Research Repository UCD

Title	Existence of universal Taylor series for non-simply connected domains
Authors(s)	Gardiner, Stephen J.
Publication date	2012-05-27
Publication information	Gardiner, Stephen J. "Existence of Universal Taylor Series for Non-Simply Connected Domains" 35, no. 2 (May 27, 2012).
Publisher	Springer-Verlag
Item record/more information	http://hdl.handle.net/10197/3842
Publisher's statement	The final publication is available at springerlink.com
Publisher's version (DOI)	10.1007/s00365-011-9133-z

Downloaded 2023-04-07T10:52:29Z

The UCD community has made this article openly available. Please share how this access benefits you. Your story matters! (@ucd_oa)

© Some rights reserved. For more information

Existence of universal Taylor series for non-simply connected domains

Stephen J. Gardiner

Abstract

It is known that, for any simply connected proper subdomain Ω of the complex plane and any point ζ in Ω , there are holomorphic functions on Ω that possess “universal” Taylor series expansions about ζ ; that is, partial sums of the Taylor series approximate arbitrary polynomials on arbitrary compacta in $\mathbb{C} \setminus \Omega$ that have connected complement. This paper shows, for non-simply connected domains Ω , how issues of capacity, thinness and topology affect the existence of holomorphic functions on Ω that have universal Taylor series expansions about a given point.

1 Introduction

Throughout this paper Ω denotes a proper subdomain of the complex plane $\mathbb{C}$ and ζ denotes a point of Ω . A holomorphic function f on Ω is said to belong to the collection $U(\Omega, \zeta)$, of functions with universal Taylor series expansions about ζ , if the partial sums

$$S_N(f, \zeta)(z) = \sum_{n=0}^N \frac{f^{(n)}(\zeta)}{n!} (z - \zeta)^n$$

have the following property:

for every compact set $K \subset \mathbb{C} \setminus \Omega$ with connected complement and every function g which is continuous on K and holomorphic on K° , there is a subsequence $(S_{N_k}(f, \zeta))$ that converges to g uniformly on K .

Nestoridis [19], [20] has shown that $U(\Omega, \zeta) \neq \emptyset$ for any simply connected domain Ω and any point $\zeta \in \Omega$. (The corresponding result, in which K is

⁰2010 *Mathematics Subject Classification* 30B30, 30E10, 31A05.

Keywords: Taylor series, universal approximation, overconvergence, logarithmic capacity, thin set.

This research was supported by Science Foundation Ireland under Grant 09/RFP/MTH2149, and is also part of the programme of the ESF Network “Harmonic and Complex Analysis and Applications” (HCAA).

required to be disjoint from $\overline{\Omega}$, had previously been established by Luh [14] and Chui and Parnes [4].) The same author further showed that possession of such universal Taylor series expansions is a generic property of holomorphic functions on simply connected domains Ω ; that is, $U(\Omega, \zeta)$ is a dense G_δ subset of the space $H(\Omega)$ of all holomorphic functions on Ω endowed with the topology of local uniform convergence. (See also [16].)

Attention has more recently been directed to the following:

General existence question: *For which non-simply connected domains Ω and points $\zeta \in \Omega$ do we have $U(\Omega, \zeta) \neq \emptyset$?*

Melas [15] has shown that $U(\Omega, \zeta) \neq \emptyset$ for any $\zeta \in \Omega$ whenever $\mathbb{C} \setminus \Omega$ is compact and connected. Another result in the same paper tells us that $U(\Omega, \zeta) \neq \emptyset$ for any $\zeta \in \Omega$ whenever $\mathbb{C} \setminus \Omega$ is a discrete set. In each of these cases $U(\Omega, \zeta)$ was shown to be a dense G_δ subset of $H(\Omega)$. On the other hand, Müller, Vlachou and Yavrian [17] have shown, for non-simply connected domains Ω , that thinness of the complementary set $\mathbb{C} \setminus \Omega$ at infinity is necessary for $U(\Omega, \zeta)$ to be non-empty. They conjectured that this condition is also sufficient, but it has recently been shown [10] that $U(\Omega, 0) = \emptyset$ when Ω is a domain of the form $\mathbb{C} \setminus (L \cup \{1\})$ and L is a non-degenerate continuum in $\mathbb{C} \setminus \overline{\mathbb{D}}$. Here $\mathbb{D} = D(0, 1)$ and $D(\xi, r) = \{z \in \mathbb{C} : |z - \xi| < r\}$. A selection of other results concerning universal Taylor series for non-simply connected domains may be found in [2], [3], [5], [6], [7], [12], [21], [23], [24], [25], [26].

The main purpose of this paper is to show how issues of (logarithmic) capacity, thinness and Euclidean topology affect the existence of functions in $U(\Omega, \zeta)$ when Ω is non-simply connected. Each of our results about universal Taylor series will appear as a corollary of a more general result. We recall that a set $A \subset \mathbb{C}$ is said to be *thin* at a point ξ if either $\xi \notin \overline{A \setminus \{\xi\}}$ or there is a subharmonic function u on a neighbourhood of ξ such that

$$\limsup_{z \rightarrow \xi, z \in A} u(z) < u(\xi).$$

The classical Wiener criterion (Theorem 5.4.1 of [22]) provides a characterization of thinness in terms of capacity. By a *polar set* we mean a set of capacity zero.

We will abbreviate $S_N(f, \zeta)$ to S_N where no confusion can arise.

