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<th><strong>Title</strong></th>
<th>Bifurcation Scenarios in Electrostatic Vibration Energy Harvesters</th>
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<tr>
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</tr>
<tr>
<td><strong>Publication date</strong></td>
<td>2012-07</td>
</tr>
<tr>
<td><strong>Publication information</strong></td>
<td>Nonlinear Dynamics of Electronic Systems, Proceedings of NDES 2012</td>
</tr>
<tr>
<td><strong>Conference details</strong></td>
<td>Nonlinear Dynamics of Electronic Systems 2012 (NDES 2012), July 11-13, 2012, Wolfenbüttel, Germany</td>
</tr>
<tr>
<td><strong>Publisher</strong></td>
<td>IEEE</td>
</tr>
<tr>
<td><strong>Link to online version</strong></td>
<td><a href="http://ieeexplore.ieee.org/stamp/stamp.jsp?tp=&amp;arnumber=6292849&amp;isnumber=6289519">http://ieeexplore.ieee.org/stamp/stamp.jsp?tp=&amp;arnumber=6292849&amp;isnumber=6289519</a></td>
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<tr>
<td><strong>Item record/more information</strong></td>
<td><a href="http://hdl.handle.net/10197/3866">http://hdl.handle.net/10197/3866</a></td>
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<tr>
<td><strong>Publisher's statement</strong></td>
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Abstract—In this paper, we present numerical bifurcation analysis of an electrostatic vibration energy harvester operating in constant-charge mode and using the in-plane gap closing transducer. We show how the system can be represented as a nonlinear oscillator and analysed using methods of nonlinear dynamics. We verify previous analytical theories and explain the behaviour of these energy harvesters, particularly in the regime between the first period doubling bifurcation and chaos.

I. INTRODUCTION

Electrostatic (capacitive) vibration energy harvesters (e-VEHs) convert kinetic energy of the environment into electrical energy using a capacitive transducer [1], [2]. One of the limits of convertible power is set by the complexity and nonlinearity of the system, as demonstrated in [3]. In particular, the desired harmonic oscillations are possible only for certain values of the system parameters. Otherwise, the system can display chaotic behaviour and the conditioning electronics will not operate properly. For these reasons, optimal design of an e-VEH requires a deep understanding of the overall system dynamics, including nonlinear effects.

While the work [4] was first to introduce an analytical tool for analysis of an e-VEH as the coupled system, reference [5] studies steady-state oscillations in the harvester applying a formal analytical approach. This allows one to determine the characteristics of steady-state regimes (for instance, the amplitude of an oscillation) and therefore to predict the power that can be generated by the device.

As was later shown in [3], [5], the steady-state regime, once it appears, eventually undergoes a period doubling bifurcation. These two events (the onset of the oscillations and the doubling bifurcation) limit the region of effective device performance. Reference [5] used the characteristic multipliers [6] to predict the bifurcation and the largest Lyapunov exponents to determine the transition to irregular chaotic oscillations.

This paper expands the analysis of nonlinear phenomena found in [5]. It was suggested in the latter work that despite the fact that one observes a doubling bifurcation, the system does not undergo the full doubling cascade and possibly displays more complex behaviour in between the first bifurcation and the chaotic regime. The aim of this paper is to report the details of this scenario. The nonlinear equation that describes the behaviour of the e-VEH represents a specific class of nonlinear oscillators, and effects found in it can be common in other systems that consist of a mechanical resonator controlled by electronics. In addition, results of numerical simulations verify the analytical results that we reported in our previous works and highlight certain details of the dynamics of the system.

II. STATEMENT OF THE PROBLEM AND WORK TO DATE

A simple electrostatic harvester consists of a high-Q resonator, a variable capacitor (transducer) $C_{tran}$, and a conditioning circuit (Fig. 1). In [5] the normalised displacement $y$ of the mass-spring-damper system driven by the external oscillations and affected by the transducer force was described by the following dimensionless equation

$$y'' + 2\beta y' + y = A_{ext} \cos(\Omega t + \theta_0) + f_t(y, y')$$  \hspace{1cm} (1)$$

where the prime denotes the derivative with respect to dimensionless time $\tau$ and the function $f_t(y, y')$ is the transducer force:

$$f_t(y, y') = \begin{cases} \frac{\nu_0}{(1-y_{max})}, & y' \leq 0 \\ 0, & y' > 0 \end{cases}$$  \hspace{1cm} (2)$$

