Frequency Quantization in First-Order Digital Phase-Locked Loops with Frequency-Modulated Input

Abstract—Frequency granularity in a digital phase-locked loop arises from quantization in the number-controlled oscillator which prevents the loop from locking exactly onto its reference signal and introduces unwanted phase jitter. Based on a nonlinear analysis of trajectories in the phase space, we have recently investigated the effect of frequency quantization in a first-order loop with a frequency-modulated input signal and have derived useful bounds on the steady-state phase jitter excursion. In this paper, we continue that work and derive the maximum modulation amplitude such that loop cycle slipping is avoided. We also examine in detail the loop behavior in acquiring phase-lock.

I. INTRODUCTION

Phase-locked loops (PLLs) have become essential building blocks in various applications, particularly in wireless and other communications systems [1]. A PLL is essentially a closed-loop system that synchronizes the phase of a controlled oscillator to the phase of an input (or reference) signal. Digital phase-locked loops (DPLLs) typically contain a number-controlled oscillator (NCO), a source of frequency granularity in the system. Figure 1 shows a phase domain model of the first-order DPLL under investigation [2], [3]. The phase detector compares the phase of the input signal, \( \phi_i \), with the phase of the NCO output signal, \( \phi_o \), and produces an output equal to the sine of the phase error \( \phi = \phi_i - \phi_o \). The phase detector output is then passed through a loop filter (the gain coefficient \( K_1 \) in our first-order loop) whose output is quantized and drives the NCO so as to minimize the phase error \( \phi \). The inherent feedback loop formed by these three blocks forces the system to track phase variations of the input signal, thus enabling the demodulation of frequency-modulated (FM) signals. The loop is said to be locked if the phase error is zero or constant. Frequency quantization, however, prevents the loop from being exactly phase-locked and introduces phase jitter: unwanted oscillatory steady-state motion of the phase error about its locked state.

Recently, tools from the theory of nonlinear dynamics have enabled an analytical treatment of the effect of frequency quantization in a DPLL. For a sinusoidal input signal, certain results obtained by Gardner in [2] using extensive simulations were rigorously verified in [3] and [4]. In our recent work [5], we investigated the steady-state phase jitter dynamics of a first-order loop (the model in Fig. 1) with an FM input signal. In particular, we showed that for a sufficiently small modulation amplitude, trajectories eventually fall into an invariant belt (a belt-shaped region in the phase space). For a large amplitude, there exists a trapping belt which, upon entering, trajectories cannot leave. In either case, the boundaries of these belts give useful bounds on the steady-state phase jitter excursion. In this paper, we shall continue that work and additionally assume a quantized gain coefficient \( K_1 \), an assumption of practical significance. We derive the maximum modulation amplitude such that loop cycle slipping is prevented and expand on the acquisition behavior of the loop.

II. FIRST-ORDER DPLL MODEL

For the phase domain model of the first-order DPLL in Fig. 1 and an FM input signal, the phase error \( \phi \) can be modeled by the nonlinear first-order difference equation [5]

\[
\phi_{n+1} = \phi_n + 2\pi\nu + A\cos(\omega(n+1) + \theta_0) - 2\pi Q_b(K_1 \sin \phi_n) \mod 2\pi. \tag{1}
\]

To simplify the discussion, we assume a b-bit midrise quantizer of the form \( Q_b(\phi) = \lfloor 2^b \phi \rfloor / 2^b \) which has a riser at zero [2]; the symbol \( \lfloor \cdot \rfloor \) refers to the floor function which gives the largest integer less than or equal to its argument. We further assume that the carrier frequency \( \nu \) is positive and that \( 2^b \nu \) is not an integer. The sinusoidal modulation signal has amplitude \( A \), initial phase \( \theta_0 \) and frequency \( \omega \), where \( \omega/(2\pi) \) is assumed to be irrational. Furthermore, we assume \( 2^b K_1 \) to be an integer (the noninteger case was considered in [5]); this assumption is of practical significance because the gain coefficient is often taken to be a power of 0.5 to allow for an efficient implementation (binary shift operations) [1].