Theorem 1 *Let $f \in H(\Omega)$ and $r = \min\{|z - \zeta| : z \in \mathbb{C} \setminus \Omega\}$, and suppose that $\mathbb{C} \setminus \Omega$ is thin at a point $\xi \in \partial\Omega \cap \partial D(\zeta, r)$. If some subsequence (S_{N_k}) of $(S_N(f, \zeta))$ is uniformly bounded on a non-polar subset of $\mathbb{C} \setminus \overline{D}(\zeta, r)$, then (S_{N_k}) converges uniformly on a neighbourhood of ξ ; in particular, f has an analytic continuation to a neighbourhood of ξ .*

Corollary 1 *Let $r = \min\{|z - \zeta| : z \in \mathbb{C} \setminus \Omega\}$. If $\mathbb{C} \setminus \Omega$ is thin at a point $\xi \in \partial\Omega \cap \partial D(\zeta, r)$ and $\mathbb{C} \setminus (\Omega \cup \overline{D}(\zeta, r))$ is non-polar, then $U(\Omega, \zeta) = \emptyset$.*

Thus, in contrast to the above result of Müller, Vlachou and Yavrian, we now see that *non-thinness* of $\mathbb{C} \setminus \Omega$ at all points of $\partial\Omega$ that are at the *minimum* distance r from ζ is a necessary condition for $U(\Omega, \zeta)$ to be non-empty (unless $\Omega \cup \overline{D}(\zeta, r)$ has polar complement). However, it is not a sufficient condition, as can be seen from either Corollary 4 below or the example given at the end of Section 2.

Since polar sets are thin everywhere, we immediately obtain the following special case.

Corollary 2 *If $D(\zeta, R) \setminus \Omega$ is non-empty and polar for some $R > 0$, and if $\mathbb{C} \setminus \Omega$ is non-polar, then $U(\Omega, \zeta) = \emptyset$.*

The result of [10], that $U(\Omega, 0) = \emptyset$ when Ω is a domain of the form $\mathbb{C} \setminus (L \cup \{1\})$ and L is a non-degenerate continuum in $\mathbb{C} \setminus \overline{\mathbb{D}}$, is clearly a particular case of Corollary 2. This corollary also allows us to write down an example of a domain Ω with bounded complement that admits no holomorphic functions with universal Taylor series expansions about even a single point:

Example 1 *Let $\Omega = \mathbb{C} \setminus (\overline{\mathbb{D}} \cup E)$, where*

$$E = \bigcup_{k=1}^{\infty} \left\{ \left(1 + 2^{-k}\right) e^{m2^{1-k}\pi i} : 1 \leq m \leq 2^k \right\}.$$

Then $U(\Omega, \zeta) = \emptyset$ for every $\zeta \in \Omega$.

Theorem 1 also reveals a link between thinness and the existence of Hadamard-Ostrowski gaps in power series. We recall that a power series $\sum a_n z^n$, with radius of convergence 1, is said to possess *Hadamard-Ostrowski gaps* if there exist natural numbers $p_1 < q_1 \leq p_2 < q_2 \leq \dots$ such that

$$\inf_k \frac{q_k}{p_k} > 1 \quad \text{and} \quad \limsup_{n \rightarrow \infty, n \in \cup_k (p_k, q_k)} |a_n|^{1/n} < 1.$$

Corollary 3 *Let $\sum a_n z^n$ be a power series that converges on $\mathbb{D}$ and is analytically continuable to a domain Ω , but not to a neighbourhood of a given point $\xi \in \partial\mathbb{D}$. If $\mathbb{C} \setminus \Omega$ is thin at ξ , then the series cannot possess Hadamard-Ostrowski gaps.*

Next, we consider the situation where a subsequence of the partial sums of a Taylor series is uniformly bounded on a disc $\overline{D}(\xi, r) \subset \mathbb{C} \setminus \{\zeta\}$ that is

not contained in the disc of convergence. It turns out that an important role is played by the quantity

$$\rho_0 = \sqrt{|\zeta - \xi|^2 - r^2},$$

which has the property that $\partial D(\zeta, \rho_0)$ and $\partial D(\xi, r)$ meet at right angles.

Theorem 2 *Let $f \in H(\Omega)$ and $\overline{D}(\xi, r) \subset \mathbb{C} \setminus \{\zeta\}$, and suppose that the set $D(\zeta, \rho) \setminus (\Omega \cup \overline{D}(\xi, r))$ is polar, where $|\zeta - \xi| - r < \rho \leq \rho_0$. If $(S_{N_k}(f, \zeta))$ is uniformly bounded on $\overline{D}(\xi, r)$, then it is locally uniformly convergent either on $D(\zeta, \rho) \setminus \overline{D}(\xi, r)$ or on a domain containing $D(\zeta, |\zeta - \xi| - r) \cup D(\xi, r)$.*

In the case where Ω is the complement of a closed disc $\overline{D}(\xi, r)$ we know from the above-cited work of Melas that $U(\Omega, \zeta)$ is a dense G_δ subset of $H(\Omega)$. Costakis and Vlachou [7] subsequently showed that there is a dense G_δ subset of functions f in $H(\Omega)$ with the stronger property that:

for every function g which is continuous on $\overline{D}(\xi, r)$ and holomorphic on $D(\xi, r)$, there is a subsequence $(S_{N_k}(f, \zeta))$ that converges to g uniformly on $\overline{D}(\xi, r)$ and to f locally uniformly on $D(\zeta, |\zeta - \xi| + r) \setminus \overline{D}(\xi, r)$.

