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Cross terms of the Wigner distribution function and aliasing in numerical simulations of paraxial optical systems

John J. Healy,^{1,2,3} William T. Rhodes,⁴ and John T. Sheridan^{1,2,3,*}

¹UCD Communications and Optoelectronic Research Centre, College of Engineering, Mathematical and Physical Sciences, University College Dublin, Belfield, Dublin 4, Ireland

²SFI Strategic Research Cluster in Solar Energy Conversion

³School of Electrical, Electronic & Mechanical Engineering, College of Engineering, Mathematical and Physical Sciences, University College Dublin, Belfield, Dublin 4, Ireland

⁴Department of Computer and Electrical Engineering and Computer Science, Florida Atlantic University, Florida, 33431, USA

*Corresponding author: john.sheridan@ucd.ie

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Sampling a function periodically replicates its spectrum. As a bilinear function of the signal, the associated Wigner distribution function contains cross terms between the replicas. Often neglected, these cross terms affect numerical simulations of paraxial optical systems. We develop expressions for these cross terms and show their effect on an example calculation. © 2010 Optical Society of America
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The cross terms of the Wigner distribution function (WDF) are often treated as noise. Many Cohen class distributions and discrete WDFs were developed to suppress cross terms [1,2]. Phase space diagrams (PSDs), a simplified representation of the WDF, often implicitly neglect them [3–5]. However, cross terms contain information about the components of the WDF, e.g., the WDF of two coherent beams interfering in a plane contains a cross term representing the interference of the beams. In this Letter, we discuss the consequences of the cross terms that arise due to sampling and, hence, spectral replication—in numerical simulations of paraxial optical systems [6].

The WDF may be defined in terms of a field's Fourier transform (FT) [1]. If $f(x)$ is sampled with sample spacing T_x , the resulting spectrum $F_s(k)$ consists of periodic replicas of $F(k)$ spaced by $2\pi/T_x$, in accord with [7]. Combining these,

$$W\{f_s(x)\}(x, k) = \frac{\sqrt{2\pi}}{T_x^2} \int_{-\infty}^{\infty} \left[\sum_{n=-\infty}^{\infty} F^* \left(k - \frac{2\pi n}{T_x} - \frac{\varepsilon}{2} \right) \right] \times \left[\sum_{m=-\infty}^{\infty} F \left(k - \frac{2\pi m}{T_x} + \frac{\varepsilon}{2} \right) \right] \exp(j\varepsilon x) d\varepsilon. \quad (1)$$

Interchanging integration and summation and considering only terms with $m=n$,

$$\frac{\sqrt{2\pi}}{T_x^2} W\{f(x)\} \left(x, k - \frac{2\pi n}{T_x} \right). \quad (2)$$

These terms, which we refer to as the phase space diagram or *PSD terms*, consist of replicas of $W\{f(x)\}(x, k)$ shifted by multiples of a fundamental period. Nyquist–Shannon assumes compact support of $W\{f(x)\}(x, k)$ in k , and indicates T_x to ensure dis-

joint support of the replicas. The continuous signal $f(x)$ can be recovered by low-pass filtering [6]. In practice, most signals do not have compact support in k , so the replicas overlap one another, resulting in aliasing.

We now consider terms in Eq. (1) with $n \neq m$, the cross terms. The result is made symmetrical by substituting $\varepsilon \rightarrow \varepsilon - 2\pi(n-m)/T_x$ in Eq. (1). Using the shift property of the WDF [8], we obtain terms of the form,

$$\frac{2\pi}{T_x} \exp \left[j2\pi x \frac{(m-n)}{T_x} \right] W\{f(x)\} \left(x, k - \frac{2\pi n + m}{T_x} \frac{1}{2} \right). \quad (3)$$

Pairing these terms assuming $m > n$, we obtain

$$\frac{4\pi}{T_x} \cos \left[2\pi x \frac{(m-n)}{T_x} \right] W\{f(x)\} \left(x, k - \frac{2\pi n + m}{T_x} \frac{1}{2} \right). \quad (4)$$