In order to reduce the number of parameters and outline only essential ones, the following normalised variables were introduced: time $\tau = \omega_0 t$, dissipation $\beta = b/(2m\omega_0)$, external frequency $\Omega = \omega_{ext}/\omega_0 = 1 + \sigma$, $y = x/d$, $\alpha = A_{ext}/(d\omega_0^2)$ and $\nu_0 = W_0/(d^2m\omega_0^3)$. Here $m$ is the mass of the resonator, $b$ is the damping factor, $\omega_0 = \sqrt{k/m}$ is the natural frequency, $k$ is the spring
By employing Floquet theory [6], we calculated the perturbations analysed by constructing an equation that describes small shifts in the Fourier series, the oscillation (3) has this constant displacement but dependent on the amplitude and \( a \). We have used the Fourier series for

\[
y_0(\tau) = y_{av,0} + a_0 \cos((1 + \sigma)\tau + \theta_0 - \psi_0) \quad (3)
\]

where \( a_0 \), \( y_{av,0} \) and \( \psi_0 \) are the steady-state amplitude, average displacement (constant shift) and phase of oscillations. The amplitude \( a_0 \) and the phase \( \psi_0 \) are found from the equations:

\[
\frac{\alpha^2}{4} = \left( \beta a_0 + \frac{b_1(y_{m,0})}{2} \right)^2 + \left( a_0 \sigma + \frac{a_1(y_{m,0})}{2} \right)^2
\]

\[
\frac{\alpha}{2} \sin \psi_0 = \beta a_0 + \frac{b_1}{2}, \quad \frac{\alpha}{2} \cos \psi_0 = -a_0 \sigma - \frac{a_1}{2}
\]

where \( y_{m,0} = y_{av,0} + a_0 \) is the maximal displacement. We have used the Fourier series for \( f(\tau) = f_0 + a_1 \cos \theta(\tau) + b_1 \sin \theta(\tau) \) for the transducer force where \( \theta(\tau) = (1 + \sigma)\tau + \theta_0 - \psi_0 \) and the functions \( f_0 \), \( a_1 \) and \( b_1 \) are the standard coefficients of the Fourier series but dependent on the amplitude \( a_0 \) and the average displacement \( y_{av,0} \). Note that due to the zero harmonic in the Fourier series, the oscillation (3) has this constant shift \( y_{av,0} = f_0 \).

For the gap closing transducer, the coefficients are

\[
f_0 = \frac{\nu_0}{2(1 - y_m)}, \quad a_1 = 0, \quad b_1 = \frac{2\nu_0}{\pi(1 - y_m)} \quad (5)
\]

In work [5], the obtained steady-state solution was analysed by constructing an equation that describes small perturbations \( \zeta(\tau) \) from the original periodic orbit \( y_0(\tau) \). The equation has the form

\[
\dddot{\zeta} + 2\beta \dot{\zeta} + F\zeta = 0, \quad (6)
\]

\[
F = (1 - f'_0(y_{m,0}) - b'_1(y_{m,0}) \sin(\Omega\tau + \theta_0 - \psi_0))
\]

and can be reduced to the well-known Mathieu equation. By employing Floquet theory [6], we calculated the bifurcation values of parameters and also indicated what type of bifurcation is undergone by the original orbit.

The plane shown in Fig. 2 summarises the results. Line 1 (blue) shows the conditions required to start steady-state harmonic oscillations in the resonator. Below this line one observes irregular small scale displacements of the resonator with many maxima detected in one period while above this line lies the area of (possible) steady-state oscillation. Line 2 (black) shows the doubling bifurcation and bounds the area of steady-state oscillations with period 2. Finally, line 3 indicates when the largest Lyapunov exponent becomes positive and the system displays chaotic oscillations.

### III. RESULTS

To plot the bifurcation diagrams below, eq. (1) was solved numerically and a Poincaré section was taken through the resulting phase portrait, across multiple values of \( A_{ext} \). The Poincaré section was established at the point where \( dy \) changed from positive to negative, i.e. at the maximum displacement.

**A. \( W_0 = 10 \text{ nJ} \)**

Figure 3 plots the bifurcation diagram for \( W_0 = 10 \text{ nJ} \) where \( A_{ext} \) is the bifurcation parameter. It was found to agree closely with the analytical theory in [5], particularly in relation to the size of the steady-state and period doubling regions and the amplitude of the steady-state oscillations. At least two doubling bifurcation are undergone and a 4T-orbit is observed (corresponding trajectory in the state space is shown in fig. 4(a)). A variety of orbits of different complexity is observed at increase of \( A_{ext} \) until a chaotic oscillations are eventually established (see the rest of fig. 4).

