Title	Random fractional Fourier transform : stochastic perturbations along the axis of propagation
Authors(s)	Abe, Sumiyoshi, Sheridan, John T.
Publication date	1999-08-01
Publication information	Abe, Sumiyoshi, and John T. Sheridan. "Random Fractional Fourier Transform : Stochastic Perturbations along the Axis of Propagation" 16, no. 8 (August 1, 1999).
Publisher	Optical Society of America
Item record/more information	http://hdl.handle.net/10197/3425
Publisher's statement	This paper was published in Journal of the Optical Society of America A and is made available as an electronic reprint with the permission of OSA. The paper can be found at the following URL on the OSA website: http://www.opticsinfobase.org/abstract.cfm?URI=JOSAA-16-8-1986 . Systematic or multiple reproduction or distribution to multiple locations via electronic or other means is prohibited and is subject to penalties under law.
Publisher's version (DOI)	10.1364/JOSAA.16.001986

Downloaded 2023-04-07T10:52:29Z

The UCD community has made this article openly available. Please share how this access benefits you. Your story matters! (@ucd_oa)

© Some rights reserved. For more information

Random fractional Fourier transform: stochastic perturbations along the axis of propagation

Sumiyoshi Abe

College of Science and Technology, Nihon University, Funabashi, Chiba 274-8501, Japan

John T. Sheridan

Physics Department, Dublin Institute of Technology, Kevin Street, Dublin 8, Ireland

Received August 25, 1998; revised manuscript received March 23, 1999; accepted April 5, 1999

The fractional Fourier transform (FRT) is known to be optically implementable with use of a medium with a perfect radial quadratic-index profile. Using the quantum-mechanical operator formalism, we examine the effects on the FRT action of such a medium that are due to small random inhomogeneities in the longitudinal direction, the direction of propagation, and we formulate the random fractional Fourier transform (RFRT). Applying the RFRT to a self-fractional Fourier function, a Gaussian function, we discuss both the total power and the variance. The random Fourier transform is examined as a special limiting case. © 1999 Optical Society of America [S0740-3232(99)00108-8]

OCIS codes: 070.2590, 000.5490, 030.4280, 270.2500.

1. INTRODUCTION

The fractional Fourier transform (FRT) has received much attention in the applied optics literature since the early 1990's.¹⁻³ It has been shown that the FRT can be implemented using both gradient-index media and free-space optics. Theoretical analysis of FRT systems to date have assumed that perfect optical systems were used in the implementation. Therefore, although the elimination of input signal noise using FRT filtering has received attention, the effects of system-induced noise have remained essentially unexplored.

Propagation of light through random media is of intense experimental and theoretical interest because of its fundamental importance for the applied sciences. (See the review paper⁴ and the references therein.) This interest has led to many examinations of the optical effects of static devices (diffusers),⁵ dynamically varying media (atmospheric turbulence),⁴ and gas lenses.^{6,7} The effects of system-induced noise in free-space optics⁸ and in waveguide systems^{9,10} have also received attention. Such analyses usually take the wave equation as their starting point and introduce some simplifying assumptions, for example, reduced dimensionality and the paraxial approximation. They then proceed to develop analytic equations for the moments of the output wave function given a particular input wave function and specific stochastic perturbation.

Many optical signal-processing systems—for example, filtering and correlation systems—have as a basic element a Fourier transforming unit.¹¹ The use of waveguides with parabolic cross-sectional index profiles as microlaser collimators is also widespread, and the operation of such gradient-index or self-focusing devices can

be modeled with the FRT. Based on these observations, it would seem appropriate to develop a transform-centered procedure that takes into account any random perturbations within the optical system.

In this paper we develop an extension of the FRT that we refer to as the random fractional Fourier transform (RFRT). Assuming a weak (first-order), white-noise (wide-frequency-band) random perturbation of the medium in the direction of propagation, we present a general expression for the RFRT output field, including intensity and phase effects. To illustrate these effects, a self-fractional Fourier function, a Gaussian function, is used as input to such a perturbed medium, and the resulting deformations of the output field are calculated. The range of validity of the method, which is limited owing to the use of the perturbative approximation, is discussed.

Our derivation is carried out with use of the quantum-mechanical operator formalism that has previously been applied in this area to systematically explore generalized linear transformations.^{12,13} Besides the obvious advantage of allowing analogies with Hamiltonian mechanics to be exploited, the application of the operator formalism may also allow the convenient extension of these results.