These cosinusoidally modulated copies of $W\{f(x)\}(x, k)$ appear at multiples of half the period of the PSD terms. We can split them into two categories: (i) those where $(n+m)/2$ is an integer, i.e., $n+m$ is an even integer; and (ii) those where it is not. Category (i) corresponds to terms that lie on top of the PSD terms, category (ii) to those that are centered exactly between the PSD terms. Henceforth, we refer to these as the *even cross terms* and the *odd cross terms*, respectively. Note, $n+m$ even $\Rightarrow m-n$ even. Conversely, $n+m$ odd $\Rightarrow m-n$ odd. Thus, Eq. (4) becomes both

$$\frac{4\pi}{T_x} \cos \left[2\pi x \frac{2q}{T_x} \right] W\{f(x)\} \left(x, k + \frac{2\pi[2r]}{T_x} \right), \quad (5a)$$

$$\frac{4\pi}{T_x} \cos \left[2\pi x \frac{2q-1}{T_x} \right] W\{f(x)\} \left(x, k + \frac{2\pi[2r-1]}{T_x} \right), \quad (5b)$$

where $1 \leq q < \infty$, $-\infty < r < \infty$.

Let us consider an example: a Gaussian of variance $\sqrt{2}$, chosen for its rotationally symmetric WDF.

$$W\{f(x)\}(x, k) = \exp(-x^2 - k^2). \quad (6)$$

Sampling with period $T_x=1$, the WDF consists of the periodic replication of three terms: (a) PSD terms, given by Eq. (2), $-\infty < n < \infty$. (b) Even cross terms, given by Eq. (5a), summed over q . (c) Odd cross terms, given by Eq. (5b), summed over q . In each of these expressions $W\{f(x)\}(x, k)$ is substituted for using Eq. (6). The replicas are shown in Fig. 1. Integration of the WDF with respect to k yields the marginal of the $W\{f(x)\}(x, k)$ [1], $|f(x)|^2$. For a discrete signal, this involves the square of a delta function, a poorly defined operation. Nevertheless, consider the marginal of a single replica of each of the three kinds of terms, truncating the infinite sum in the cross terms, Fig. 2. Unlike the marginal of a WDF, these functions may take negative values. Figure 2(a), the PSD term, resembles the continuous Gaussian. Figure 2(b), the even cross term, consists of narrow pulses at half-integer intervals under a Gaussian envelope, due to the term $\cos[2\pi x(2q/T_x)]$ in Eq. (5a). When x is an integer multiple of $T_x/2$ (recall $T_x=1$), this becomes

Fig. 1. Modified phase space diagram (PSD) of a sampled Gaussian, indicating the cross terms. This example's WDF is circularly symmetric and so may be represented by a circle, diameter chosen as in [5]. The sampling rate is twice the Nyquist rate. At this rate, the PSD terms lie at intervals of twice their diameter. The even cross terms, indicated by fine, dashed circles, lie atop the PSD terms. The odd cross terms, represented by circles of alternating thick dashes and dots, lie between the PSD terms. For a lower sampling rate, the odd cross terms overlap the PSD terms. The figure is infinitely periodic in k .

Fig. 2. Plots of the integral over k of the three basic units making up the WDF of a sampled Gaussian. (a) The PSD term (2); (b) The even cross-term Eq. (5a) (truncating the sum to $1 \leq q \leq 20$); (c) The odd cross-term Eq. (5b) (truncating the sum to $1 \leq q \leq 20$); (d) The sum of the previous three terms [(a)–(c)]. The periodically sampled structure appears to emerge.

$\cos(4\pi w) = 1$, where w is any integer. Figure 2(c), the odd cross terms, consists of narrow, alternately positive and negative pulses under a Gaussian envelope, arising from $\cos[2\pi x(2q-1/T_x)]$ in Eq. (5b). When x is a multiple of T_x , this evaluates as +1, and when x is an odd multiple of $T_x/2$, it becomes -1 . Figures 2(b) and 2(c) appear to converge to zero elsewhere for sufficiently large sums over q . Figure 2(d) shows the sum of the other three parts of Fig. 2. Note how the pulses at half integer intervals cancel, leaving a Gaussian envelope sampled at unit intervals.

