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DISCONTINUOUS PIECEWISE-LINEAR DISCRETE-TIME DYNAMICS – MAPS WITH GAPS IN ELECTRONIC SYSTEMS

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Abstract

Purpose

Many important electronic systems are modelled by discrete-time equations with nonlinearities that are discontinuous and piecewise-linear, often arising as a result of quantization. Approximations based on linearization – the standard engineering response to nonlinearity – are often quite unhelpful in these systems, because of the form of the nonlinearity. Certain methods and results have been developed over a number of years for the analysis of discontinuous piecewise-linear discrete-time dynamics. The aim of this tutorial paper is to review that body of knowledge, and to show how it can be applied to representative electronic systems.

Approach

The paper uses an important electronic circuit – the sigma-delta modulator – as a central example, and considers the dynamical behaviour exhibited by this circuit and related circuits.

Findings

The circuits under investigation exhibit complex forms of behaviour that can be explained by the application of methods of nonlinear discrete-time dynamics.

Value

This paper is intended to provide a brief introduction to the body of research that exists into the behaviour of nonlinear discrete-time circuits and systems with discontinuous piecewise-linear nonlinearities.

1. Introduction

The mathematical science of nonlinear dynamics has gone through a number of distinct phases of development, from the work of pioneers such as Lyaponov (1907) and Poincaré (1893) around the end of the nineteenth century, through the developments of Lorenz (1963), Smale (1967), Sarkovskii (1964) and others in the 1960s, leading to an explosion in interest and awareness that was facilitated by the visualisation tools of modern computing (Gleick, 1987; Hilborn, 2000). Modern developments in nonlinear dynamics came to the attention of the electronic engineering community most notably with the invention in 1985 of Chua's circuit (Matsumoto *et al*, 1985) – recognised as the first mathematically-proven physical implementation of chaos. Over the intervening twenty-five years, a significant body of research has been developed to explain and manage nonlinear behaviour across a range of electronic circuits and systems.

Most of this research has concentrated on the nonlinear dynamics of continuous-time circuits and systems. The complementary body of systems that operate – or are best modelled – in discrete time has not received equivalent attention, despite the growing technical dominance of such systems. Moreover, the nonlinearities that appear in many of the most important discrete-time circuits are discontinuous and piecewise-linear – a class of nonlinearity that has received comparatively little attention within the nonlinear dynamics literature.

The purpose of this tutorial paper is to review elementary methods of nonlinear dynamics, with particular reference to discontinuous piecewise-linear nonlinearities, and to illustrate the application of these methods within electronic engineering. The primary vehicle for this illustration will be one particular important circuit – the sigma-delta modulator and its variants – but at each stage reference will be made to other circuits and systems that display similar behaviour. It is hoped that this will allow non-specialists to gain an understanding of the complex dynamics that can be exhibited by discrete-time electronic circuits, and their implications for electronic engineers.

2. Nonlinear discrete-time dynamics

In this section we will briefly review some theory of discrete-time nonlinear dynamics, highlighting in particular special aspects of the theory as it applies to the case of discontinuous piecewise-linear systems. Discrete-time nonlinear dynamics have been the focus of much attention and are well covered in the scientific literature – from expositions of the underlying mathematics to applications throughout the physical and social sciences. Many of the classic papers from this field are highly accessible to non-experts (May, 1976), and the ease with which

complex and fascinating concepts can be demonstrated by simple low-dimensional systems has led to coverage of this material in popular science books (Gleick, 1987). Our treatment will be necessarily brief, and readers interested in a deeper treatment are referred to any of the considerable number of textbooks in the area, such as (Hilborn, 2000).

A nonlinear discrete-time dynamical system is one whose state is updated at discrete time instants according to a nonlinear mapping of the form.

$$x_{n+1} = f(x_n, n) \tag{1}$$

x_n is the state at (integer) time n, lying in the *state space*, and may be a vector. The components of x_n are the *state variables*, and the dimension of x_n is the dimension of the dynamical system.