Theorem 2 shows that, for *every* function f in $U(\Omega, \zeta)$, where $\Omega = \mathbb{C} \setminus \overline{D}(\xi, r)$, subsequences of the partial sums of the Taylor series that converge on $\overline{D}(\xi, r)$ necessarily also overconverge to f on $D(\zeta, \rho_0) \setminus \overline{D}(\xi, r)$. It also yields a stronger version of Corollary 2 for a certain family of domains:

Corollary 4 *Let $\Omega \subset \mathbb{C} \setminus \overline{D}(\xi, r)$ and $0 < \rho \leq \rho_0$. If $D(\zeta, \rho) \setminus (\Omega \cup \overline{D}(\xi, r))$ is non-empty and polar, then $U(\Omega, \zeta) = \emptyset$.*

This last result brings us much closer to a complete solution of the existence question for universal Taylor series in the particular case where $\mathbb{C} \setminus \Omega$ consists of a disc and an additional point. To summarise the current state of knowledge, let

$$\Omega = \mathbb{C} \setminus (\overline{D}(\xi, r) \cup \{\xi_0\}), \text{ where } 0 \in \Omega \text{ and } \xi_0 \notin \overline{D}(\xi, r). \quad (1)$$

If $|\xi_0| > |\xi| + r$, then we know from [7] that $U(\Omega, 0)$ is a dense G_δ subset of $H(\Omega)$. Recent work of Tsirivas and Vlachou [24] shows that this remains the case when $|\xi_0| = |\xi| + r$. In contrast, the above corollary shows that $U(\Omega, 0)$ is empty when $|\xi_0| < \sqrt{|\xi|^2 - r^2}$. So we are left with the following question.

Open problem *If Ω is given by (1) and $\sqrt{|\xi|^2 - r^2} \leq |\xi_0| < |\xi| + r$, is $U(\Omega, 0)$ non-empty?*

Subharmonic functions are an important tool in this paper. Their role will become clearer in Theorem 3 below. This result itself yields, as a corollary, a necessary topological condition on Ω for the existence of universal Taylor series. Theorems 1 and 2 will then be proved in the remaining two sections, along with their respective corollaries. The possibility that Theorem 2 might hold was initially suggested by earlier work of the author on the extension of superharmonic functions (see [8], or Chapter 6 of [9]). For the necessary potential theoretic background we refer to the books [22] and [1].

2 A preliminary result

We denote the extended complex plane $\mathbb{C} \cup \{\infty\}$ by $\widehat{\mathbb{C}}$. Let $G_\omega(\cdot, \xi)$ denote the Green function of an open set $\omega \subset \widehat{\mathbb{C}}$ with pole at $\xi \in \omega$, when it exists, that is, when the complement of ω is non-polar. The function $G_\omega(\cdot, \xi)$ should be interpreted as 0 outside the connected component of ω that contains ξ . The following theorem, which strengthens and extends Theorem 1 of Gehlen [11], will be useful in proving our main results. It captures information about the behaviour of a subsequence of the partial sums of a Taylor series in terms of the subharmonicity of a certain function, which we can later exploit.

Theorem 3 *Let $f \in H(\Omega)$, where $0 \in \Omega$, let (S_{N_k}) be a subsequence of $(S_N(f, 0))$, and let U be the largest domain containing 0 on which (S_{N_k}) is locally uniformly convergent. Further, suppose that $\Omega \setminus U \neq \emptyset$ and that (S_{N_k}) is uniformly bounded on a compact set K (possibly empty) contained in $\mathbb{C} \setminus U$. Then*

- (i) U is bounded and simply connected;
- (ii) $(\overline{U})^\circ \cap \Omega = U \cap \Omega$; and
- (iii) the upper semicontinuous regularization v^* of the function

$$v(z) = \begin{cases} -G_U(z, 0) & (z \in U) \\ G_V(z, \infty) & (z \in V) \\ 0 & (\text{elsewhere in } \mathbb{C}) \end{cases},$$

where $V = \widehat{\mathbb{C}} \setminus (\overline{U} \cup K)$, is subharmonic on $\Omega \cup (\mathbb{C} \setminus \partial U)$ and continuously vanishes on ∂U .

Proof. If the domain U were unbounded, then there would exist a closed set $E \subset U$, non-thin at infinity, such that $\limsup_{k \rightarrow \infty} |S_{N_k}|^{1/N_k} \leq 1$ on E . By Theorem 1 of Müller and Yavrian [18], this would then imply that $\Omega \subseteq U$. Since we have assumed that $\Omega \setminus U \neq \emptyset$, the domain U must be bounded. Also, it follows from the maximum modulus principle that U is simply connected, so part (i) of the theorem is established. Since both U and $\widehat{\mathbb{C}} \setminus \overline{U}$ are regular for the Dirichlet problem, the function v is continuous

at points of ∂U , and so v^* continuously vanishes there. In the light of this it remains only to prove the subharmonicity in part (iii), for (ii) will then follow by the maximum principle. Since f has an analytic continuation to $U \cup \Omega$, we may assume from now on that $U \subset \Omega$.

The sequence (S_{N_k}) is uniformly bounded on some closed disc centred at 0, so $(|S_{N_k}|^{1/N_k})$ is locally uniformly bounded on $\mathbb{C}$, by a simple case of Bernstein's lemma (Theorem 5.5.7 in [22]). Since

$$|S_{N_k} - f|^{1/N_k} \leq 2^{1/N_k} \max \left\{ |S_{N_k}|^{1/N_k}, |f|^{1/N_k} \right\} \quad \text{on } \Omega$$

by the triangle inequality, we see that $(|S_{N_k} - f|^{1/N_k})$ is locally uniformly bounded in Ω . It follows that the upper semicontinuous regularization u^* of the function

$$u = \limsup_{k \rightarrow \infty} \log |S_{N_k} - f|^{1/N_k}$$

is subharmonic on Ω (see Theorem 3.4.3(a) of [22]), and

$$u \leq \left(\limsup_{k \rightarrow \infty} \log |S_{N_k}|^{1/N_k} \right)^+ \quad \text{on } \Omega. \quad (2)$$