The difference equation in (1) is nonautonomous due to the explicit presence of the discrete-time variable \( n \). By introducing the argument of the cosine as a new variable, we can convert (1) into the nonlinear
autonomous system of two difference equations
\[ \theta_{n+1} = \theta_n + \omega \quad \text{mod} \ 2\pi \]
\[ \phi_{n+1} = \Phi(\theta_{n+1}, \phi_n) \quad \text{mod} \ 2\pi \]
where
\[ \Phi(\theta, \phi) = \phi + 2\pi \nu + A \cos \theta + \frac{2\pi}{2^b} \lfloor 2^bK_1 \sin \phi \rfloor. \] (4)

These equations define a map of the torus \( S_{2\pi} \times S_{2\pi} \) onto itself, where \( S_{2\pi} = \mathbb{R} \mod 2\pi \) denotes the circle of length \( 2\pi \). For our analysis, it will be convenient to write (2) – (4) in a different form. The discontinuities of \( \Phi \), which do not depend on \( \theta \), are the quantizer threshold values. Figure 2 shows the term \( 2^bK_1 \sin \phi \) (dashed) and its quantized version \( \lfloor 2^bK_1 \sin \phi \rfloor \) (solid). Denoting \( k_{\max} = 2^bK_1 \) and letting \( -k_{\max} + 1 \leq k \leq k_{\max} - 1 \), these discontinuities are given by
\[ \sigma_k^+ = \sin^{-1} \left( \frac{k}{2^bK_1} \right) \]
for \( \phi \in (-\pi/2, \pi/2) \) (the ascending sine branch) and
\[ \sigma_k^- = \pi - \sigma_k^+ \]
for \( \phi \in (\pi/2, 3\pi/2) \) (the descending sine branch). Note that the single point \( \phi = \pi/2 \), for which \( \sigma_{k_{\max}}^+ = \sigma_{k_{\max}}^- = \pi/2 \), is ignored because it is reached by a trajectory with probability zero. These discontinuities divide the circle in \( \phi \) into the \( 4k_{\max} - 2 \) intervals
\[ \Delta_k = \left\{ \sigma_k^+, \sigma_k^- \right\} \cup \left( \sigma_{k+1}^-, \sigma_k^+ \right), \quad -k_{\max} + 1 \leq k \leq k_{\max} - 1 \]
where each corresponds to one of the \( 2k_{\max} \) quantizer output values. Hence, for \( \phi \in \Delta_k \), \( -k_{\max} + 1 \leq k \leq k_{\max} - 1 \), we can write (2) – (4) as the map
\[ F_k : \left( \theta, \phi \right) \mapsto \left( \theta + \omega, \phi + c_k + A \cos \theta + \omega \right) \]
where \( c_k = 2\pi \nu - (2\pi/2^b)k \). The 2D phase space is therefore divided into \( 4k_{\max} - 2 \) strips, where each two adjacent ones are separated by the discontinuity line \( D_{k+1}^* = S_{2\pi} \times \{ \sigma_k^+ \} \) on the ascending sine branch, and by \( D_k^* = S_{2\pi} \times \{ \sigma_k^- \} \) on the descending sine branch. In addition, we define the inverse map of \( F_k \) as
\[ F_k^{-1} : \left( \theta, \phi \right) \mapsto \left( \theta - \omega, \phi - c_k - A \cos \theta \right) \]
such that \( F_k^{-1}(F_k(\theta, \phi)) = F_k(F_k^{-1}(\theta, \phi)) \equiv (\theta, \phi) \).

III. INVARIANT BELT, TRAPPING BELT AND INVARIANT CURVES

In our previous work [5], we investigated the limit behavior of trajectories \( \{(\theta_n, \phi_n)\}_{n=0}^{\infty} \) of the map (2) – (4) in the 2D phase space. In this section, we will summarize those results that we will use in later sections. Let us consider the images of the \( k \)th discontinuity line \( D_k^* \) under its map (8) from below and above, i.e.,
\[ U_k^+ : \phi = \sigma_k^+ + c_{k-1} + A \cos \theta \]
\[ L_k^+ : \phi = \sigma_k^+ + c_k + A \cos \theta \]
where \( U_k^+ = F_{k-1}(D_k^*) \) and \( L_k^+ = F_k(D_k^*) \). For \( A_0 < A < A_1 \), where
\[ A_0 = \frac{2\pi}{2^b} \min \left\{ \left\lfloor 2^b\nu \right\rfloor, 1 - \left\lfloor 2^b\nu \right\rfloor \right\} \]
\[ A_1 = \frac{2\pi}{2^b} (2^bK_1 - 1) - 2\pi \nu \]
there exists a trapping belt which, upon entering, trajectories cannot leave. The trapping belt is the bounded set
\[ T = \left\{ (\theta, \phi) | L_k^+(\theta) \leq \phi < U_k^+(\theta), \theta \in S_{2\pi} \right\} \] (14)
where the upper boundary \( U_k^+ \) and the lower boundary \( L_k^+ \) are the curves (10) and (11) with \( k \) replaced by the integers
\[ k^* = \left\lfloor 2^b\nu + \frac{2^bA}{2\pi} \right\rfloor + 1 \]
\[ k_* = \left\lfloor 2^b\nu - \frac{2^bA}{2\pi} \right\rfloor + 1 \]
respectively. An example of the trapping belt is shown in Fig. 3, which plots 1000 points of a trajectory with the first 300 discarded. Since \( A_0 < A < A_1 \) for the given parameters, the trajectory is trapped in the region between the solid curves \( U_k^+ \) and \( L_k^+ \). The dashed lines in the figure are the curves \( U_k^+ \) and \( L_k^+ \), \( k_* \leq k \leq k^* \) (two of them coincide with the belt boundaries). The series of dotted lines are the discontinuity lines \( D_k^* \) and \( D_{k+1}^* \). The solid curve above the belt is the repelling contour which will be described in more detail in Sec. V (see Fig. 6).

