Motion detection, the Wigner distribution function, and the optical fractional Fourier transform

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Received December 13, 2002

It is shown that both surface tilting and translational motion can be independently estimated by use of the speckle photographic technique by capturing consecutive images in two different fractional Fourier domains. A geometric interpretation, based on use of the Wigner distribution function, is presented to describe this application of the optical fractional Fourier transform when little prior information is known about the motion. The Wigner representation of a field \( u(x) \) can be defined as

\[
W(x, k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dy \exp(-jky)u^*(x - y/2)u(x + y/2).
\]  

(1)

It is a pseudo distribution function, and \( W(x, k) \) can have negative values. To find the intensity, \( I(x) = |u(x)|^2 \), we integrate \( W(x, k) \) over \( k \). Similarly, to find the spatial frequency distribution, \( \tilde{I}(k) = |\text{FT}[u(x)]|^2 = |\tilde{U}(k)|^2 \), we integrate \( W(x, k) \) over \( x \).

Previous publications \({8,9,10}\) have reported that both tilt and translation could be simultaneously determined because it was assumed either (i) that a fixed linear relationship existed between the tilting motion and the translation motion such that measurement in a single fractional domain could be specified in the speckle photography setup or (ii) that a reference field was available (from, for example, a hologram) that permitted the retrieval of phase information. In these ways the translation and tilting motions could both be completely determined.

In this Letter we show that, even if no fixed relationship between tilt and translation is known and no reference field is available, the motion of a surface can still be found by use of a variable FRT-based speckle photographic system. For simplicity in our analysis we assume one-dimensional fields and we use the WDF to provide a geometrical interpretation of the results.

In Fig. 1 a WDF, \( W(x, k) \), which we use to designate a field reflected from a surface, is designated by a contour centered at the origin \((0, 0)\). If the surface moves slightly, the corresponding WDF may be presented as simply a shifted version of the initial WDF to coordinates \((\xi, \kappa)\) in phase space, becoming \( W(x - \xi, k - \kappa) \), to correspond to a translation of magnitude \( \xi \) combined with a shift in spatial frequency of size \( \kappa \). This motion can be described in many exactly equivalent
Comparing Eqs. (3) and (4), we can see that the FRT of the original field, $U_0(q)$, has been multiplied by a phase factor and then shifted by an amount $Q' = \xi \cos \theta + \kappa \sin \theta$. From Fig. 1 we identify this shift distance in $q$ as the projection onto $q$ of the actual shift distance $\sqrt{\xi^2 + \kappa^2}$. We can now write in addition to Eqs. (2) that

$$U_0(q) \rightarrow U(q - Q) \exp[+j\Phi(q)],$$

(5a)

where

$$\Phi(q) = q \cot \theta \left( Q - \frac{\xi}{\cos \theta} \right) + \frac{\cot \theta}{2} \left( \xi^2 - Q^2 \right) + \kappa \xi.$$

(5b)

We note that $Q' = \kappa \cos \theta - \xi \sin \theta$; see Fig. 1. We further note that $Q' = \partial Q/\partial \theta$ and that $\sqrt{\xi^2 + \kappa^2} = \sqrt{\xi^2 + \kappa^2}$. In the special case when the shift in the WDF is parallel to $q$, tan $\theta = \xi/\kappa$, $Q = \sqrt{\xi^2 + \kappa^2}$, and $Q' = 0$. In this case $\Phi(q) = \kappa \xi/2$, which was previously identified as significant in fractional-Fourier-based holographic interferometry.

Following the usual speckle photographic procedure, we subtract the resultant intensities, i.e., the absolute values of Eqs. (2) and (4) captured in the FRT plane, and take the FT of the result, which yields

$$|\text{FT}([U_0(q)]^2) - |U_0(q - Q)|^2| \exp[j\Phi(q)]|^2| = 2 \text{FT}[I_0(q)]|q'\cos(qq'/2),$$

(6)

where $I_0(q) = |U_0(q)|^2$ and is equal to the integration of $W(x,k)$ over $q'$, the axis that is perpendicular to $q$; see Fig. 1.