Many electronic systems operate by updating their internal state at clock instants, and so they can be modelled by a system of the form (1). In many important cases, the nonlinearities in these systems are discontinuous and are often in addition piecewise-linear, with their domain space partitioned into a finite number of regions in each of which the mapping is linear or affine (linear plus shift). Such nonlinearities arise, for example, as a consequence of quantization within the system. It is of interest to us, therefore, to consider systems of the form (1) in which the function f is discontinuous and piecewise-linear. This takes us outside the main body of study of nonlinear discrete-time dynamics, which typically concentrates on continuous nonlinearities. There have been some studies (Hogan *et al*., 2007; Kollar *et al*., 2004) of the complex behaviour that can arise in discontinuous piecewise-linear discrete-time systems, systems given the helpful title "maps with gaps" by Hogan *et al*. (2007). This complexity may be somewhat of a surprise – in each of the regions of linearity the systems exhibit simple and well-understood dynamics, but the switching between these regions can lead to surprisingly rich dynamical behaviour.

If the nonlinear function f in (1) is independent of the time variable n, we have an *autonomous* system of the form

$$x_{n+1} = f(x_n) \tag{2}$$

A non-autonomous system may be converted to autonomous form by adding the time variable as an additional component of the state vector. This is especially useful in the case where the dependence on time n is periodic, which can arise as a result of a periodic input. We will focus for simplicity on autonomous systems of the form (2), including those that arise from periodically-driven non-autonomous systems, in this tutorial.

Given an initial condition x_0 at time $n = 0$, the sequence of iterates $x_1 = f(x_0)$, $x_2 = f(x_1)$, etc. forms the *trajectory* or *orbit* of x_0, and much of the effort of nonlinear dynamics is concerned with

predicting the form of these trajectories either quantitatively or qualitatively. The simplest form of steady-state behaviour that a trajectory can exhibit is to remain fixed at a single point – a *fixed point*. The point x^* is a fixed point of the map (2) if it satisfies the equation $x^* = f(x^*)$. Once a trajectory lands on a fixed point it stays there for all time, in theory. In a practical implementation of the map, noise or rounding errors mean that a fixed point will be observed only if it is stable. A fixed point is stable if trajectories originating nearby stay nearby, and is asymptotically stable if nearby trajectories stay nearby and converge to the fixed point as the iteration process proceeds. The stability of a fixed point of a nonlinear system can usually be ascertained by examining the linearized system $\tilde{x}_{n+1} = A\tilde{x}_n$, where A is the Jacobian of f evaluated at the fixed point. If the eigenvalues of A lie inside the unit circle, trajectories of the linearized system converge to the origin and the fixed point of the nonlinear system is asymptotically stable. In the case of piecewise-linear mappings, the linearization is trivial – at all points that do not lie on boundaries between affine regions, the mapping is already linear in a neighbourhood of that point, and so the dynamics and stability locally are prescribed by simple linear equations. The extent of the region governed by this simple linear behaviour is defined by the boundaries between the affine regions.

The next form of steady-state behaviour is *periodic* behaviour, where a trajectory moves among a finite set of points in a periodic orbit. $\hat{x}$ is a period-k point of the map f if k is the lowest integer such that $\hat{x} = f^k(\hat{x})$, where f^k denotes the kth iterate of the map f. If x_0 is a period-k point of the map f, the trajectory originating at x_0 is of the form $x_0\ x_1\ x_2\ \ldots\ x_{k-1}\ x_0\ x_1 \ldots$ The stability of a periodic orbit of a map is investigated by noting that a period-k point of a map f is a fixed point of the kth iterate f^k. The stability of $\hat{x}$ viewed as a period-k point of f is the same as that of $\hat{x}$ viewed as a fixed point of f^k.

In the particular case of piecewise-linear systems, to locate fixed- and periodic points of a map we use the fact that the map is linear (or affine) over each of a finite number of regions of the domain space. If we know the region in which the fixed point resides (or the sequence of regions in which successive points of the periodic orbit reside) then the fixed (or periodic) point can be found by solving a set of linear equations. This is only a valid solution, however, if it (along with its iterates, in the periodic case) resides in the assumed region. This imposes a set of inequalities to be checked. This intuitive method was widely used in the analysis of relay control systems (Jury, 1964), where it was attributed to Tsypkin.