We now consider an example that highlights the significance of the cross terms in numerical propagation [6]. To numerically approximate the fractional Fourier transform (FRT) of order 0.5 of a continuous Gaussian, one follows these steps: (1) sample the Gaussian, as before, (2) take the (discrete-space) FRT of these samples, (3) sample the output [(2) and (3) are performed in one step (a discrete FRT), but splitting this up can provide additional insight], (4) truncation and filtering to extract a continuous result from the discrete output. We now discuss how the cross terms created by Step (1) may degrade this calculation.

The WDF after Step (1) was established in the previous example. The WDF after Step (2) may be obtained from the expressions already derived by coordinate substitution, $x \rightarrow x \cos \theta - k \sin \theta$ and $k \rightarrow x \sin \theta + k \cos \theta$, where $\theta = 45^\circ$. This rotates the WDF in Fig. 1 by θ . The marginal of this WDF, illustrated in Fig. 3(a), shows the magnitude squared of the function we are to sample in Step (3). This latter WDF again consists of the sum of shifted replicas of the PSD terms and the cross terms. The PSD terms are periodic along a line in phase space at 45° to the output spatial axis. If we have chosen the sampling rate to be sufficiently high, the overlap between these terms is small. The marginal of the PSD terms consists of a sum of periodically replicated Gaussians [which looks much like Fig. 3(a), as the other terms that make up that figure are relatively small], the error in that approximation being given by the margin-

Fig. 3. (a) Marginal of the order $\frac{1}{2}$ FRT of a Gaussian of variance $\sqrt{2}$, sampled at unit intervals. Circles indicate where the even (thin dash) and odd (thick dash dot) cross terms are located, even terms on the peaks, and odd terms in the troughs. (b) Marginal of an even term, Eq. (7a). (c) Marginal of an odd term, Eq. (7b), with a shift to center it on $\sqrt{\pi/2}$.

als of the even and odd cross terms, which are of the form

$$\frac{4\pi}{T_x} \sum_{q=1}^{\infty} \sqrt{\pi} \exp\left(-x^2 - \left[\frac{2\pi q \sin(\theta)}{T_x}\right]^2\right) \cos\left(\frac{4\pi q x \cos(\theta)}{T_x}\right), \quad (7a)$$

$$\frac{4\pi}{T_x} \sum_{q=1}^{\infty} \sqrt{\pi} \exp\left(-x^2 - \left[\frac{\pi(2q-1)\sin(\theta)}{T_x}\right]^2\right) \cos\left(\frac{2\pi(2q-1)x \cos(\theta)}{T_x}\right). \quad (7b)$$

As before, we truncate the sums in Eqs. (7a) and (7b) to $1 \leq q \leq 20$. As $-q^2$ appears in the exponential term, the contribution of terms for large q is very small, except as $\theta \rightarrow 0$. Equations (7a) and (7b) are plotted in Figs. 3(b) and 3(c), respectively. We emphasize that no “new” aliasing occurs, separate from the overlap of replicas predicted by linear canonical transform sampling theory [6]. The cross terms contain information about (1) the relative position of the PSD terms in k , the marginals being insensitive to shifts in frequency; and (2) a correction for the difference between the marginal of the sum of PSD terms and the sum of the marginals of those PSD terms.

The even marginal, Fig. 3(b), is 9 orders of magnitude smaller than the PSD term, as it is due to the interference of terms separated by at least two periods. The peak of odd marginal is about 1% of the PSD

term. The error between the analytical FRT of a continuous Gaussian and the discrete FRT of the discrete Gaussian is dominated by the overlap of the adjacent replica, but the cross term increases or decreases this error.

We have established a relationship between the cross terms created by sampling and aliasing in discrete linear transforms used in the modeling of paraxial optical systems. Much remains to be done to fully understand the implications for such numerical simulations.

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