Clearly $u^* \leq 0$ on U . On Ω we have

$$\log |S_{N_k}(z) - f(z)|^{1/N_k} = \log |z| + w_k(z),$$

where w_k is a subharmonic function on Ω . Hence $u^*(z) = \log |z| + w(z)$ on Ω , where w is a subharmonic function on Ω . Since $G_U(\cdot, 0)$ is the least nonnegative superharmonic function on U of the form $-\log |\cdot| - w$, where w is subharmonic on U , and $u^* \leq 0$ on U , we conclude that

$$u^* \leq -G_U(\cdot, 0) \quad \text{on } U. \quad (3)$$

Next, let (L_n) be an exhaustion of U by compact sets, and let $\omega_n = \widehat{\mathbb{C}} \setminus (L_n \cup K)$. By Bernstein's lemma,

$$\left(\frac{|S_{N_k}|}{\sup_{L_n \cup K} |S_{N_k}|} \right)^{1/N_k} \leq e^{G_{\omega_n}(\cdot, \infty)} \quad \text{on } \mathbb{C} \quad (n \geq 1; k \geq 1).$$

Since (S_{N_k}) is uniformly bounded on $L_n \cup K$, we see from (2) that $u \leq G_{\omega_n}(\cdot, \infty)$ on Ω for all n . Hence

$$u \leq \lim_{n \rightarrow \infty} G_{\omega_n}(\cdot, \infty) = G_V(\cdot, \infty) \quad \text{on } \Omega \quad (4)$$

and, in particular, $u \leq 0$ on $\Omega \setminus V$. (One way of seeing the equality in (4) is to use Theorems 5.7.3(iv) and 5.7.4(iii) in [1] together with the fact that U is non-thin at each point of ∂U .) It follows from (3) and (4) that

$$u^* \leq v^* \quad \text{on } \Omega.$$

Let $\xi \in \Omega \cap \partial U$ and $\overline{D}(\xi, r) \subset \Omega$. Then $u^*(\xi) = 0$, for otherwise $u^*(\xi) < 0$ and we would arrive at the contradictory conclusion that $S_{N_k} \rightarrow f$ uniformly on a neighbourhood of ξ and so $\xi \in U$. Hence

$$v^*(\xi) = 0 = u^*(\xi) \leq A(u^*; \xi, r) \leq A(v^*; \xi, r),$$

where $A(w; \xi, r)$ denotes the area mean value of a function w over $D(\xi, r)$. Since v^* is clearly subharmonic on $\mathbb{C} \setminus \partial U$, we conclude that v^* is subharmonic on $\Omega \cup (\mathbb{C} \setminus \partial U)$, as required. ■

Corollary 5 *Let $r = \min\{|z - \zeta| : z \in \mathbb{C} \setminus \Omega\}$ and suppose that $\Omega \setminus \overline{D}(\zeta, r)$ has a bounded component ω . If $U(\Omega, \zeta) \neq \emptyset$, then $(\omega \cup \overline{D}(\zeta, r))^\circ \cap \Omega$ is simply connected.*

Before proving Corollary 5 we note that it yields another family of domains Ω such that $U(\Omega, \zeta) = \emptyset$ for certain ζ . The example below illustrates this, and also the remark we made following Corollary 1.

Example 2 *Suppose that $\mathbb{C} \setminus \Omega$ consists of $\partial D(3/2, 1) \setminus \mathbb{D}$ and $\overline{D}(3/2, 1/2)$. Then $U(\Omega, 0) = \emptyset$.*

Proof of Corollary 5. Without loss of generality we may suppose that $\zeta = 0$. If Ω is simply connected, the conclusion is trivial, so we assume otherwise. The existence of functions in $U(\Omega, 0)$ then implies that Ω is unbounded. Let r and ω be as in the statement of the corollary, let $R > \sup\{|z| : z \in \omega\}$ and $K = \overline{D}(0, R) \setminus \Omega$. Then K has connected complement. Let $f \in U(\Omega, 0)$ and q be an arbitrary polynomial. Then we can find a subsequence (S_{N_k}) of $(S_N(f, 0))$ that converges uniformly to q on K . Let U and v^* be as in the statement of Theorem 3 and let $W = (\omega \cup \overline{D}(0, r))^\circ \cap \Omega$. Then W is connected, $D(0, r) \subset U$ and so $\partial W \subset \overline{U} \cup K$. Thus $v^* < 0$ on W by Theorem 3 and the maximum principle, and so $W \subset U$. If $\mathbb{C} \setminus W$ had a bounded component L , then $L \subset U$ by the simple connectedness of U and so (S_{N_k}) would converge uniformly on L to a limit function that is uniquely determined by the analytic continuation of f to U . This contradicts the fact that (S_{N_k}) was chosen to converge to the arbitrary polynomial q on K , which intersects L . ■

3 Proof of Theorem 1 and Corollaries

Proof of Theorem 1. Let $f \in H(\Omega)$ and $r = \min\{|z - \zeta| : z \in \mathbb{C} \setminus \Omega\}$, and let $\mathbb{C} \setminus \Omega$ be thin at a point $\xi \in \partial \Omega \cap \partial D(\zeta, r)$. Further, suppose that there is a subsequence (S_{N_k}) of $(S_N(f, \zeta))$ that is uniformly bounded on some non-polar subset K of $\mathbb{C} \setminus \overline{D}(\zeta, r)$. Without loss of generality we may assume that $\zeta = 0$ and $r = 1$, and that K is compact.