A nonlinear map can have multiple stable fixed and periodic points (along with other stable forms of steady-state behaviour or *attractors*, to be discussed shortly). The set of points that converge to a particular attractor reside in the *basin of attraction* for that attractor. It is intuitively clear that

two nearby points in a system such as ours will converge to different stable attractors only if after some number of iterations their iterates land in different affine regions of the piecewise-linear map, i.e. on different sides of a boundary between such regions. Thus these boundaries and their preimages under the map play an important role in bounding the basins of attraction of the fixed and periodic points. A detailed exploration of the role of the images and preimages of the boundaries – termed *critical curves* – is contained in (Mira *et al.*,1996).

A third form of behaviour closely linked to periodicity is *quasiperiodicity*. A sampled version of a periodic continuous-time signal may not itself be periodic in discrete time: the sequence $\sin(2\pi\nu n)$ is periodic if ν is rational, but if ν is irrational the sequence never repeats and is said to be quasiperiodic. Quasiperiodic orbits arise naturally in many of the systems we study.

The fourth and most complex form of bounded steady-state behaviour that can be exhibited by a nonlinear system is *chaos*. Chaotic trajectories appear to wander randomly, despite the fact that they arise from a deterministic system. When viewed appropriately, however, there is in fact significant and often very surprising structure to this "randomness", with successive magnifications of trajectories revealing ever finer levels of nested swirls and folds. Another key feature of chaos is the phenomenon of sensitive dependence on initial conditions, which means that two chaotic trajectories whose starting points are infinitesimally close will eventually diverge from one another.

Since it will prove relevant to our studies later in this paper, we will briefly consider the special case of functions that map a point on the unit circle to another point on the unit circle. Since a point on the unit circle can be specified by its angle, these circle maps are of the form $\theta_{n+1} = f(\theta_n) \bmod 2\pi$, where θ_n is the angle at time instant n. The simplest circle map is just the rotation map that adds a constant amount $2\pi\Omega$ to the angle with each iteration. It is easy to show that all trajectories are periodic if Ω is rational, and quasiperiodic, winding indefinitely around the circle but never closing on themselves, if Ω is irrational. Next, we make the map slightly more complex by the addition of a nonlinear term to form the sine circle map:

$$\theta_{n+1} = \theta_n + 2\pi\Omega + K\sin\theta_n \qquad \bmod 2\pi \tag{3}$$

The interaction of the simple rotation represented by the Ω term and the periodic sinusoidal function gives rise to periodic behaviour somehow depending on both of these terms. If we plot the *rotation number* – the average amount (relative to the circumference) by which trajectories are rotated per iteration of the map (3) – versus Ω, we get a graph of the form shown in Figure 1. The plateaus in the graph correspond to rational rotation numbers, or periodic orbits. Note how the