**B. \( W_0 = 20 \text{ nJ} \)**

Figure 5 plots the bifurcation diagram for \( W_0 = 20 \text{ nJ} \) and the corresponding plot of the largest Lyapunov

<table>
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<th>TABLE I</th>
<th>PARAMETERS OF THE SYSTEM</th>
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<tr>
<td>m</td>
<td>200 \cdot 10^{-6} kg</td>
</tr>
<tr>
<td>b</td>
<td>\sqrt{2} \cdot 10^{-3} Nm^{-1}</td>
</tr>
<tr>
<td>k</td>
<td>300 Nm \cdot m^{-1}</td>
</tr>
<tr>
<td>d</td>
<td>20 \cdot 10^{-3} m</td>
</tr>
<tr>
<td>S</td>
<td>10 \cdot 10^{-3} m^2</td>
</tr>
<tr>
<td>( W_0 )</td>
<td>0.5 - 3 \cdot 10^{-3} J</td>
</tr>
<tr>
<td>( A_{ext} )</td>
<td>1 - 10 ms^{-2}</td>
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Fig. 2. Plane of parameter \((W_0, A_{ext})\) obtained in [5] where the different areas correspond to different regimes displayed by the system. Line 1 is shows the onset (necessary condition) of steady-state oscillations, line 2 is the doubling bifurcation and line 3 is transition to chaos.
Equation (3) describes the oscillations well in this area. After the period-1 regime oscillations (see Fig. 7(a)) period double, where appear a bubble in the bifurcation diagram that represents a quasi periodic regime(Fig. 7(b)). The corresponding quasi-periodic orbit resembles the period-2 oscillations, however, in a longer time scale one can see that the trajectory does not form the closed orbit in the state space. The largest Lyapunov exponential is very close to zero in the area of $A_{ext}$ corresponding to this bubble which evidence quasi-periodicity of the orbit. On the contrary to $W_0 = 10 \text{ nJ}$, for this values of $W_0$, the doubling bifurcation scenario ‘collapses’: further increase of $A_{ext}$ leads to quasi-periodic oscillations (Fig. 7(c)). The quasi-period motion (Fig. 7(c)) is destructed and chaotic regimes

The magnitude of the steady-state oscillations and the value of $A_{ext}$ at which bifurcation occurs are again as predicted in [5]. An unexpected feature, however, is the lower branch evident in Fig. 5(b) but not Fig. 3, due to the appearance of ‘notches’ in the oscillations. The presence of the lower branch, between $A_{ext} = 3 \text{ m/s}^2$ and $4 \text{ m/s}^2$ in Fig. 5(a), reduces the region of the desired harmonic oscillations. (Indeed, in the latter reference, the boundary line 1 from Fig. 2 is called the necessary condition.) This in turn reduces the range of values of $A_{ext}$ for which the e-VEH will operate efficiently and is due to assumptions that had to be made as part of the analysis in [5]. For $A_{ext} = 3.5 \text{ m/s}^2$ the velocity is not sinusoidal with numerous maxima, and minima, around the $\dot{y} = 0$ axis.

The reason that we cannot operate the e-VEH for multiple maxima is due to the inefficiency of the e-VEH caused by the use of switches in the conditioning circuitry in Fig. 1. The desired harmonic oscillations are represented by the plots shown in Fig. 7(a) at $A_{ext} = 5.5 \text{ m/s}^2$. As a consequence, the region in which the harvester can collect energy effectively is reduced.
Fig. 7. Trajectories in the state space and spectra that correspond to different points in the bifurcation diagram in fig. 5. (a) $A_{ext} = 5.5$, (b) $A_{ext} = 9.1$, (c) $A_{ext} = 9.55$ and (d) $A_{ext} = 9.7 \text{ m/s}^2$ for $W_0 = 10 \text{ nJ}$.

appear (Fig. 7(d)).

The spectra of the displacement are shown for every waveform in Fig. 7, and one can see the main driving frequency ($\Omega = 1 + \sigma \sim 1.03$) in each of them. Typical for a doubling bifurcation, the two subharmonic satellites appear with respect to $\Omega$. However note the asymmetry in the spectrum: new harmonics tend to appear only in the area of lower frequencies is seen from fig. 7(d).

C. $W_0 = 30 \text{ nJ}$

In the $W_0 = 30 \text{ nJ}$ bifurcation diagram, Fig. 8, the region of steady-state oscillations has been significantly reduced by the lower branch that tends to join with the similar lower branch starting from $A_{ext} \approx 9.6 \text{ m/s}^2$. Practically, for large $W_0$ where will be no area of $A_{ext}$ with stable harmonic oscillations.

Chaotic oscillations develop after the destruction of quasi-periodic regime. Figure 9 shows a phase portraits and a spectrum of an quasi-periodic oscillations example appeared at $A_{ext} = 11.08 \text{ m/s}^2$.

IV. CONCLUSIONS

The numerical results shown here support the analytical theory put forward in [5]. The few discrepancies are primarily due to assumptions made in [5]. The results in this paper provide a better insight into the bifurcation scenario in an electrostatic vibration energy harvester, for different values of $W_0$.

ACKNOWLEDGMENTS

This work was supported by Science Foundation Ireland.

REFERENCES