A generalization of universal Taylor series in simply connected domains

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Abstract: Let $\Omega$ be a simply connected proper subdomain of the complex plane and $z_0$ be a point in $\Omega$. It is known that there are holomorphic functions $f$ on $\Omega$ for which the partial sums $(S_n(f, z_0))$ of the Taylor series about $z_0$ have universal approximation properties outside $\Omega$. In this paper we investigate what can be said for the sequence $(\beta_n S_n(f, z_0))$ when $(\beta_n)$ is a sequence of non-zero complex numbers. We also study a related analogue of a classical Theorem of Seleznev concerning the case where the radius of convergence of the universal power series is zero.


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1. Introduction

We begin with the abstract definition of universality [6].

Definition 1.1. Let $(X, T_X)$, $(Y, T_Y)$ be topological vector spaces over a field $\mathbb{K}$ and $T_n : X \to Y$, $n = 1, 2, \ldots$ be a sequence of continuous linear operators. We say that the sequence $(T_n)$ is universal if there exists some $x \in X$ such that $Y = \bigcup_{n} T_n(x)$. Any $x \in X$ with the above property is called a universal vector of $X$ with respect to $(T_n)$ and we denote the set of universal vectors of the space $X$

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with respect to \((T_n)\) by

\[
U(T_n) := \left\{ x \in X \mid Y = \bigcup_n T_n(x) \right\}.
\]

In the case where \(X = Y\) we call the sequence \((T_n)\) hypercyclic.

Let \((X, T_X)\) be a topological vector space and \((Y_i, T_{Y_i}), i \in I\), be a family of topological vector spaces over \(\mathbb{K}\). For every \(i \in I\) let \(T_n^i : X \to Y_i, n = 1, 2, \ldots\) be a sequence of continuous linear operators. Also let \((\beta_n)\) be a sequence of complex numbers. Let \(U(\beta_n T_n^i), i \in I\) be the sets of universal vectors in \(X\) with respect to the families \((\beta_n T_n^i), i \in I\), as in Definition 1.1. The question is whether the families \((\beta_n T_n^i), i \in I\) share a common universal vector, that is, if \(\bigcap_i U(\beta_n T_n^i) \neq \emptyset\).

This subject is closely related to the notion of Cesàro hypercyclicity [5]. Below we formulate an important particular case of this question and then describe its complete solution.

Let \(\Omega\) be a simply connected proper subdomain of \(\mathbb{C}\), let \(H(\Omega)\) denote the space of holomorphic functions on \(\Omega\), and let \(z_0 \in \Omega\) be fixed. We endow \(H(\Omega)\) with the topology \(T_u\) of uniform convergence on compact subsets of \(\Omega\). Let \(f \in H(\Omega)\). We denote by \(S_n(f, z_0)\) the \(n\)-th partial sum of the Taylor development of \(f\) about \(z_0\); that is,

\[
S_n(f, z_0)(z) = \sum_{k=0}^{n} \frac{f^{(k)}(z_0)}{k!} (z - z_0)^k, \quad n = 0, 1, 2, \ldots, \quad z \in \mathbb{C}.
\]

Where no misunderstanding can arise, we write \(S_n\) instead of \(S_n(f, z_0)\).

Let \(\mathcal{M}_{\Omega^c}\) be the collection of compact subsets of \(\Omega^c\) with connected complement. For every \(K \in \mathcal{M}_{\Omega^c}\) we consider the space \(A(K)\) of continuous functions on \(K\) that are holomorphic in \(K^0\), endowed with the supremum norm, which is a Banach Algebra. Let \(\beta = (\beta_n)_{n \in \mathbb{N}_0}\) be a sequence in \(\mathbb{C}\setminus\{0\}\). For each \(K \in \mathcal{M}_{\Omega^c}\) we consider the sequence of continuous linear operators \(S_n^K : H(\Omega) \to A(K)\), where

\[
S_n^K(f)(z) = S_n(f, z_0)(z) \quad \text{for every} \quad f \in H(\Omega), \quad z \in K, \quad n = 0, 1, 2, \ldots.
\]

Now we apply the above terminology after Definition 1.1 of universality with \(X := H(\Omega), I := \mathcal{M}_{\Omega^c}, Y^K := A(K)\) for every \(K \in \mathcal{M}_{\Omega^c}\) and \(T_n^K := S_n^K\) for every \(K \in \mathcal{M}_{\Omega^c}\) and \(n = 0, 1, 2, \ldots\). We define

\[
U(\Omega, z_0, \beta) := \bigcap_{K \in \mathcal{M}_{\Omega^c}} U(\beta_n T_n^K).
\]
Thus a holomorphic function $f$ on $\Omega$ belongs to $U(\Omega, z_0, \beta)$ if, for each $K \in \mathcal{M}_\Omega$, and $h \in A(K)$, there is a sequence $\lambda = (\lambda_n)$ of natural numbers such that $\beta_{\lambda_n} S^K_{\lambda_n}(f) \to h$ as $n \to \infty$ uniformly on $K$.

Our main aim in this paper is to completely characterize the sequences $\beta$ for which $U(\Omega, z_0, \beta) \neq \emptyset$. The solution to this problem is given below.

**Theorem 1.2.** The set $U(\Omega, z_0, \beta)$ is non-empty if and only if $(\sqrt[n]{|\beta_n|})$ has 1 as a limit point. In this case $U(\Omega, z_0, \beta)$ is a $G_\delta$ dense subset of $\mathcal{H}(\Omega)$ that contains a dense vector subspace of $\mathcal{H}(\Omega)$ except 0. 