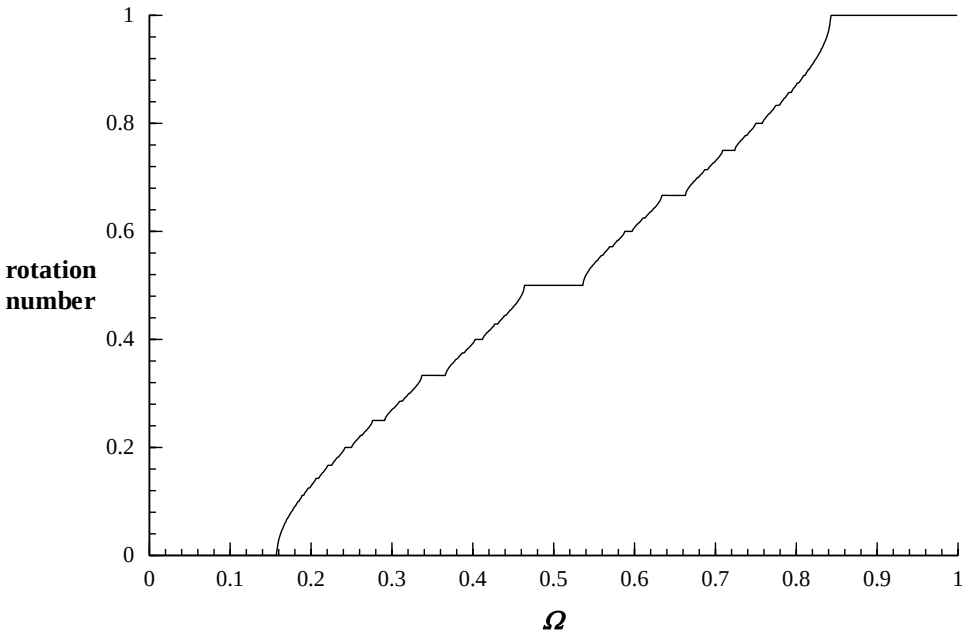

Figure 1 Rotation number of the sine circle map (3) plotted versus Ω for $K = 0.99$.

presence of the nonlinear term in the map has widened out the region of existence of each periodic orbit from a single value of Ω when $K = 0$ (the rotation map) to a whole range of values of Ω. This phenomenon is known as *mode-locking*, and arises in the study of coupled oscillations in many disparate fields. The graph in Figure 1 has been called the *devil's staircase*, and it is a widely studied example of a *fractal* curve, whose form is qualitatively similar regardless of the level of resolution at which it is viewed.

Having briefly surveyed relevant aspects of nonlinear discrete-time dynamics, highlighting some aspects specific to discontinuous and piecewise-linear nonlinearities, we will now see how this material can be applied to certain important electronic circuits and systems. We will use the sigma-delta modulator as an example system, for simplicity and continuity, but will at each stage highlight other electronic systems that display similar behaviour.

3. Application to sigma-delta modulation and related electronic systems

Sigma-delta ($\Sigma\Delta$, also known as delta-sigma) modulation is a technique very widely used throughout electronic systems (Norsworthy *et al*., 1996) It first found use as a method of conversion between analogue and digital signals, but the technique is such a useful one that it has found use in many other systems, often well beyond the area of data conversion (Galton *et al*,

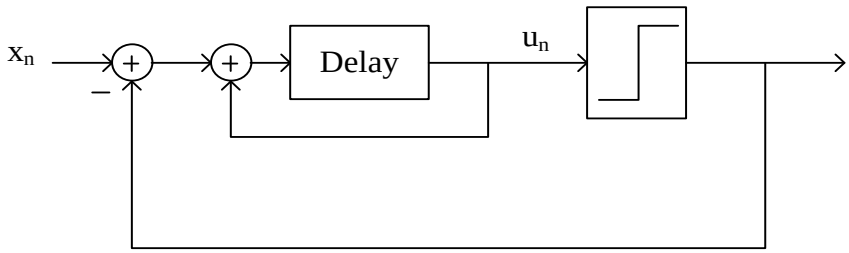

Figure 2 Block diagram of the first-order ΣΔ modulator modelled by (4)

1998; Callegari and Bizzarri, 2010). Because of the simplicity and widespread applicability of the method, we will use the ΣΔ modulator and its variants as a vehicle to illustrate many of the methods of the previous section.

The most basic ΣΔ modulator is the first-order loop of Figure 2. It consists of a discrete-time integrator and a quantizer within a feedback loop, and is modelled by the equation