Examining Eq. (6), we see that the value of shift $Q$ along $q$ can be found from the resultant speckle fringe pattern. Inasmuch as $\theta$ is also known, the magnitude of the total shift in phase space can be estimated as $\sqrt{\xi^2 + \kappa^2} = Q/\cos \theta$. However, the values of $\xi$ and $\kappa$ are still not independently known. In general, we require two projections to be able to completely determine the two components of the shift vector. Clearly these projections do not have to be on orthogonal axes.

To acquire two projections we assume that we can vary our FRT angle by an amount $\Delta \theta$ between measurements. In this case $\theta \rightarrow \theta + \Delta \theta$; i.e., we are now projecting the same WDFs onto a different FRT domain. In this case there will be a change in value of $Q$ as the FRT order changes; i.e., $Q \rightarrow Q + \Delta Q = \xi \cos(\theta + \Delta \theta) + \kappa \sin(\theta + \Delta \theta)$. By use of the speckle photographic technique, both $Q$ and $Q + \Delta Q$ can be determined as described above, yielding two simultaneous equations in two unknowns. Solving these, we get that

$$\xi = \frac{Q \sin(\theta + \Delta \theta) - (Q + \Delta Q) \sin(\theta)}{\sin(\Delta \theta)},$$

(7a)

$$\kappa = \frac{-Q \cos(\theta + \Delta \theta) + (Q + \Delta Q) \cos(\theta)}{\sin(\Delta \theta)}.$$

(7b)
If we assume that $\Delta \theta$ is small, carrying out Taylor series expansions will yield

\[ \xi = -\frac{\Delta Q}{\Delta \theta} \sin(\theta) + Q \cos(\theta) \]
\[ -\frac{\Delta \theta}{6} (3Q + \Delta Q) \sin(\theta) + O(\Delta \theta)^3 \]
\[ = -\frac{dQ}{d\theta} \sin(\theta) + Q \cos(\theta), \quad (8a) \]

\[ \kappa = \frac{\Delta Q}{\Delta \theta} \cos(\theta) + Q \sin(\theta) \]
\[ + \frac{\Delta \theta}{6} (3Q + \Delta Q) \cos(\theta) + O(\Delta \theta)^3 \]
\[ \approx \frac{dQ}{d\theta} \cos(\theta) + Q \sin(\theta). \quad (8b) \]

Clearly we are free to choose a value of $\theta$ that simplifies the implementation of our system. If we choose $\theta = \pi/2$, which corresponds to an optical FT, then $\kappa = Q$ and $\xi = -(dQ/d\theta)$.

In conclusion, it has been shown that, by optically generating FRT planes of suitable order within a speckle photographic system, one can estimate both the tilting and the translation of the surface. The method requires the capture of four speckle images, two in one fractional domain and two in a second, i.e., $I_\theta(q_1; t_1)$, $I_{\theta+\Delta \theta}(q_2; t_2)$, $I_\theta(q_3; t_3)$, and $I_{\theta+\Delta \theta}(q_2; t_4)$, where for example the time sequence may be of the form $t_1 < t_2 \ll t_3 < t_4$. No discussion of possible techniques to implement a variable-order FRT has been presented here. However, clearly a practical system would require access to an accurate, fast electronically controlled method of FRT order variation. Furthermore, no discussion of speckle size or decorrelation or of the effect of the optics used on the operation of the system, e.g., noise introduction by the optical FRT itself, has been presented. These parameters will be critically important in determining the capabilities of any such system.

We believe that the geometrical method presented here provides a new way to describe and analyze optical metrology systems. Furthermore, it provides physical insights that have allowed us to propose new metrology systems. Initial experimental results have already been presented in the literature, and the practicality of these systems is currently being examined.

The authors acknowledge the support of Enterprise Ireland through the Research Innovation Fund. J. T. Sheridan's e-mail address is john.sheridan@ucd.ie.

**References**