Let U , V and v^* be as in Theorem 3. Then $\mathbb{D} \subset U$ and clearly

$$v^*(z) = -G_U(z, 0) \leq -G_{\mathbb{D}}(z, 0) = \log |z| \quad (z \in \overline{\mathbb{D}}).$$

Further, since $V \subset \widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$, and $\mathbb{C} \setminus (\overline{\mathbb{D}} \cup V)$ contains the non-polar set K , we see from the maximum principle that

$$v^*(z) < G_{\mathbb{C} \setminus \overline{\mathbb{D}}}(z, \infty) = \log |z| \quad (z \in \mathbb{C} \setminus \overline{\mathbb{D}}).$$

Let Ω_0 be the component of $\Omega \cap [D(0, 2) \setminus \overline{D}(0, 1/2)]$ that meets $\mathbb{D}$, and let $H_{v^*}^{\Omega_0}$ denote the solution to the Dirichlet problem on Ω_0 with boundary data v^* . Since the set $\partial\Omega_0 \setminus \overline{\mathbb{D}}$ has positive harmonic measure for Ω_0 , we obtain a strictly positive harmonic function on Ω_0 by defining

$$w(z) = \log |z| - H_{v^*}^{\Omega_0}(z) \quad (z \in \Omega_0).$$

The thinness of $\mathbb{C} \setminus \Omega$ at ξ tells us that ξ is an irregular boundary point for the Dirichlet problem on Ω_0 (see Theorems 6.6.7 and 7.5.1 of [1]), so by Theorem 7.5.5 of [1] there exists a set A , thin at ξ , and a number $\ell \geq 0$, such that

$$w(z) \rightarrow \ell \quad (z \rightarrow \xi, z \notin A).$$

Further, by the continuity of w on Ω_0 , we may assume that the set A is closed. We must have $\ell > 0$, for otherwise w would be a barrier for the open set $\Omega_0 \setminus A$ at the irregular boundary point ξ , which is impossible (see Theorem 6.6.5 of [1]). By a well-known property of thin sets (Theorem 7.3.9 of [1]) we can choose $r_0 \in (0, e^{\ell/2} - 1)$ such that $\partial D(\xi, r_0) \subset \Omega_0 \setminus A$ and $w \geq \ell/2$ on $\partial D(\xi, r_0)$. We now use the subharmonicity of v^* on Ω (by Theorem 3) to see that

$$v^*(z) \leq H_{v^*}^{\Omega_0}(z) = \log |z| - w(z) \leq \log(1 + r_0) - \ell/2 < 0 \quad (z \in \partial D(\xi, r_0)),$$

whence $\partial D(\xi, r_0) \subset U$. Since U is simply connected, we conclude that (S_{N_k}) converges uniformly on $\overline{D}(\xi, r_0)$, as required. ■

Proof of Corollary 1. Let $r = \min\{|z - \zeta| : z \in \mathbb{C} \setminus \Omega\}$, suppose that $\mathbb{C} \setminus \Omega$ is thin at a point $\xi \in \partial\Omega \cap \partial D(\zeta, r)$, and that $\mathbb{C} \setminus (\Omega \cup \overline{D}(\zeta, r))$ is non-polar. Then there exists a compact non-polar subset K of $\mathbb{C} \setminus (\Omega \cup \overline{D}(\zeta, r))$, and we can arrange further that $\mathbb{C} \setminus K$ is connected. If there exists f in $U(\Omega, \zeta)$, then we can find a subsequence (S_{N_k}) of $(S_N(f, \zeta))$ that is uniformly bounded on K and that converges to some arbitrarily chosen limit ℓ at ξ . Theorem 1 would then tell us that f has an analytic continuation to a neighbourhood of ξ and that $S_{N_k}(\xi) \rightarrow f(\xi)$. This contradicts the arbitrary nature of ℓ , so $U(\Omega, \zeta) = \emptyset$. ■

Proof of Corollary 3. Let $\sum a_n z^n$ be a power series that converges on $\mathbb{D}$ and is analytically continuable to a domain Ω , but not to any neighbourhood

of a given point $\xi \in \partial\mathbb{D}$. If the series possesses Hadamard-Ostrowski gaps, then a classical theorem of Ostrowski tells us that there is a subsequence of $(S_N(f, 0))$ that converges uniformly on a neighbourhood of each point of $\Omega \cap \partial\mathbb{D}$ (cf. Theorem 16.7.1 in [13]). Theorem 1 would then yield the conclusion that $\mathbb{C} \setminus \Omega$ cannot be thin at ξ . ■

4 Proof of Theorem 2 and Corollary 4

In proving Theorem 2 we may assume that $\zeta = 0$. Let $f \in H(\Omega)$ and $\overline{D}(\xi, r) \subset \mathbb{C} \setminus \{0\}$, let $\rho_0 = \sqrt{|\xi|^2 - r^2}$ and suppose that $D(0, \rho) \setminus (\Omega \cup \overline{D}(\xi, r))$ is polar, where $|\xi| - r < \rho \leq \rho_0$. Further, suppose that $(S_{N_k}(f, 0))$ is uniformly bounded on $\overline{D}(\xi, r)$, and let U be the largest domain containing 0 on which $(S_{N_k}(f, 0))$ is locally uniformly convergent. Since $D(0, |\xi| - r) \setminus \Omega$ is polar, and so everywhere thin, Theorem 1 shows that the Taylor series of f about 0 has radius of convergence at least $|\xi| - r$, and so $D(0, |\xi| - r) \subset U$.

If $\overline{D}(\xi, r) \cap U \neq \emptyset$, then $D(\xi, r) \subset U$ by Montel's theorem, and there is nothing more to prove. If $\Omega \subset U$, then $D(0, \rho) \setminus \overline{D}(\xi, r) \subset U$, since the polar set $[D(0, \rho) \setminus \overline{D}(\xi, r)] \setminus \Omega$ is totally disconnected and U is simply connected, and again there is nothing more to prove. So we suppose from now on that $\overline{D}(\xi, r) \subset \mathbb{C} \setminus U$ and $\Omega \setminus U \neq \emptyset$.

Let V and v^* be as defined in Theorem 3, where K is taken to be the disc $\overline{D}(\xi, r)$. From that result we see that U is bounded. Let

$$R = \sup\{t : D(0, t) \subset U \cup \overline{D}(\xi, r)\}.$$

Then $R \geq |\xi| - r$. We suppose, for the sake of contradiction, that $R < \rho$, and choose a point η in the set $\partial D(0, R) \cap \partial(U \cup \overline{D}(\xi, r))$. By Theorem 3 the function v^* is subharmonic on $D(0, \rho) \setminus \overline{D}(\xi, r)$ because polar sets are removable singularities for locally upper-bounded subharmonic functions.

