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Wigner Cross-Terms in Sampled and other Periodic Signals

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Abstract: If we sample a scalar wave field, it becomes periodic in frequency. We examine the cross-terms which occur between these periodic replicas in the Wigner-Ville distribution function of such a signal. We present analytic results for Gaussian signals. The results also have implications for physical systems which contain periodic gratings.

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1. Introduction

The cross-terms inherent in the Wigner-Ville distribution function (WDF) are often treated as an unwanted noise, something to be ignored and preferably suppressed. However, they are related to the physical phenomenon of interference, and deserve closer study. Cross-terms appear in the WDF of a sampled signal. What role do they play in reconstruction? Can we simply ignore them?

For simplicity, we consider 1D wave fields. A 1D scalar wave field may be represented by a complex function of a real spatial variable, $f(x)$. The WDF of $f(x)$, with a Fourier transform (FT) $F(k)$, may be given by the following [1]:

$$W\{f(x)\}(x, k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F^*\left(k - \frac{\varepsilon}{2}\right) F\left(k + \frac{\varepsilon}{2}\right) \exp(j\varepsilon x) d\varepsilon. \quad (1)$$

2. Sampling

We wish to uniformly sample $f(x)$ with period of T_x . The sampled signal is expressed as the product of the unsampled signal and a comb of Dirac delta functions [2]. Consequently, the FT of this is therefore the convolution of the FT of the unsampled signal with the FT of the comb, giving,

$$\frac{\sqrt{2\pi}}{T_x} \sum_{n=-\infty}^{\infty} F\left(k - \frac{2\pi n}{T_x}\right). \quad (2)$$

Consider the WDF of the sampled signal, given by,

$$W\{g(x)\}(x, k) = \frac{\sqrt{2\pi}}{T_x^2} \int_{-\infty}^{\infty} \left[\sum_{n=-\infty}^{\infty} F^*\left(k - \frac{2\pi n}{T_x} - \frac{\varepsilon}{2}\right) \right] \left[\sum_{m=-\infty}^{\infty} F\left(k - \frac{2\pi m}{T_x} + \frac{\varepsilon}{2}\right) \right] \exp(j\varepsilon x) d\varepsilon. \quad (3)$$

We take the sums outside of the integral, and obtain a sum over the variables n and m of terms of a particular form. Consider first those terms for $n = m$. These are of the form

$$\frac{2\pi}{T_x^2} W\{f(x)\}\left(x, k - \frac{2\pi n}{T_x}\right) \quad (4)$$

by the frequency shift property of the WDF [1]. These terms are shifted replicas of the WDF of the unsampled signal, $f(x)$. We will refer to these as the natural terms. The remaining terms, i.e. those for which $n \neq m$, are the cross-terms created by interference between the natural terms. These are given by,

$$\frac{\sqrt{2\pi}}{T_x^2} \int_{-\infty}^{\infty} F^*\left(k - \frac{2\pi n}{T_x} - \frac{\varepsilon}{2}\right) F\left(k - \frac{2\pi m}{T_x} + \frac{\varepsilon}{2}\right) \exp(j\varepsilon x) d\varepsilon. \quad (5)$$

We wish to write this in terms of the WDF of the signal, $f(x)$. Substitute the following for ε ,

$$\varepsilon \rightarrow \varepsilon + \frac{2\pi(n-m)}{T_x}, \quad (6)$$

Additionally assuming $m > n$, Eq. (5) becomes,

$$2 \cos \left[2\pi x \frac{(n-m)}{T_x} \right] W \left\{ f(x) \right\} \left(x, k + \frac{\pi(n+m)}{T_x} \right). \quad (7)$$

These are cosinusoidally modulated replicas of the WDF of $f(x)$, shifted to half way between the interfering natural terms (i.e., at integer and half integer multiples of $1/T_x$) and multiplied by a linear phase which depends on the separation between them. Eqs. (4) and (7) determine the general form of the WDF of a sampled function. We consider the special case of a Gaussian for the purposes of illustration. We construct a signal from a Gaussian and several periodic replicas. Using an infinite number would produce a signal which was truly discrete. However, we truncate this series to simplify analysis. This signal is shown in Fig. 1(a). Fig. 1(b) shows the WDF of the signal. The ridges at the extremes of the distribution are replicas free of cross-terms. The remaining replicas have substantial cross-terms overlapping them.

Figure 1. (a) ‘Semi-discrete’ Gaussian, formed by adding six shifted replicas to the continuous Gaussian. (b) WDF of this signal.

3. Propagation and Coherent Aliasing

Fresnel propagation [3] of the wave field alters the WDF by skewing it in the x direction [4]. Sufficient propagation will separate the replicas in the space domain, though they will retain their relative frequency shifts. We argue that the cross-terms characterize the interference terms between these separated Gaussians, and close study of their behavior may provide a insight into interference from the perspective of Fourier optics.

Furthermore, considerable attention has been devoted in recent years to sampling of the linear canonical transforms which describe paraxial propagation in quadratic phase systems. An overview of this may be found in [5]. Regarding discrete-time Fourier transforms, the cross-terms created by periodicity in the Fourier domain are known to integrate to zero in the Fourier domain. However, further study is required to determine the nature of the effect that these cross-terms have in the more general linear canonical domains. Stern’s derivation [6] of the sampling theorem for signals bounded in some linear canonical transform domain provides an argument against the existence of some sort of coherent aliasing due to the cross-terms in addition to the aliasing described by Shannon sampling.

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