The paper is organized as follows. In Section 2 the derivation of the RFRT is presented, and operator formalism notation and concepts are detailed where necessary. In Section 3 the physical significance of the RFRT is examined with use of a ground-state Hermite–Gaussian mode as input. Since this function acts as a self-fractional Fourier function, a simple comparison of the input and output fields is possible. Section 4 contains concluding remarks and possible future extensions of the present result.

2. RANDOM FRACTIONAL FOURIER TRANSFORM DERIVATION WITH USE OF THE OPERATOR FORMALISM

The FRT is known to be generated by the following quantum-mechanical harmonic-oscillator Hamiltonian operator¹²:

$$\hat{H}_0 = \frac{1}{2}\hat{p}^2 + \frac{1}{2}\hat{x}^2. \quad (1)$$

In this equation, $\hat{p}$ and $\hat{x}$ are the usual momentum and position operators. Unlike in the quantum-mechanical case, we substitute z , longitudinal distance, for t , time. For simplicity all variables are made dimensionless. The basic paraxial wave equation in the operator formalism is

$$i\frac{\partial}{\partial z}|\psi_0(z)\rangle = \hat{H}_0|\psi_0(z)\rangle, \quad (2)$$

where k , the wave number in the z direction, is set equal to unity. This differential equation has the formal solution

$$|\psi_0(z)\rangle = \exp[-i\hat{H}_0(z - z_0)]|\psi(z_0)\rangle. \quad (3)$$

Here $|\psi(z_0)\rangle$ and $|\psi_0(z)\rangle$ are the abstract wave vectors. In the position representation they are written as usual as $\psi(x, z_0) = \langle x|\psi(z_0)\rangle$ and $\psi_0(x, z) = \langle x|\psi_0(z)\rangle$, respectively. In this representation, Eq. (2) is expressed in the following familiar form:

$$i\frac{\partial\psi_0(x, z)}{\partial z} = -\frac{1}{2}\frac{\partial^2\psi_0(x, z)}{\partial x^2} + \frac{1}{2}x^2\psi_0(x, z). \quad (4)$$

$\psi(x, z_0)$ is the input to the system at $z = z_0$ (initial condition), and $\psi_0(x, z)$ is the output, i.e., the FRT of $\psi(x, z_0)$, after a distance of $z - z_0$ has been propagated. [See Eq. (24) below.]

In this paper we wish to generalize to the case in which the system Hamiltonian is a function of z :

$$\hat{H}(z) = \frac{1}{2}\hat{p}^2 + \frac{1}{2}\omega^2(z)\hat{x}^2. \quad (5)$$

We perturb ω so as to include a randomly varying component $f(z)$ in the framework of the FRT:

$$\omega^2(z) = 1 + 2f(z). \quad (6)$$

We assume that $f(z)$ is the Gaussian white noise that satisfies the following conditions:

$$\langle f(z) \rangle = 0, \quad \langle (fz)f(z') \rangle = D\delta(z - z'), \quad (7)$$

where the brackets represent averaging over the noise and D denotes the diffusion constant, which is assumed to be sufficiently small to make a perturbative analysis valid. This will be discussed below. The basic wave equation that must now be satisfied has the stochastic form

$$i\frac{\partial}{\partial z}|\psi(z)\rangle = \hat{H}(z)|\psi(z)\rangle. \quad (8)$$

Again in the position representation this equation is expressed as follows:

$$i\frac{\partial\psi(x, z)}{\partial z} = -\frac{1}{2}\frac{\partial^2\psi(x, z)}{\partial x^2} + \frac{1}{2}\omega^2(z)x^2\psi(x, z). \quad (9)$$

To solve Eq. (8) we apply time-dependent perturbation theory¹⁴ and introduce the unitary evolution operator $\hat{W}(z, z_0)$, which yields the solution of this equation in the form

$$|\psi(z)\rangle = \hat{W}(z, z_0)|\psi(z_0)\rangle. \quad (10)$$

Clearly, $\hat{W}(z, z_0)$ satisfies the operator equation

$$i\frac{\partial}{\partial z}\hat{W}(z, z_0) = \hat{H}(z)\hat{W}(z, z_0), \quad (11)$$

with the initial condition $\hat{W}(z_0, z_0) = 1$. The solution to this equation is known to be given by¹⁴

$$\hat{W}(z, z_0) = T\exp\left[-i\int_{z_0}^z d\tau\hat{H}(\tau)\right]. \quad (12)$$

Here T represents the time-ordering symbol; e.g., $T[\hat{A}(z)\hat{B}(z')] = \Theta(z - z')\hat{A}(z)\hat{B}(z') + \Theta(z' - z)\hat{B}(z')\hat{A}(z)$, and $\Theta(z)$ is the Heaviside unit-step function.