We first consider the case where $\eta \in \partial D(0, R) \cap \partial U \setminus \partial \overline{D}(\xi, r)$. We define

$$w(z) = \begin{cases} -G_{D(0, R) \setminus \overline{D}(\xi, r)}(z, 0) & (|z| < R) \\ G_{\widehat{\mathbb{C}} \setminus (\overline{D}(0, R) \cup \overline{D}(\xi, r))}(z, \infty) & (|z| > R) \\ 0 & (|z| = R) \end{cases}$$

and

$$h(z) = \begin{cases} -G_{D(0, R) \setminus \overline{D}(\xi, r)}(z, 0) & (|z| < R) \\ G_{D(0, R) \setminus \overline{D}(\xi, r)}(R^2/\bar{z}, 0) & (|z| > R) \\ 0 & (|z| = R) \end{cases}.$$

The latter function is harmonic on $\mathbb{C} \setminus (\overline{D}(\xi, r) \cup \{0\})$ by the reflection principle (and a conformal mapping from the upper half-plane to $D(0, R)$). Since

$$U \supset D(0, R) \setminus \overline{D}(\xi, r) \quad \text{and} \quad V \subset \widehat{\mathbb{C}} \setminus (\overline{D}(0, R) \cup \overline{D}(\xi, r)),$$

we see that $v^* \leq w$ on $\mathbb{C}$. Also, it follows from inversion in $\partial D(0, R)$, and the fact that $R < \rho_0$, that

$$G_{D(0, R) \setminus \overline{D}(\xi, r)}(R^2/\bar{z}, 0) > G_{\widehat{\mathbb{C}} \setminus (\overline{D}(0, R) \cup \overline{D}(\xi, r))}(z, \infty)$$

when $z \in \mathbb{C} \setminus (\overline{D}(0, R) \cup \overline{D}(\xi, r))$, so $w < h$ on that set. Now let $\delta > 0$ be small enough that $\overline{D}(\eta, \delta) \subset D(0, \rho) \setminus \overline{D}(\xi, r)$. Since $\eta \in \partial U$, we arrive at the contradictory conclusion that

$$0 = v^*(\eta) \leq A(v^*; \eta, \delta) \leq A(w; \eta, \delta) < A(h; \eta, \delta) = h(\eta) = 0.$$

It remains to consider the more delicate case where $\eta \in \partial D(0, R) \cap \partial U \cap \partial D(\xi, r)$. We will use the following special case of the boundary Harnack principle, which can be deduced from Lemma 8.5.1 in [1] using inversion.

Lemma 4 *There is an absolute constant $C_0 > 1$ with the following property. If $\eta \in \partial D(\xi, r)$ and $\sigma \in (0, r]$, and if h_0 is a positive harmonic function on $D(\eta, \sigma) \setminus \overline{D}(\xi, r)$ that vanishes continuously on $D(\eta, \sigma) \cap \partial D(\xi, r)$, then*

$$C_0^{-1} \frac{h_0(\eta_\sigma)}{\log(1 + \sigma/2r)} \leq \frac{h_0(z)}{\log(|z - \xi|/r)} \leq C_0 \frac{h_0(\eta_\sigma)}{\log(1 + \sigma/2r)} \quad (z \in D(\eta, \sigma/2) \setminus \overline{D}(\xi, r)),$$

where $\eta_\sigma = \xi + (1 + \frac{\sigma}{2r})(\eta - \xi)$.

We can write $\eta = Re^{i\phi}$ and $\xi - \eta = re^{i\theta}$ for some ϕ and θ . Then

$$|\xi|^2 = R^2 + r^2 + 2Rr \cos(\theta - \phi),$$

so

$$\rho_0^2 = R^2 + 2Rr \cos(\theta - \phi),$$

and since $R < \rho_0$, we see that $\cos(\theta - \phi) > 0$. It follows that U contains a sector of vertex η and angle exceeding $\pi/2$. We will assume for the present that 0 , η and ξ are not collinear. This implies that, for small s , the set $V \cap D(\eta, s)$ is contained in a sector of vertex η and angle less than $\pi/2$. More precisely, we can choose sectors of the form

$$S_j = \begin{cases} \{\eta + se^{it} : 0 < s < s_0, |t - \pi - (\theta + \phi)/2| < \alpha_j\} & (j = 1, 2) \\ \{\eta + se^{it} : 0 < s < s_0, |t - t_0| < \alpha_j\} & (j = 3, 4) \end{cases},$$

where $s_0 \in (0, r]$, $t_0 \in [0, 2\pi]$,

$$\alpha_1 > \alpha_2 > \frac{\pi}{4} > \alpha_3 > \alpha_4, \quad \overline{S}_1 \setminus U = \{\eta\}, \quad \overline{D}(\eta, s_0) \cap V \subset \overline{S}_4$$

and $S_1 \cap S_3 = \emptyset$. We can define a positive harmonic function on S_1 by

$$p_1(\eta + se^{it}) = s^{\frac{\pi}{2\alpha_1}} \cos\left(\frac{\pi}{2\alpha_1} \left(t - \pi - \frac{\theta + \phi}{2}\right)\right).$$

Since $\overline{S}_1 \setminus U = \{\eta\}$, the function $G_U(\cdot, 0)$ has a positive lower bound on the set

$$\left\{ \eta + s_0 e^{it} : \left| t - \pi - \frac{\theta + \phi}{2} \right| \leq \alpha_1 \right\},$$

so by the minimum principle there is a positive constant c_1 such that

$$G_U(\cdot, 0) \geq c_1 p_1 \quad \text{on } S_1. \quad (5)$$

Similarly, we can define a positive harmonic function on S_3 by

$$p_2(\eta + s e^{it}) = s^{\frac{\pi}{2\alpha_3}} \cos \left(\frac{\pi}{2\alpha_3} (t - t_0) \right).$$

This function has a positive lower bound on $\partial D(\eta, s_0) \cap V$, since the latter set is contained in $\overline{S}_4$, so by the maximum principle there is a positive constant c_2 such that

$$c_2 p_2 \geq G_V(\cdot, \infty) \quad \text{on } S_3. \quad (6)$$

We assign p_1 (respectively, p_2) the value 0 on $D(\eta, s_0) \setminus S_1$ (respectively, on $D(\eta, s_0) \setminus S_3$).

