Motion detection, the Wigner distribution function, and the optical fractional Fourier transform

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Speckle photography is a practical means of extracting in-plane translation and tilting motion information by use of captured intensity information.\(^1\,^2\) Tilt measurement, for example, involves the capture of the optical Fourier transform of the reflected surface field. Adding or subtracting two sequential images and numerically calculating the Fourier transform (FT) of the result produces a set of interference fringes whose period is inversely proportional to any constant shift in field spatial frequency and so in surface tilt. In this case no information is known about surface translation. Angular velocity and acceleration can be found based on a series of sequentially captured images. The resolution and the dynamical range of detectable movements are fixed and depend on the wavelength used, the reflected field speckle size (including decorrelation effects), and the measurement system used, e.g., the system’s point-spread function and the CCD’s pixel size and sensitivity. The time resolution depends on the speed of the camera used.

The fractional Fourier transform (FRT), which is a generalization of the FT, has received a great deal of attention in the optics literature.\(^3\,^4\) The combination of the h Bootstrap text...
Comparing Eqs. (3) and (4), we can see that the FRT of the original field, $U_\theta(q)$, has been multiplied by a phase factor and been shifted by an amount $Q = \xi \cos \theta + \kappa \sin \theta$. From Fig. 1 we identify this shift distance in $q$ as the projection onto $q$ of the actual shift distance $\sqrt{\xi^2 + \kappa^2}$. We can now write in addition to Eqs. (2) that

$$U_\theta(q) \rightarrow U(q - Q)\exp[j\Phi(q)],$$

where

$$\Phi(q) = q \cot \theta \left( Q - \frac{\xi}{\cos \theta} \right) + \cot \theta \left( \frac{\xi^2 - Q^2}{2} \right) + \kappa \xi$$

$$= \frac{qQ' - QQ'}{2} + \kappa \xi.$$  

We note that $Q' = \kappa \cos \theta - \xi \sin \theta$; see Fig. 1. We further note that $Q' = dQ/d\theta$ and that $\sqrt{\xi^2 + \kappa^2} = \{Q' + Q^2\}^{1/2}$. In the special case when the shift in the WDF is parallel to $q$, $\tan \theta = \xi/\kappa$, $Q = \sqrt{\xi^2 + \kappa^2}$, and $Q' = 0$. In this case $\Phi(q) = \kappa \xi/2$, which was previously identified as significant in fractional-Fourier-based holographic interferometry.

Following the usual speckle photographic procedure, we subtract the resultant intensities, i.e., the absolute values of Eqs. (2) and (4) captured in the FRT plane, and take the FT of the result, which yields

$$|\text{FT}[U_\theta(q)]|^2 + |U_\theta(q - Q)\exp[j\Phi(q)]|^2 = 2 \text{FT}[I_\theta(q)](q')\cos(Qq'/2),$$

where $I_\theta(q) = |U_\theta(q)|^2$ and is equal to the integration of $W(x,k)$ over $q'$, the axis that is perpendicular to $q$; see Fig. 1.

Examining Eq. (6), we see that the value of shift $Q$ along $q$ can be found from the resultant speckle fringe pattern. Inasmuch as $\theta$ is also known, the magnitude of the total shift in phase space can be estimated as $\sqrt{\xi^2 + \kappa^2} = Q/\cos \theta$. However, the values of $\xi$ and $\kappa$ are not independently known. In general, we require two projections to be able to completely determine the two components of the shift vector. Clearly these projections do not have to be on orthonormal axes.

To acquire two projections we assume that we can vary our FRT angle by an amount $\Delta\theta$ between measurements. In this case $\theta \rightarrow \theta + \Delta\theta$; i.e., we are now projecting the same WDFs onto a different FRT domain. In this case there will be a change in value of $Q$ as the FRT order changes, i.e., $Q = Q + \Delta Q = \xi \cos(\theta + \Delta\theta) + \kappa \sin(\theta + \Delta\theta)$. By use of the speckle photographic technique, both $Q$ and $\Delta Q$ can be determined as described above, yielding two simultaneous equations in two unknowns. Solving these, we get

$$\xi = \frac{Q \sin(\theta + \Delta\theta) - (Q + \Delta Q)\sin(\theta)}{\sin(\Delta\theta)}$$

$$\kappa = \frac{-Q\cos(\theta + \Delta\theta) + (Q + \Delta Q)\cos(\theta)}{\sin(\Delta\theta)}.$$
If we assume that $\Delta \theta$ is small, carrying out Taylor series expansions will yield

$$\xi = -\frac{\Delta Q}{\Delta \theta} \sin(\theta) + Q \cos(\theta) - \frac{\Delta \theta}{6} (3Q + \Delta Q) \sin(\theta) + O(\Delta \theta)^3$$

$$= -\frac{dQ}{d\theta} \sin(\theta) + Q \cos(\theta), \quad (8a)$$

$$\kappa = \frac{\Delta Q}{\Delta \theta} \cos(\theta) + Q \sin(\theta) + \frac{\Delta \theta}{6} (3Q + \Delta Q) \cos(\theta) + O(\Delta \theta)^3$$

$$= \frac{dQ}{d\theta} \cos(\theta) + Q \sin(\theta). \quad (8b)$$

Clearly we are free to choose a value of $\theta$ that simplifies the implementation of our system. If we choose $\theta = \pi/2$, which corresponds to an optical FT, then $\kappa = Q$ and $\xi = -(dQ/d\theta)$.

In conclusion, it has been shown that, by optically generating FRT planes of suitable order within a speckle photographic system, one can estimate both the tilting and the translation of the surface. The method requires the capture of four speckle images, two in one fractional domain and two in a second, i.e., $I_0(q_1; t_1), I_{\phi+\Delta\phi}(q_2; t_2), I_\phi(q_3; t_3)$, and $I_{\phi+\Delta\phi}(q_2; t_4)$, where for example the time sequence may be of the form $t_1 < t_2 \ll t_3 < t_4$.

No discussion of possible techniques to implement a variable-order FRT has been presented here. However, clearly a practical system would require access to an accurate, fast electronically controlled method of FRT order variation. Furthermore, no discussion of speckle size or decorrelation or of the effect of the optics used on the operation of the system, e.g., noise introduction by the optical FRT itself, has been presented. These parameters will be critically important in determining the capabilities of any such system.

We believe that the geometrical method presented here provides a new way to describe and analyze optical metrology systems. Furthermore, it provides physical insights that have allowed us to propose new metrology systems. Initial experimental results have already been presented in the literature, and the practicality of these systems is currently being examined.

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References