Using this form, we now wish to separate the noise contribution from the pure FRT operator part. To do this we decompose $\hat{W}(z, z_0)$ as follows:

$$\hat{W}(z, z_0) = \exp[-i\hat{H}_0(z - z_0)]\hat{U}(z, z_0), \quad (13)$$

where $\hat{U}(z, z_0)$ is a unitary operator that contains the z -direction perturbations. To calculate this operator, we substitute Eq. (13) into Eq. (11). The solution of the resulting equation for $\hat{U}(z, z_0)$ has the form

$$\begin{aligned} \hat{U}(z, z_0) = T\exp\left\{-i\int_{z_0}^z d\tau f(\tau)\exp[i\hat{H}_0(z - z_0)]\hat{x}^2 \right. \\ \left. \times \exp[-i\hat{H}_0(z - z_0)]\right\}. \end{aligned} \quad (14)$$

To separate the pure FRT from the perturbation effects, we write

$$\hat{W}(z, z_0) = \hat{V}(z, z_0)\exp[-i\hat{H}_0(z - z_0)], \quad (15)$$

where

$$\begin{aligned} \hat{V}(z, z_0) = \exp[-i\hat{H}_0(z - z_0)]\hat{U}(z, z_0) \\ \times \exp[\hat{H}_0(z - z_0)]. \end{aligned} \quad (16)$$

$\hat{V}(z, z_0)$ now represents the perturbation operator. Substituting from above for $\hat{U}(z, z_0)$, applying the time-ordered product on the interval $[z_0, z]$, and using the operator identity $\exp(-\hat{A})g(\hat{B})\exp(\hat{A}) = g[\exp(-\hat{A})\hat{B}\exp(\hat{A})]$ for an operator-valued function g , the perturbative expansion of $\hat{V}(z, z_0)$ with respect to $f(z)$ can be found:

$$\begin{aligned}\hat{V}(z, z_0) = & 1 - i \int_{z_0}^z d\tau f(\tau) \hat{X}^2(\tau - z) \\ & + \frac{(-i)^2}{2!} \int_{z_0}^z d\tau \int_{z_0}^z d\tau' f(\tau) f(\tau') \\ & \times T[\hat{X}^2(\tau - z) \hat{X}^2(\tau' - z)] \\ & + (\text{terms of higher order in } f), \quad (17)\end{aligned}$$

where

$$\begin{aligned}\hat{X}(\tau - z) = & \exp[i\hat{H}_0(\tau - z)] \hat{x} \exp[-i\hat{H}_0(\tau - z)] \\ = & \hat{x} \cos(\tau - z) + \hat{p} \sin(\tau - z). \quad (18)\end{aligned}$$

We now define the mutual-intensity function,^{5,15} which in quantum mechanics is called the density matrix^{14,15}:

$$\rho(x', x; z) = \langle x' | \psi(z) \rangle \langle \psi(z) | x \rangle = \psi(x', z) \psi^*(x, z). \quad (19)$$

We note that if $x = x'$, then $\rho(x, x; z) = |\psi(x, z)|^2$, the intensity at the point (x, z) . In the perturbed FRT case $\rho(x', x; z)$ has the form

$$\begin{aligned}\rho(x', x; z) = & \langle x' | \hat{V}(z, z_0) \exp[-i\hat{H}_0(z - z_0)] | \psi(z_0) \rangle \\ & \times \langle \psi(z_0) | \exp[i\hat{H}_0(z - z_0)] \hat{V}^\dagger(z, z_0) | x \rangle, \quad (20)\end{aligned}$$

where the dagger represents Hermitian conjugation. $\rho(x', x; z)$ does not represent a measured quantity. Measurement will involve averaging $\rho(x', x; z)$ over the noise fluctuations in the system, i.e., incoherent reduction. We refer to the resulting quantity as the reduced density matrix,

$$\rho_R(x', x; z) = \langle \rho(x', x; z) \rangle, \quad (21)$$

where the brackets represent averaging over the noise that is carried out with the expressions given in Eq. (7). This quantity can actually be regarded as the second-order correlation function at equal z .