Now let $W_\sigma = D(\eta, \sigma) \setminus \overline{D}(\xi, r)$, and let $H_g^{W_\sigma}$ denote the solution to the Dirichlet problem on W_σ with given boundary data g on $\partial D(\eta, \sigma) \setminus \overline{D}(\xi, r)$ and 0 elsewhere. If $\sigma < \min\{s_0, \rho - R\}$, then we see from (5) and (6) that

$$H_{v^*}^{W_\sigma} = H_{G_V(\cdot, \infty)}^{W_\sigma} - H_{G_U(\cdot, 0)}^{W_\sigma} \leq c_2 H_{p_2}^{W_\sigma} - c_1 H_{p_1}^{W_\sigma}.$$

Since

$$p_1(z) \geq |z - \eta|^{\frac{\pi}{2\alpha_1}} \cos \left(\frac{\pi\alpha_2}{2\alpha_1} \right) \quad \text{on } S_2$$

and

$$p_2(z) \leq |z - \eta|^{\frac{\pi}{2\alpha_3}} \quad \text{on } S_3,$$

we deduce that

$$H_{v^*}^{W_\sigma} \leq c_2 \sigma^{\frac{\pi}{2\alpha_3}} h_{2,\sigma} - c_1 \sigma^{\frac{\pi}{2\alpha_1}} h_{1,\sigma} \quad \text{on } W_\sigma,$$

where

$$h_{1,\sigma} = \cos \left(\frac{\pi\alpha_2}{2\alpha_1} \right) H_{\chi_{S_2}}^{W_\sigma}, \quad h_{2,\sigma} = H_{\chi_{S_3}}^{W_\sigma}$$

and χ_A denotes the characteristic function of a set A . We note that $h_{1,\sigma} > 0$ since $\alpha_1 > \alpha_2$. If $|z - \eta| < \sigma/2$, then we deduce, using Lemma 4, that

$$H_{v^*}^{W_\sigma}(z) \leq \left\{ C_0 c_2 \sigma^{\frac{\pi}{2\alpha_3}} h_{2,\sigma}(\eta_\sigma) - C_0^{-1} c_1 \sigma^{\frac{\pi}{2\alpha_1}} h_{1,\sigma}(\eta_\sigma) \right\} \frac{\log(|z - \xi|/r)}{\log(1 + \sigma/2r)}.$$

Since

$$H_{\chi_{S_2}}^{W_\sigma}(\eta_\sigma) \geq H_{\chi_{S_2}}^{S_2 \cap D(\eta, \sigma)}(\eta_\sigma)$$

and $\alpha_2 > \pi/4$, it follows by dilation that there is a constant $c_3 > 0$, independent of σ , such that $h_{1,\sigma}(\eta_\sigma) \geq c_3$. Also, trivially $h_{2,\sigma}(\eta_\sigma) \leq 1$. Hence

$$H_{v^*}^{W_\sigma}(z) \leq \sigma^{\frac{\pi}{2\alpha_1}} \left\{ C_0 c_2 \sigma^{\frac{\pi}{2}(\frac{1}{\alpha_3} - \frac{1}{\alpha_1})} - C_0^{-1} c_1 c_3 \right\} \frac{\log(|z - \xi|/r)}{\log(1 + \sigma/2r)} \quad \left(z \in D\left(\eta, \frac{\sigma}{2}\right) \right).$$

Since $\alpha_3 < \alpha_1$, we conclude from the subharmonicity of v^* on $D(0, \rho) \setminus \overline{D}(\xi, r)$ that

$$v^* \leq H_{v^*}^{W_\sigma} < 0 \quad \text{on } D(\eta, \sigma/2) \setminus \overline{D}(\xi, r),$$

provided σ is chosen small enough. This implies that $D(\eta, \sigma/2) \setminus \overline{D}(\xi, r) \subset U$, which contradicts the fact that $\eta \in \partial(U \cup \overline{D}(\xi, r))$.

Finally, if $0, \eta$ and ξ are collinear, then $D(0, R)$ and $D(\xi, r)$ are externally tangent. A similar argument applies in this case, except that *two* disjoint sectors with small aperture and vertices at η are now required to cover $D(\eta, s_0) \cap V$. The resulting contradiction shows that we cannot have $R < \rho$, so Theorem 2 is established.

Proof of Corollary 4. Let $\Omega \subset \mathbb{C} \setminus \overline{D}(\xi, r)$ and suppose that the set $D(\zeta, \rho) \setminus (\Omega \cup \overline{D}(\xi, r))$ is non-empty and polar, where $0 < \rho \leq \rho_0$. If $\rho \leq |\zeta - \xi| - r$, then $D(\zeta, \rho) \setminus \Omega$ is non-empty and polar, and the result follows from Corollary 2. We now assume, for the sake of contradiction, that $|\zeta - \xi| - r < \rho \leq \rho_0$ and that there exists f in $U(\Omega, \zeta)$. Let $\xi_0 \in D(\zeta, \rho) \setminus (\Omega \cup \overline{D}(\xi, r))$ and $\ell \in \mathbb{C} \setminus \{0\}$. Then we can choose a subsequence (S_{N_k}) of $(S_N(f, \zeta))$ such that $S_{N_k} \rightarrow 0$ uniformly on $\overline{D}(\xi, r)$ and $S_{N_k}(\xi_0) \rightarrow \ell$. Since $f \not\equiv 0$, we see from Theorem 2 that f has an analytic continuation to $D(\zeta, \rho) \setminus \overline{D}(\xi, r)$, which is valued ℓ at ξ_0 . This contradicts the arbitrary nature of ℓ . ■

We conclude with an example which relies on elements of both Theorem 2 and Corollary 5.