$$u_{n+1} = u_n + x_n - \operatorname{sgn} u_n \tag{4}$$

where sgn u_n = 1 when $u_n \geq 0$ and -1 when $u_n < 0$. When used in analogue-to-digital conversion, the input signal x_n is converted to a stream of bits (corresponding to sgn u_n) at the output that can subsequently be decimated and filtered to retrieve a good approximation to the input. In the case of a dc input x, the map $u_n \mapsto u_{n+1}$ defined by (4) is a discontinuous piecewise-linear map with discontinuity at the origin and unit slope in each of its affine regions. The dynamics of this system can easily be understood. When $|x| < 1$, the only non-trivial case, the set $[x\text{-}1, x\text{+}1)$ is invariant and absorbs all trajectories in the system. If the end points of the invariant interval $[x\text{-}1, x\text{+}1)$ are glued together to make a circle of length 2, the variable u is rotated around this circle by the distance x+1 (which scales to give the rotation number $(x+1)/2$) with each iteration. Thus in this special case the discontinuous map of the real line can be transformed into a continuous map on the circle – in fact the simple and widely-studied rotation of the circle described in Section 2. It follows that for rational input x the output bit stream is periodic, with average value equal to x, and for irrational input x the trajectory is quasiperiodic, winding indefinitely around the circle, but never closing on itself. The bit streams that correspond to admissible periodic orbits of this system can be found by application of an algorithm due to Euclid (Friedman, 1988), and have the property that the ones and zeros (representing levels of 1 and -1) that make up the orbit are distributed as evenly as possible (Gray, 1987). This concept of approximating a value by alternating between two (or more) possibilities, and doing so in a way that has certain desirable

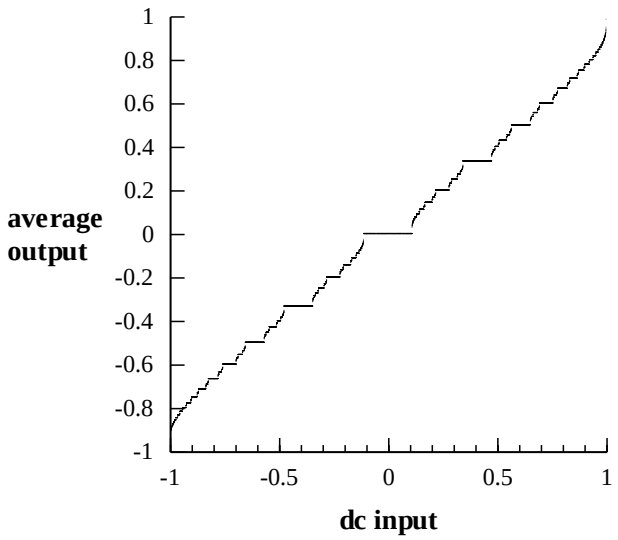

Figure 3 Average output versus dc input for the map (5) with p = 0.8.

features, is at the core of the ΣΔ concept, and has seen variants on the basic scheme implemented in a wide variety of applications.

Circuit non-idealities within the implementation of the basic ΣΔ modulator (4) result in the so-called leaky ΣΔ modulator modelled by

$$u_{n+1} = pu_n + x_{n+1} - \operatorname{sgn} u_n \tag{5}$$

and studied in (Feely and Chua, 1991). When $p < 1$, as is the case in practical implementations, the map is discontinuous even when transformed to a circle map. It is natural to examine whether the periodic bit streams observed for $p = 1$ persist for $p < 1$. With the aid of Tsypkin's method (Jury, 1964), this is easy to achieve – assuming a bit sequence at the output, and remembering that each bit corresponds to one of the two regions of linearity of the mapping, we solve the linear equations for the corresponding sequence of affine mappings and apply the appropriate inequality constraints to the resulting periodic orbit to check that it is a valid solution. We then find that the periodic sequences from the ideal system (4) do indeed persist for $p < 1$, but that instead of corresponding only to a single input value, each one exists over a range of x values, and so we have mode-locking. If we then plot the average output as a function of the input value, we get the graph shown in Figure 3 for $p = 0.8$, which we recognise as a devil's staircase. The steps of the staircase can be found analytically, quantifying the highly nonlinear loss of resolution of the modulator.

