodic signal, as shown in Fig. 2(d). The output of this discrete-space Fourier transform must be sampled, an operation that creates replicas in the k direction [Fig. 2(e)]. Finally, the second chirp operation is applied [Fig. 2(f)]. Thus we have shown that the sampling processes are orthogonal, i.e., independent of each other. This orthogonality constitutes an additional advantage over [4], as the additional constraints of [9] need not be considered. In [16], we presented a similar analysis for a fast LCT (FLCT) algorithm that is not based on the decomposition of the optical system, noting the orthogonality of the replicas created by the two sampling operations (input and output domains). Here, we have established that there is a decomposition-based algorithm that also has this property.

In this Letter, we have modified the analysis in [4] using the discrete LCT proposed in [13]. We have shown advantages of this sampling methodology for the DM over that of [4]. Its two sampling operations are orthogonal, which simplifies the determination of sampling requirements [9]. We have also shown that no interpolation is required at intermediate stages in the algorithm. We have demonstrated that fewer samples are required for calculations using this algorithm, making calculations with it faster. The DM was previously known to enjoy a significant practical advantage over other algorithms in that it requires only a pair of chirps that are of order N complexity, and therefore not significant to the run time, and a single FFT that is of order $N \log N$ complexity, which enjoys substantial optimization of run times. By way of compensation, nondecomposition-based FLCT algorithms [12,16] have been thought to enjoy an advantage in terms of the number of samples required [16]; however, based on the results presented here, this is no longer the case. While such FLCT algorithms are also of order $N \log N$ complexity, their implementations have not yet received the kind of optimization that the FFT algorithms have, and as such it is currently reasonable to claim that the DM

should be preferred for calculating the discrete LCT except where $B \rightarrow 0$.

We acknowledge the support of Enterprise Ireland; Science Foundation Ireland through the Research Frontiers Programme under the National Development Plan; the Irish Research Council for Science, Engineering and Technology; and FÁS, The Irish National Training and Employment Authority. The authors thank the reviewers for helpful comments.

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