For \( A < A_0 \), the map of the trajectory in steady-state becomes that of a first-order sigma-delta modulator with a sampled periodic input signal, and the results from [6] imply that the trapping belt in (14) reduces to an invariant belt of constant vertical thickness \( 2\pi/2^b \). For \( A > A_1 \), the trajectory will not reside in a trapping belt, but extend into the strip \( S_{2\pi} \times \Delta_{k_{\max}-1} \) and cycle slipping may occur; choosing \( A < A_1 \) gives a sufficient condition for cycle slipping to be prevented [5]. A more detailed analysis will be given in Sec. IV.
Finally, because of the sinusoidal forcing term, the invariant curves of the map (8) can be derived in closed form. Using the result from [6] for the case $c_k = 0$, the invariant curves inside the strip $S_{2\pi} \times \Delta k$, $-k_{\text{max}} + 1 \leq k \leq k_{\text{max}} - 1$, are

$$S_k : \quad \phi = \frac{c_k}{\omega} \theta + A \sin \left( \theta + \frac{\omega}{2} \right) + C(\theta_0, \phi_0)$$  \hspace{1cm} (17)

where $A' = A/(2\sin(\omega/2))$ and $C(\theta_0, \phi_0) = \phi_0 - c_k \theta_0 / \omega - A' \sin(\theta_0 + \omega/2)$.

IV. MAXIMUM A TO PREVENT CYCLE SLIPPING

In this section, we will consider the case where $A$ exceeds the sufficient bound in (13) and derive the maximum value of $A$, denoted by $A_2$, such that cycle slipping is prevented. An example trajectory is depicted in Fig. 4. Since $A > A_1$ for the given parameters, the trajectory does not fit in a trapping belt, but extends above the line $U^k_{\phi}$. If $A$ becomes larger than $A_2$, the trajectory will cross the repelling contour and cycle slipping will occur, with the trajectory making a full rotation about the torus in the $\phi$ direction. It follows from [6] that the trajectory in the strip $S_{2\pi} \times \Delta k_{\text{max}}$ is bounded from above by a continuous curve that consists of pieces of a finite number of critical curves. This continuous curve is depicted bold in Fig. 4 and referred to as the boundary.

To derive $A_2$, we begin with a precise construction of the boundary. Let us denote the critical curves by $V_m$, $m \geq 1$; they are the consecutive images of the uppermost discontinuity line $D^+_{k_{\text{max}}}$ under the map (8). In particular, the first critical curve $V_1$ is the image of this line under its map below, i.e., $V_1 = F_{k_{\text{max}}-2}(D^+_{k_{\text{max}}-1})$. This image is obtained from (10) with $k = k_{\text{max}} - 1$ as

$$V_1 = U^+_{k_{\text{max}}-1} : \quad \phi = \sigma^+_{k_{\text{max}}-1} + c_{k_{\text{max}}-2} + A \cos \theta.$$  \hspace{1cm} (18)

The subsequent critical curves are the images of $V_1$ under the map $F_{k_{\text{max}}-1}$ from above this discontinuity line. Thus, we have

$$V_1 = U^+_{k_{\text{max}}-1} \quad \text{and} \quad V_m = F_{k_{\text{max}}-1}(V_{m-1}), \quad m \geq 2.$$  \hspace{1cm} (19, 20)

This construction implies that the maximum of the boundary is given by the maximum of one of these critical curves; we will refer to this particular curve as the maximum critical curve $V_{m_0}$, for a certain integer $m_0$. It now becomes clear how to find $A_2$: we first determine $m_0$ along with an explicit expression for $V_{m_0}$, and then compute the value of $A$ for which the maximum of $V_{m_0}$ touches the discontinuity line $D^+_{k_{\text{max}}-1}$, which is part of the repelling contour.