Substituting the expansion for $\hat{V}(z, z_0)$ given in Eq. (17) into Eq. (21), neglecting the higher order terms in f ,

and then using the fact that $T[\hat{X}^2(\tau - z) \hat{X}^2(\tau - z)] = \hat{X}^4(\tau - z)$, we obtain the following form for the reduced density matrix:

$$\begin{aligned}\rho_R(x', x; z) = & \langle x' | \psi_0(z) \rangle \langle \psi_0(z) | x \rangle \\ & + D \int_{z_0}^z d\tau \langle x' | \hat{X}^2(\tau - z) | \psi_0(z) \rangle \langle \psi_0(z) | \hat{X}^2(\tau - z) | x \rangle \\ & - \frac{D}{2} \int_{z_0}^z d\tau [\langle x' | \psi_0(z) \rangle \langle \psi_0(z) | \hat{X}^4(\tau - z) | x \rangle \\ & + \langle x' | \hat{X}^4(\tau - z) | \psi_0(z) \rangle \langle \psi_0(z) | x \rangle], \quad (22)\end{aligned}$$

where, as in Eq. (3), $|\psi_0(z)\rangle = \exp[-i\hat{H}_0(z - z_0)] \times |\psi(z_0)\rangle$ is the unperturbed FRT of the input $|\psi(z_0)\rangle$, and therefore the first term on the right-hand side represents the unperturbed FRT density matrix.

We recall that we wish to write our final expression using the integral FRT form. To do so, we use Eq. (18) and introduce the following definition of a weighted FRT integral:

$$\begin{aligned}[F_\theta^{(n)}\psi](x) = & \frac{1}{\sqrt{2\pi}|\sin\theta|} \exp\left(-i\phi + \frac{i}{2}x^2 \cot\theta\right) \\ & \times \int_{-\infty}^{\infty} dx_0 x_0^n \exp\left(\frac{i}{2}x_0^2 \cot\theta - ixx_0 \operatorname{cosec}\theta\right) \psi(x_0), \quad (23)\end{aligned}$$

where $[F_\theta^{(0)}\psi](x) \equiv [F_\theta\psi](x)$ is the standard FRT of $\psi(x, z_0)$ of angular order θ , and ϕ is a constant phase. In the present paper, since all physical quantities are normalized to be dimensionless, the angular order is explicitly defined in terms of the propagation distance as follows:

$$\theta = z - z_0. \quad (24)$$

Using Eq. (23) in Eq. (22), we calculate the reduced density matrix to be

$$\begin{aligned}\rho_R(x', x; z) \equiv & \rho_\theta(x', x) \\ = & \left\{ 1 - \frac{D}{2} [3 \cot\theta - \theta(1 + 3 \cot^2\theta)] - \frac{iD}{2} (x'^2 - x^2)(2 + 3 \cot^2\theta - 3\theta \cot\theta \operatorname{cosec}^2\theta) \right. \\ & + \frac{D}{16} (x'^2 - x^2)^2 (5 \cot\theta + 3 \cot^3\theta - 3\theta \operatorname{cosec}^4\theta) \left. \right\} [F_\theta\psi](x') [F_\theta\psi]^*(x) \\ & + \left\{ -iDx [3 \cot\theta - \theta(1 + 3 \cot^2\theta)] \operatorname{cosec}\theta \right. \\ & + \frac{D}{4} x(x'^2 - x^2)(2 + 3 \cot^2\theta - 3\theta \cot\theta \operatorname{cosec}^2\theta) \operatorname{cosec}\theta \left. \right\} [F_\theta\psi](x') [F_\theta^{(1)}\psi]^*(x) \\ & + \left\{ iDx' [3 \cot\theta - \theta(1 + 3 \cot^2\theta)] \operatorname{cosec}\theta \right. \\ & - \frac{D}{4} x'(x'^2 - x^2)(2 + 3 \cot^2\theta - 3\theta \cot\theta \operatorname{cosec}^2\theta) \operatorname{cosec}\theta \left. \right\} [F_\theta^{(1)}\psi](x') [F_\theta\psi]^*(x)\end{aligned}$$