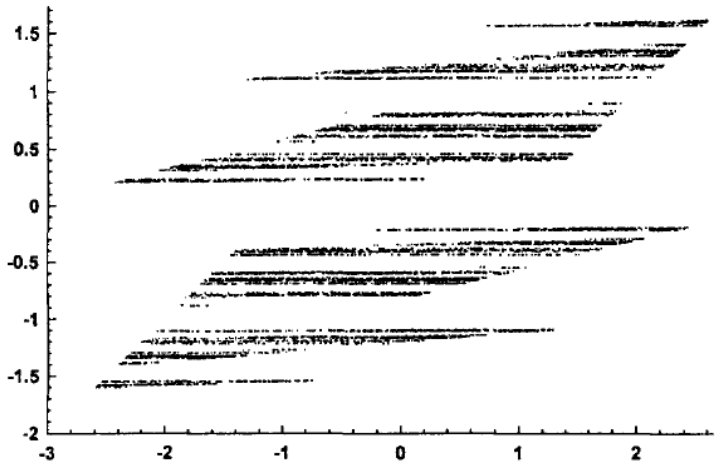

Figure 4 Trajectory of (6) with $p_1 = 0.5$ and $p_2 = 1.4$.

The repeated appearance of this form of mode-locking in a variety of electronic circuits, such as (Maity *et al.*, 2007), has resulted in the devil's staircase being termed "the electrical engineer's fractal" (Kennedy *et al*., 1989). More broadly, it is interesting to note the appearance of the map (5) in a very different context as the model for a genetic regulatory network (Coutinho *et al*., 2006), with the identical mode-locking behaviour noted and studied in that context. The system of (Dominguez *et al*., 2003) is also of interest, as it takes ideas from ΣΔ modulation and applies them in a new domain, that of microelectromechanical systems (MEMS), applying force pulses to a cantilever based on the sign of its position relative to a reference. In that case, unlike the system (5), different periodic orbits can coexist, and steps of the staircase can overlap (Teplinsky and Feely, 2008). A more detailed exploration of the dynamics of the MEMS pulsed digital oscillator can be found in (Blokhina *et al*, 2010).

If $p > 1$ in the map (5), all periodic orbits are unstable and chaotic attractors are observed. Figure 4 shows a chaotic orbit of a double-loop ΣΔ modulator, modelled by the equations

$$\begin{aligned} u_{n+1} &= p_2 u_n + v_{n+1} - \operatorname{sgn} u_n \\ v_{n+1} &= p_1 v_n + x_n - \operatorname{sgn} u_n \end{aligned} \tag{6}$$

The discontinuous piecewise-linear nature of the map can be used to explain the structure of the attractor, and the bifurcation in which it ultimately disappears (Feely, 1994; Fournier-Prunaret et al., 2001).

Another two-dimensional ΣΔ modulator is the second-order bandpass modulator studied in (Feely and Fitzgerald, 1996; Feely *et al.* 2000), which is modelled for zero-input by the equations

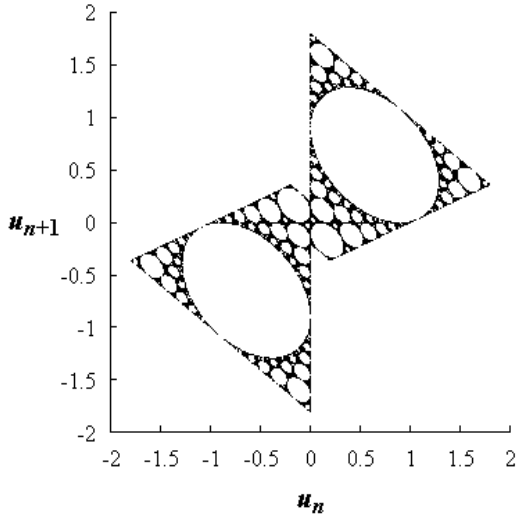

Figure 5 A trajectory of the bandpass ΣΔ modulator (7)

$$
\begin{aligned}
v_{n+1} &= u_n \\
u_{n+1} &= 2\cos\theta\,(u_n - \operatorname{sgn} u_n) - (v_n - \operatorname{sgn} v_n)
\end{aligned}
\tag{7}
$$

Fixed and periodic points of this map can be found as before. In this case the map is area-preserving, and the linear dynamics that apply close to the periodic points result in trajectories that move elliptically around those points in a periodic or quasiperiodic manner. This behaviour applies only as long as none of the ellipses crosses the *u*- or *v*-axis – the lines of discontinuity. Outside the ellipses, other, much more complex, trajectories exist, such as that shown in Figure 5. There has been extensive further study of the dynamics of the bandpass ΣΔ system (Ashwin *et al.* 2001; Ashwin *et al*, 2003). With the addition of leakage, the elliptical motion around fixed and periodic points is replaced by spiral motion in towards those points for nearby trajectories. The role of the critical curves in bounding the basins of attraction in this case is discussed in (Feely *et al*., 2000).