The integer $m_0$ gives the number of iterates required to build the maximum critical curve $V_{m_0}$. To find this number, let us take a closer look at the onset of the boundary. The $\theta$ value at the onset is the intercept of the discontinuity line $D^+_{k_{\text{max}}-1}$ with its image under its map from above, i.e., $L^+_{k_{\text{max}}-1} = F_{k_{\text{max}}-1}(D^+_{k_{\text{max}}-1})$. Since this image is obtained from (11) with $k = k_{\text{max}} - 1$, solving $L^+_{k_{\text{max}}-1} = D^+_{k_{\text{max}}-1}$ gives two solutions for $\theta \in S_{2\pi}$, where

$$\theta_o = 2\pi - \cos^{-1} \left( \frac{A_1}{A} \right).$$  \hspace{1cm} (21)

This corresponds to the onset and $A_1 = -c_{k_{\text{max}}-1}$. The intercept of the vertical line at $\theta_o$ (shown in the figure by the rightmost dotted line) and the first critical curve $V_1$ gives the corresponding value $\phi_o$. Now, the piece $V_1(\theta_o, \theta_o + \omega)$ is the first piece of the boundary, and the point $(\theta_o, \phi_o)$ is its leftmost end. The boundary can be precisely constructed by iterating this piece according to (20). The $n$th iterate of the point $(\theta_o, \phi_o)$ will then be the connection point between the two pieces of the adjacent critical curves $V_{m-1}$ and $V_m$, $m \geq 2$, and the sequence of these points will lie on the invariant curve $S^+_{k_{\text{max}}-1}$ in (17). These two facts imply that the piece of the maximum critical curve, and thus the maximum of the boundary, will be close to the maximum of $S^+_{k_{\text{max}}-1}$. The $\theta$ value corresponding to the maximum of $S^+_{k_{\text{max}}-1}$ can be readily obtained from (17) as

$$\theta_{\text{max}} = \cos^{-1} \left( \frac{d_1 A_1}{A^2} - \frac{\omega}{2} \right)$$  \hspace{1cm} (22)

which, after inserting (21) and (22) and taking into account that $\theta$ is taken mod $2\pi$, yields

$$m_0 = \left[ \frac{\theta_{\text{max}} - \theta_o}{\omega} \right] + 1.$$  \hspace{1cm} (23)

Combining (19) and (20) and setting $m = m_0$ gives the maximum critical curve

$$V_{m_0} = F^{k_{\text{max}}-1}_{k_{\text{max}}-1}(V_1).$$  \hspace{1cm} (25)

By inserting (18), applying $(m_0 - 1)$-times the map (8) and using trigonometric identities, this curve can be explicitly expressed as

$$V_{m_0} : \quad \phi = \sigma^+_{k_{\text{max}}-1} + c_{k_{\text{max}}-1} + c_{k_{\text{max}}-2} + 2A' \sin \left( \frac{\omega}{2} m_0 \right) \cos \left( \frac{\theta - \theta_o}{2} (m_0 - 1) \right)$$  \hspace{1cm} (26)

and is shown in Fig. 4 by the dash-dotted curve. Given (24) and (26), $A_2$ is then the value of $A$ for which the maximum of $V_{m_0}$ touches the discontinuity line $D^+_{k_{\text{max}}-1}$. More specifically, setting the maximum of (26) equal to $\sigma^+_{k_{\text{max}}-1} = \pi - \sigma^+_{k_{\text{max}}-1}$ and multiplying by $\omega/(2c_{k_{\text{max}}-1})$ gives after some algebra

$$f(A) = \frac{\omega}{2} m_0(A) - \frac{A}{d_1 A_1} \sin \left( \frac{\omega}{2} m_0(A) \right) - d_2$$  \hspace{1cm} (27)

where $d_2 = \omega(\pi - 2\sigma^+_{k_{\text{max}}-1} - c_{k_{\text{max}}-2} + c_{k_{\text{max}}-1})/(2c_{k_{\text{max}}-1})$ and $m_0(A)$ denotes the explicit dependence of $m_0$ on $A$. The function $f(A)$ expresses the normalized vertical distance from the maximum of $V_{m_0}$ to $D^+_{k_{\text{max}}-1}$ as a function of $A$, with the required $A_2$ fulfilling $f(A_2) = 0$. The result $A_2$ as a function of $\omega$ is plotted in Fig. 5. The dashed line corresponds to the sufficient condition $A_1$ in (13). It can be seen that the maximum modulation amplitude to prevent cycle...
slipping increases with increasing \( \omega \). However, if \( \omega \) becomes larger than some maximum value (given by the right end of the curve in the figure), the onset of the boundary will be displaced and we do not get the maximum amplitude; a general construction of the boundary for any \( \omega \) is part of our ongoing investigations.