$$\begin{aligned}
& + \left\{ -\frac{iD}{2}(1 - 3 \operatorname{cosec}^2 \theta + 3 \theta \cot \theta \operatorname{cosec}^2 \theta) \right. \\
& - \frac{D}{8}(x'^2 - 3x^2)[3 \cot \theta - \theta(1 + 3 \cot^2 \theta)] \operatorname{cosec}^2 \theta \left. \right\} [F_\theta \psi](x') [F_\theta^{(2)} \psi]^*(x) \\
& + \left\{ \frac{iD}{2}(1 - 3 \operatorname{cosec}^2 \theta + 3 \theta \cot \theta \operatorname{cosec}^2 \theta) \right. \\
& - \frac{D}{8}(x^2 - 3x'^2)[3 \cot \theta - \theta(1 + 3 \cot^2 \theta)] \operatorname{cosec}^2 \theta \left. \right\} [F_\theta^{(2)} \psi](x') [F_\theta \psi]^*(x) \\
& - \frac{D}{2} x' x [3 \cot \theta - \theta(1 + 3 \cot^2 \theta)] \operatorname{cosec}^2 \theta [F_\theta^{(1)} \psi](x') [F_\theta^{(1)} \psi]^*(x) \\
& + \frac{D}{4} x(1 - 3 \operatorname{cosec}^2 \theta + 3 \theta \cot \theta \operatorname{cosec}^2 \theta) \operatorname{cosec} \theta [F_\theta \psi](x') [F_\theta^{(3)} \psi]^*(x) \\
& + \frac{D}{4} x'(1 - 3 \operatorname{cosec}^2 \theta + 3 \theta \cot \theta \operatorname{cosec}^2 \theta) \operatorname{cosec} \theta [F_\theta^{(3)} \psi](x') [F_\theta \psi]^*(x) \\
& - \frac{D}{4} x'(1 - 3 \operatorname{cosec}^2 \theta + 3 \theta \cot \theta \operatorname{cosec}^2 \theta) \operatorname{cosec} \theta [F_\theta^{(1)} \psi](x') [F_\theta^{(2)} \psi]^*(x) \\
& - \frac{D}{4} x(1 - 3 \operatorname{cosec}^2 \theta + 3 \theta \cot \theta \operatorname{cosec}^2 \theta) \operatorname{cosec} \theta [F_\theta^{(2)} \psi](x') [F_\theta^{(1)} \psi]^*(x) \\
& - \frac{D}{8}(2 \cot \theta - 3 \cot \theta \operatorname{cosec}^2 \theta + 3 \theta \operatorname{cosec}^4 \theta) [F_\theta^{(2)} \psi](x') [F_\theta^{(2)} \psi]^*(x) \\
& + \frac{D}{16}(2 \cot \theta - 3 \cot \theta \operatorname{cosec}^2 \theta + 3 \theta \operatorname{cosec}^4 \theta) [F_\theta \psi](x') [F_\theta^{(4)} \psi]^*(x) \\
& + \frac{D}{16}(2 \cot \theta - 3 \cot \theta \operatorname{cosec}^2 \theta + 3 \theta \operatorname{cosec}^4 \theta) [F_\theta^{(4)} \psi](x') [F_\theta \psi]^*(x). \quad (25)
\end{aligned}$$

In particular, the diagonal element of the density matrix is given by

$$\begin{aligned}
\rho_\theta(x, x) = & \left\{ 1 - \frac{D}{2}[3 \cot \theta - \theta(1 + 3 \cot^2 \theta)] \right\} |[F_\theta \psi](x)|^2 \\
& + 2Dx[3 \cot \theta - \theta(1 + 3 \cot^2 \theta)] \operatorname{cosec} \theta \operatorname{Im}\{[F_\theta \psi](x)[F_\theta^{(1)} \psi]^*(x)\} \\
& + D[1 - 3 \operatorname{cosec}^2 \theta + 3 \theta \cot \theta \operatorname{cosec}^2 \theta] \operatorname{Im}\{[F_\theta \psi](x)[F_\theta^{(2)} \psi]^*(x)\} \\
& + \frac{D}{2}x^2[3 \cot \theta - \theta(1 + 3 \cot^2 \theta)] \operatorname{cosec}^2 \theta \operatorname{Re}\{[F_\theta \psi](x)[F_\theta^{(2)} \psi]^*(x)\} \\
& - \frac{D}{2}x^2[3 \cot \theta - \theta(1 + 3 \cot^2 \theta)] \operatorname{cosec}^2 \theta |[F_\theta^{(1)} \psi](x)|^2 \\
& + \frac{D}{2}x(1 - 3 \operatorname{cosec}^2 \theta + 3 \theta \cot \theta \operatorname{cosec}^2 \theta) \operatorname{cosec} \theta (\operatorname{Re}\{[F_\theta \psi](x)[F_\theta^{(3)} \psi]^*(x)\} - \operatorname{Re}\{[F_\theta^{(1)} \psi](x)[F_\theta^{(2)} \psi]^*(x)\}) \\
& - \frac{D}{8}(2 \cot \theta - 3 \cot \theta \operatorname{cosec}^2 \theta + 3 \theta \operatorname{cosec}^4 \theta) (|[F_\theta^{(2)} \psi](x)|^2 - \operatorname{Re}\{[F_\theta \psi](x)[F_\theta^{(4)} \psi]^*(x)\}). \quad (26)
\end{aligned}$$