Example 3 Let $\Omega = \mathbb{C} \setminus (E \cup \{-5\})$, where

$$E = \overline{D}(5, 4) \cup [\partial D(-5, 4) \setminus D(0, 3)].$$

Then $U(\Omega, 0) = \emptyset$.

To see this, suppose that there exists f in $U(\Omega, 0)$. Then we can choose a subsequence (S_{N_k}) of $(S_N(f, 0))$ such that $S_{N_k} \rightarrow 0$ uniformly on E and $S_{N_k}(-5) \rightarrow 1$. We can apply Theorem 2, where $\rho = \rho_0 = \sqrt{5^2 - 4^2} = 3$, to see that (S_{N_k}) is locally uniformly convergent on $D(0, 3) \setminus \overline{D}(5, 4)$, and then argue as in the proof of Corollary 5 to obtain the contradictory conclusion that the set

$$[D(-5, 4) \cup D(0, 3)] \setminus [\overline{D}(5, 4) \cup \{-5\}]$$

is simply connected.

References

- [1] D. H. Armitage and S. J. Gardiner, *Classical Potential Theory*, Springer, London, 2001.
- [2] A. G. Bacharoglou, “Universal Taylor series on doubly connected domains”, *Results Math.* 53 (2009), 9–18.
- [3] F. Bayart, “Universal Taylor series on general doubly connected domains”, *Bull. London Math. Soc.* 37 (2005), 878–884.
- [4] C. Chui and M. N. Parnes, “Approximation by overconvergence of power series”, *J. Math. Anal. Appl.* 36 (1971), 693–696.
- [5] G. Costakis, “Some remarks on universal functions and Taylor series”, *Math. Proc. Camb. Philos. Soc.* 128 (2000), 157–175.
- [6] G. Costakis, “Universal Taylor series on doubly connected domains in respect to every center”, *J. Approx. Theory* 134 (2005), 1–10.
- [7] G. Costakis and V. Vlachou, “Universal Taylor series on non-simply connected domains”, *Analysis* 26 (2006), 347–363.
- [8] S. J. Gardiner, “Superharmonic extension and harmonic approximation”, *Ann. Inst. Fourier (Grenoble)* 44 (1994), 65–91.
- [9] S. J. Gardiner, *Harmonic Approximation*, London Mathematical Society Lecture Notes, Cambridge Univ. Press, 1995.
- [10] S. J. Gardiner and N. Tsirivas, “Universal Taylor series for non-simply connected domains”, *C. R. Math. Acad. Sci. Paris* 348 (2010), 521–524.
- [11] W. Gehlen, “Overconvergent power series and conformal maps”, *J. Math. Anal. Appl.* 198 (1996), 490–505.
- [12] W. Gehlen, W. Luh and J. Müller, “On the existence of O-universal functions”, *Complex Variables Theory Appl.* 41 (2000), 81–90.
- [13] E. Hille, *Analytic Function Theory. Vol. II.* Ginn, Boston, 1962.
- [14] W. Luh, “Universal approximation properties of overconvergent power series on open sets”, *Analysis* 6 (1986), 191–207.
- [15] A. Melas, “Universal functions on nonsimply connected domains”, *Ann. Inst. Fourier (Grenoble)* 51 (2001), 1539–1551.
- [16] A. Melas and V. Nestoridis, “Universality of Taylor series as a generic property of holomorphic functions”, *Adv. Math.* 157 (2001), 138–176.

- [17] J. Müller, V. Vlachou and A. Yavrian, “Universal overconvergence and Ostrowski-gaps”, *Bull. London Math. Soc.* 38 (2006), 597–606.
- [18] J. Müller and A. Yavrian, “On polynomial sequences with restricted growth near infinity”, *Bull. London Math. Soc.* 34 (2002), 189–199.
- [19] V. Nestoridis, “Universal Taylor series”, *Ann. Inst. Fourier (Grenoble)* 46 (1996), 1293–1306.
- [20] V. Nestoridis, “An extension of the notion of universal Taylor series”, in *Computational Methods and Function Theory 1997 (Nicosia)*, pp.421–430, Ser. Approx. Compos., 11, World Sci. Publ., River Edge, NJ, 1999.
- [21] V. Nestoridis and C. Papachristodoulos, “Universal Taylor series on arbitrary planar domains”, *C. R. Math. Acad. Sci. Paris* 347 (2009), no. 7-8, 363–367.
- [22] T. Ransford, *Potential Theory in the Complex Plane*, Cambridge Univ. Press, 1995.
- [23] N. Tsirivas, “Universal Faber and Taylor series on an unbounded domain of infinite connectivity”, *Complex Var. Theory Appl.*, to appear.
- [24] N. Tsirivas and V. Vlachou, “Universal Faber series with Hadamard-Ostrowski gaps”, *Comput. Methods Funct. Theory* 10 (2010), 155–165.
- [25] V. Vlachou, “A universal Taylor series in the doubly connected domain $\mathbb{C} \setminus \{1\}$ ”, *Complex Var. Theory Appl.* 47 (2002), 123–129.
- [26] V. Vlachou, “Universal Taylor series on a non-simply connected domain and Hadamard-Ostrowski gaps”, in *Complex and Harmonic Analysis*, pp.221–229, DEStech Publ., Inc., Lancaster, PA, 2007.

School of Mathematical Sciences,
University College Dublin,
Belfield, Dublin 4, Ireland.

e-mail: stephen.gardiner@ucd.ie