V. REPPELLING CONTOUR

In this section, we will investigate the loop behavior in acquiring phase-lock. We will see that, depending on their origin in the phase space, some trajectories fall into the trapping (or invariant) belt after significantly more iterates than others.

It is illustrative to begin with the map (8) for \( A = 0 \) (unmodulated case) and consider trajectories in the 2D phase space. Of particular importance in our analysis is the discontinuity line \( D_{k^* + 1} \), where \( k^* = \lfloor 2^\nu' \rfloor \). It follows from (8) that a trajectory starting on or just below this line will decrease with each iteration since \( c_{k^* + 1} = (2\pi/2^\nu')(\lfloor 2^\nu' \rfloor - 1) < 0 \). In contrast, a trajectory starting just above this line will increase with each iteration since \( c_{k^*} = (2\pi/2^\nu')(\lfloor 2^\nu' \rfloor) > 0 \). This discontinuity line is therefore repelling (for the case \( A = 0 \) only) and corresponds to the unstable fixed point \( \phi = \pi - \sin^{-1}((\nu/K_1) \) in a DPLL without frequency quantization.

Let us now investigate the map (8) for \( 0 < A < A_1 \). From Sec. III we know that trajectories will eventually fall into either an invariant or a trapping belt. We again consider a trajectory originating close to the discontinuity line \( D_{k^* + 1} \). Now, depending on the particular value of \( A \), a trajectory starting within a certain distance above this line for a range of \( \theta \in S_{2\pi} \) will move downwards and enter the belt from above. Conversely, a trajectory starting within a certain distance below this line for a range of \( \theta \in S_{2\pi} \) will move in the opposite direction around the torus and enter the belt from below. The result is a repelling contour, which could be thought of as bending the discontinuity line \( D_{k^* + 1} \) along the \( \theta \)-axis.

In the following, we outline the precise construction of this contour. Taking the contour in Fig. 3 as an example, a more detailed picture of it is shown in Fig. 6 (the solid curve). Contrary to the construction of the belt, we now have to consider the preimages of the discontinuity lines. In particular, the preimages of the \( k \)th discontinuity line \( D_{k^*} \) under its inverse map (9) from below and above are

\[
U_k^* : \quad \phi = \sigma_k^* - c_{k^*} - A \cos(\theta + \omega) \tag{28}
\]

\[
L_k^* : \quad \phi = \sigma_k^* - c_{k^*} - A \cos(\theta + \omega) \tag{29}
\]

where \( U_k^* = F_{k^*}^{-1}(D_k^*) \) and \( L_k^* = F_{k^*}^{-1}(D_k^*) \) (the dashed curves in Fig. 6). These preimages can now be used to determine the number of strips the contour will extend into. If, for some \( -k_{\max} + 1 \leq k \leq k_{\max} - 1 \), the curve \( U_k^* \) has an intercept with its corresponding discontinuity line \( D_k^* \), then the contour will extend into the strip \( S_{2\pi} \times \Delta_k \). The intercepts are distinct points needed in the construction of the contour and are depicted by circle markers in Fig. 6. Depending on \( A \), we can distinguish between the following cases. For \( 0 < A < A_0 \), it can be shown that there is no intercept, and the repelling contour will reduce to the repelling discontinuity line \( D_{k^* + 1} \) (the dash-dotted line), as in the case \( A = 0 \). For \( A > A_0 \), we first consider an intersection segment of the discontinuity line \( D_{k^* + 1} \) with the strip between \( U_{k^* + 1} \) and \( L_{k^* + 1} \) and choose that segment’s left endpoint (the filled circle marker) as the starting point for our construction. Similar to the way we built the boundary in Sec. IV, the repelling contour is constructed by iterating a piece of a critical curve, which here is the preimage of a discontinuity line. In the figure, this piece is part of the preimage crossing our starting point. The construction proceeds by iterating this piece backwards under the inverse map within this strip. When it reaches the next discontinuity line, we continue with the next intercept (the empty circle markers) as our new starting point and follow the same steps, again with the inverse map corresponding to this strip. This procedure continues until we return to the initial point, wherein the construction completes.

VI. CONCLUSIONS

In this paper, we have continued our previous work [5] on the effect of frequency quantization in a first-order DPLL with an FM input. For the case of a quantized gain coefficient, we have derived the maximum modulation amplitude such that loop cycle slipping is prevented and have also investigated the phase acquisition behavior of the loop.

REFERENCES