Detailed technical information regarding all the derivations presented here is available on request from the authors.¹⁶

3. PHYSICAL INTERPRETATION OF THE RESULT

In order to examine the physical significance of these results, we employ the ground-state Hermite-

Gaussian mode

$$\psi_G(x_0) = A \exp(-x_0^2/2), \quad (27)$$

where A is a real constant. This function is a self-fractional Fourier function, that is, $[F_\theta \psi_G](x) = \exp(-i\phi') \psi_G(x)$, where ϕ' is a constant phase. For this function, the weighted FRT integrals employed in the reduced density matrix simplify as follows:

$$[F_{\theta}^{(0)}\psi_G](x) = [F_{\theta}\psi_G](x) = B\sqrt{\frac{2\pi}{\alpha}}, \quad (28)$$

$$[F_{\theta}^{(1)}\psi_G](x) = -\beta[F_{\theta}\psi_G](x), \quad (29)$$

$$[F_{\theta}^{(2)}\psi_G](x) = \left(\frac{1}{\alpha} + \beta^2\right)[F_{\theta}\psi_G](x), \quad (30)$$

$$[F_{\theta}^{(3)}\psi_G](x) = -\beta\left(\frac{3}{\alpha} + \beta^2\right)[F_{\theta}\psi_G](x), \quad (31)$$

$$[F_{\theta}^{(4)}\psi_G](x) = \left(\frac{3}{\alpha^2} + \frac{6\beta^2}{\alpha} + \beta^4\right)[F_{\theta}\psi_G](x), \quad (32)$$

where

$$B = \frac{A}{\sqrt{2\pi|\sin\theta|}} \exp\left(-i\phi' - \frac{x^2}{2}\right), \quad (33)$$

$$\alpha = -i \exp(i\theta) \operatorname{cosec} \theta, \quad (34)$$

$$\beta = -x \exp(-i\theta). \quad (35)$$

If we substitute these results into Eq. (26), it can be shown that

$$\begin{aligned} \rho_{G,\theta}(x, x)/|[F_{\theta}\psi_G](x)|^2 \\ = 1 + \frac{D}{4} [12 \cot \theta + 6 \sin^3 \theta \cos \theta - 17 \sin \theta \cos \theta \\ + \theta(11 - 6 \cot^2 \theta - 6 \operatorname{cosec}^2 \theta)] \\ + Dx^2 \left[-15 \cot \theta - 6 \sin^3 \theta + 19 \sin \theta \cos \theta \right. \\ \left. + \frac{\theta}{2} (-23 + 15 \cot^2 \theta + 15 \operatorname{cosec}^2 \theta) \right] \\ + Dx^4 [(3 \cot \theta - 2 \sin \theta \cos \theta)(1 + \cos^2 \theta) \\ + \theta(1 - 6 \cot^2 \theta)]. \end{aligned} \quad (36)$$

On examining this equation we note that the periodicity or cyclic behavior of the FRT output, which for a general input function repeats itself every 2π rad, has been destroyed by the inclusion of the random noise. In Eqs. (25), (26), and (36), this destruction appears as terms of the form $D\theta$. These terms are known as secular terms, and the perturbative technique requires that the magnitude of such terms be small, that is, $D\theta \ll 1$. In the present work the range of θ is chosen to be

$$0 \leq \theta \leq \pi/2, \quad (37)$$

which simplifies the fixing of the secular term size, thus ensuring the validity of the perturbative technique.