It is interesting to note that the behaviour exhibited by the bandpass ΣΔ system (7) is qualitatively very similar to that of the digital filter with overflow nonlinearity studied by Chua and Lin (1988). In that system too, trajectories can lie on one or more ellipses around a fixed or periodic point, and can also evolve in a complex fashion in the space carved out between the basic elliptical regions, as shown in Figure 6. The work of Chua and Lin represented one of the first applications of methods of discrete-time nonlinear dynamics to explain the complex behaviour of a discrete-

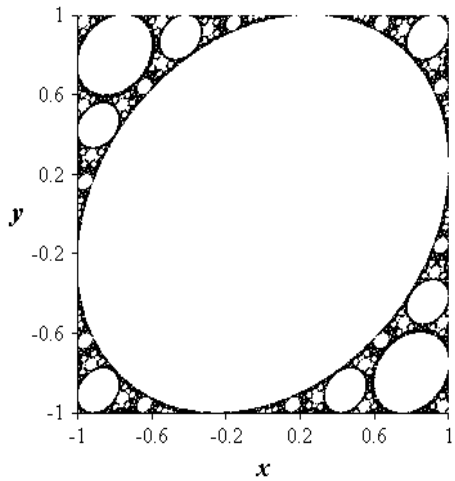

Figure 6 A trajectory of the digital filter with overflow nonlinearity from (Chua and Lin, 1988)

time electronic system, and spawned much later work in the area (see for example (Kocarev *et al*., 1996; Ogorzalek, 1992).

Finally, we consider the application of a periodic driving term to the first-order ΣΔ modulator (3). Transforming to autonomous form gives the equations

$$\begin{aligned} \theta_{n+1} &= \theta_n + \rho \quad \bmod 2\pi \\ u_{n+1} &= u_n + f(\theta_{n+1}) - \operatorname{sgn} u_n \end{aligned} \tag{8}$$

In this case all trajectories of the system are absorbed by a bounded invariant set – a belt in this case. In the simplest case where $|f(\theta)| < 1$ for all θ, the invariant belt B is bounded by the curves $f(\theta) \pm 1$. These are critical curves – the images of the sets $u_n = 0^+$ and $u_n = 0^-$ obtained as we approach either side of the discontinuity line – and their appearance in bounding the invariant set is typical of such systems. It is interesting to note that the bang-bang phase-locked loop (Walker, 2003; Da Dalt, 2005; Tertinek *et al*., 2010) can under certain conditions be described by a set of equations of the form (8), and the theory used to describe the limiting behaviour of the driven first-order ΣΔ modulator can also explain the slew-rate limiting observed in the bang-bang phase-locked loop (Teplinsky et al., 2005).

4. Conclusions

Many important electronic systems are modelled by discrete-time maps. Methods and concepts of the mathematical science of nonlinear dynamics can be helpful in the analysis of these systems: attractors correspond to (desirable or undesirable) steady states of the system; their basins of attraction tell us how likely they are to be observed in practice; the bifurcations in which they lose

stability limit the usable range of operation of the system. However, there has been relatively little treatment in the mathematics literature of a class of nonlinearity often encountered in electronic systems – those that are discontinuous and piecewise-linear. The aim of this paper has been to review the basic theory of nonlinear discrete-time dynamics, with particular reference to discontinuous piecewise-linear systems, and to highlight some of the ways in which this material has been applied to certain important electronic systems, taking the sigma-delta modulator as a central example. It is hoped that this brief introduction, and the references given below, can provide readers with some insight into this fascinating area of research.

5. Acknowledgement

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