Let us discuss two limiting cases of particular interest: First, in the case where the angular order of the FRT is small ($\theta \ll 1$), the infinitesimal RFRT, it can be shown that

$$\rho_{G,\theta}(x, x) \cong [1 - D\theta(x^4 - 10x^2 + 2)][|F_{\theta}\psi_G](x)|^2. \quad (38)$$

Clearly, $\rho_{G,\theta}(x, x)/|[F_{\theta}\psi_G](x)|^2 \rightarrow 1$ ($\theta \rightarrow +0$), which indicates that the input field is identical to the output field, as it should be. Second, we examine the case where the

angular order of the FRT is $\pi/2$. For this value the FRT becomes the Fourier transform, and it can be shown that

$$\begin{aligned} \rho_{G,\pi/2}(x, x) = \left\{ 1 + \frac{\pi D}{2} \left[x^4 - 4 \left(1 + \frac{3}{\pi} \right) x^2 + \frac{5}{4} \right] \right\} \\ \times |[F_{\pi/2}\psi_G](x)|^2. \end{aligned} \quad (39)$$

This is an interesting result since it indicates that points exist in the output plane at which, to the first order, the quality of the Fourier transform is unaffected by the perturbative noise. It would also seem that through the use of suitable scaling, these zero noise points can be chosen.

Let us now examine the total output power in the system. This quantity can be found as follows:

$$\begin{aligned} P_{\text{out}} &= \int_{-\infty}^{\infty} dx \rho_{G,\theta}(x, x) \\ &= \sqrt{\pi} A^2 [1 - 3D \sin^3 \theta (1 - \cos \theta)]. \end{aligned} \quad (40)$$

Clearly, power appears to be lost in the system since the output power predicted is less than the input power $P_{\text{in}} = \sqrt{\pi} A^2$. This is because only forward propagation is taken into account here. Examining the small-angle approximation of Eq. (40), we find that $P_{\text{out}} \cong \sqrt{\pi} A^2 (1 - (\frac{3}{2}) D \theta^5)$. Perhaps the most remarkable facts in relation to result (40) are that no secular terms appear in the expression and that it is periodic.

We can also examine the variance of the output, which is defined as follows:

$$(\Delta x)^2 = \int_{-\infty}^{\infty} dx x^2 \rho_{G,\theta}(x, x) / \int_{-\infty}^{\infty} dx \rho_{G,\theta}(x, x). \quad (41)$$

This can be shown to be equal to

$$\begin{aligned} (\Delta x)^2 &= \frac{1}{2} \{ 1 + D [\cot \theta + 2 \cot \theta \cos^2 \theta \\ &\quad + 9 \sin^3 \theta (\cos \theta - 1)] \\ &\quad - D \theta (1 + 3 \cot^2 \theta) \} / \\ &\quad [1 - 3D \sin^3 \theta (1 - \cos \theta)]. \end{aligned} \quad (42)$$

We note that the variance of the input Gaussian wave function has a value of $1/2$, and if the noise is eliminated we return to this variance value in our output beam. Examining this expression we see that as the angular order increases to $\pi/2$, the variance decreases monotonically. Therefore the intensity distribution in x narrows, or is squeezed.

4. CONCLUDING REMARKS

We have derived the random fractional Fourier transform (RFRT), using the quantum-mechanical operator formalism. The reduced density matrix has been presented for a general input function. The total power and the variance of the output wave function have been examined for the special case of an input ground-state Hermite-Gaussian wave function. The perturbative nature of the calculation is indicated by the existence of secular terms in the reduced density matrix. The random Fourier transform has also been examined, as a special limiting case of the RFRT. It has been shown that in this case,

points exist in the output plane at which the effect of the first-order perturbation vanishes. The off-diagonal terms of the RFRT reduced density matrix presented here are necessary for calculating phase diffusion, i.e., the reduction in coherence of the input wave function caused by the random fluctuations in the medium. These problems relating coherent, partially coherent, and incoherent inputs and outputs are still to be explored.

Two obvious generalizations of the results presented here are (i) examination of the effects of other types of noise, e.g., colored noise, and (ii) extension to general linear canonical transformations.^{12,13} These generalizations remain to be examined.

Finally, we make a comment on the potential significance of applying renormalization group theory to this problem. It has previously been shown¹⁷ that when this theory is used, the effects of the secular terms, the $D\theta$ terms in our case, that arise when perturbation techniques are used can be eliminated. In the present analysis θ is limited to the finite range in Eq. (37), which is of primary interest in the FRT theory. Therefore the condition $D\theta \ll 1$ is in fact realized if D is typically $\sim 10^{-3}$, making $D\theta$ comparable with, e.g., the fine-structure constant. Under this condition, the results obtained by using the renormalization group asymptotically coincide with the present perturbative ones. However, if one would wish to extend the range of θ arbitrarily large, then elimination of the secular terms by renormalization group theory would be of great benefit.

ACKNOWLEDGMENT

Part of this work was carried out while J. T. Sheridan was a visiting researcher at Nihon University, supported by funding from FORBAIRT Ireland under the auspices of the National Research Support Fund Board's International Collaboration Programme.

REFERENCES

1. V. Namias, "The fractional order Fourier transform and its application to quantum mechanics," *J. Inst. Math. Appl.* **25**, 241–265 (1980).

2. H. M. Ozaktas and D. Mendlovic, "Fractional Fourier optics," *J. Opt. Soc. Am. A* **12**, 743–751 (1995).
3. A. W. Lohmann, "Image rotation, Wigner rotation, and the fractional Fourier transform," *J. Opt. Soc. Am. A* **10**, 2181–2186 (1993).
4. M. I. Charnotskii, J. Gozani, V. I. Tatarskii, and V. U. Zavorotny, "Wave propagation theories in random media based on the path-integral approach," in *Progress in Optics XXXII*, E. Wolf, ed. (North-Holland, Amsterdam, The Netherlands, 1993), Chap. IV, pp. 203–266.
5. M. Born and E. Wolf, *Principles of Optics*, 6th ed. (Pergamon, Oxford, UK, 1980).
6. M. Imai, S. Kikuchi, T. Matsumoto, and Y. Kinoshita, "Mode conversion due to fluctuations in a lens-like medium," *J. Opt. Soc. Am.* **59**, 904–913 (1969).
7. T. Asakura, Y. Kinoshita, and M. Suzuki, "Further correlation studies of Gaussian-beam fluctuations caused by a random medium," *J. Opt. Soc. Am.* **59**, 913–920 (1969).
8. G. C. Papanicolaou, D. McLaughlin, and R. Burridge, "A stochastic Gaussian beam," *J. Math. Phys.* **14**, 84–89 (1973).
9. M. Eve and J. H. Hannay, "Ray theory and random mode coupling in an optical fibre wave guide," *Opt. Quantum Electron.* **8**, "Part I," 503–508; "Part II," 509–512 (1976).
10. A. Sharma, I. C. Goyal, N. K. Bansal, and A. K. Ghatak, "Propagation of Gaussian beams through parabolic index optical waveguides with random dielectric constant gradient," *Fiber Integr. Opt.* **2**, 299–314 (1979).
11. J. W. Goodman, *Introduction to Fourier Optics*, 2nd ed. (McGraw-Hill, New York, 1996).
12. S. Abe and J. T. Sheridan, "Generalization of the fractional Fourier transformation to an arbitrary linear lossless transformation: an operator approach," *J. Phys. A Math. Gen.* **27**, 4179–4187 (1994); Corrigenda **27**, 7937 (1994).
13. S. Abe and J. T. Sheridan, "Optical operations on wave functions as the Abelian subgroups of the special affine Fourier transformation," *Opt. Lett.* **19**, 1801–1803 (1994).
14. M. O. Scully and M. S. Zubairy, *Quantum Optics* (Cambridge U. Press, Cambridge, UK, 1997).
15. L. Mandel and E. Wolf, *Optical Coherence and Quantum Optics* (Cambridge U. Press, Cambridge, UK, 1995).
16. Information is available from Sumiyoshi Abe, College of Science and Technology, Nihon University, 7-24-1 Narashinodai, Funabashi Chiba 274-8501, Japan, or John T. Sheridan, Physics Department, Dublin Institute of Technology, Kevin Street, Dublin 8, Ireland.
17. L.-Y. Chen, N. Goldenfeld, and Y. Oono, "Renormalization group and singular perturbations: multiple scales, boundary layers, and reductive perturbation theory," *Phys. Rev. E* **54**, 376–394 (1996).