2. Proof of Theorem 1.2

The conclusion of Theorem 1.2 follows easily from known results if $(\beta_n)$ has a finite non-zero limit ([8], [10], [11]), or if $(\beta_n)$ has a finite non-zero limit point (see [2, page 420 Theorem 1]).

When these are not the cases new arguments are required. We use the following lemma ([8], [10]).

**Lemma 2.1.** There is a sequence $(K_n)_{n \in \mathbb{N}}$ in $\mathcal{M}_\Omega$ such that, for every $K \in \mathcal{M}_\Omega$, there exists $m \in \mathbb{N}$ such that $K \subset K_m$.

The space $(\mathcal{H}(\Omega), T_u)$ is a complete metric space, so Baire’s Category theorem is at our disposal. We will write $U(\Omega, z_0, \beta)$ in the form $\bigcap_n V_n$, where the sets $V_n$ are open and dense in $\mathcal{H}(\Omega)$. Now we describe the sets $V_n$.

Let $(f_j)_{j \geq 1}$ be an enumeration of all polynomials of one complex variable with coefficients in $\mathbb{Q} + i\mathbb{Q}$, where $\mathbb{Q}$ is the set of rational numbers. Let $(K_n)_{n \in \mathbb{N}}$ be a sequence of compact sets as in Lemma 2.1. Now for each $j, s, m \in \mathbb{N}$ and $n \in \mathbb{N}_0$ we consider the set:

$$E(m, j, s, n) := \left\{ f \in \mathcal{H}(\Omega) \left| \beta_n S_n(f) - f_j \right| < \frac{1}{s} \text{ on } K_m \right\}.$$ 

**Lemma 2.2.** With the above notation,

$$U(\Omega, z_0, \beta) = \bigcap_m \bigcap_{j=1}^{\infty} \bigcap_{s=1}^{\infty} \bigcup_{n=0}^{\infty} E(m, j, s, n).$$

**Lemma 2.3.** For each $m, j, s \in \mathbb{N}$ and $n \in \mathbb{N}_0$ the set $E(m, j, s, n)$ is open in the space $(\mathcal{H}(\Omega), T_u)$. 

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The above two lemmas hold without any restriction on the sequence $\beta$. They can be proved by following the arguments in Lemma 2.4 and Proposition 2.5 of [7], and using Mergelyan’s Theorem [13]. We now assume that 1 is a limit point of $(\sqrt[n]{|\beta_n|})$.

**Lemma 2.4.** Suppose that 1 is a limit point of $(\sqrt[n]{|\beta_n|})$. For each $m, j, s \in \mathbb{N}$, the set $\bigcup_{n=0}^{\infty} E(m, j, s, n)$ is dense in $(\mathcal{H}(\Omega), T_u)$.

**Proof.** In view of the known cases, and the fact that we need only work with a subsequence of $(\sqrt[n]{|\beta_n|})$, it enough to consider what happens when $|\beta_n|$ tends to infinity, or when $\beta_n$ tends to zero where $\beta_n$ is different to zero for each $n$.

Case a) $\lim \limits_{n \to \infty} |\beta_n| = +\infty$

Let $m_0, j_0, s_0 \in \mathbb{N}$, let $p_0$ be a polynomial, let $\varepsilon_0 > 0$ and $L \subseteq \Omega$ be a compact set. It suffices to find $N_0 \in \mathbb{N}$, and a holomorphic function $f \in \mathcal{H}(\Omega)$, such that

$$|f - p_0| < \varepsilon_0 \text{ on } L, \text{ and } \left| \beta_{N_0} S_{N_0}(f) - f_{j_0} \right| < \frac{1}{s_0} \text{ on } K_{m_0}. \quad (*)$$

Because $\Omega$ is a simply connected domain we can find connected compact sets $C_1, C_2$ that have connected complements and boundaries that are simple smooth loops (see [4, p. 24]), disjoint open sets $G_1, G_2$ and simple smooth loops $\gamma_1, \gamma_2$ such that

$$L \subset \mathring{C}_1 \subset C_1 \subset \text{Int}(\gamma_1) \subset \overline{\text{Int}(\gamma_1)} \subset G_1, \quad K_{m_0} \subset \mathring{C}_2 \subset C_2 \subset \text{Int}(\gamma_2) \subset \overline{\text{Int}(\gamma_2)} \subset G_2.$$

Here Int$(\gamma_1)$ denotes the interior of the curve $\gamma_1$ as usual, and we can further arrange that Ind$(\gamma_1(C_1)) = 1$, Ind$(\gamma_1(C_2)) = 0$, Ind$(\gamma_2(C_1)) = 0$, and Ind$(\gamma_2(C_2)) = 1$ [4, Exercise 10.10].

Now let $m \in \mathbb{N}$ and let $F_m : G_1 \cup G_2 \to \mathbb{C}$ be defined by

$$F_m(z) := \begin{cases} p_0(z) & \text{if } z \in G_1 \\ \frac{1}{\beta(m)} f_{j_0}(z) & \text{if } z \in G_2. \end{cases}$$

Also, let $n \in \mathbb{N}$, where $n \geq 2$ and let $q_n$ be a Fekete polynomial of degree at most $n$ for the set $C_3 := C_1 \cup C_2$, (see [12, Definition 5.5.3]). We define the function $p_n(m) : C_3 \to \mathbb{C}$ defined by the formula

$$p_n(m)(w) := \frac{1}{2\pi i} \int_{\gamma_1} \frac{F_m(z)}{q_n(z)} \cdot \frac{q_n(w) - q_n(z)}{w - z} \, dz + \frac{1}{2\pi i} \int_{\gamma_2} \frac{F_m(z)}{q_n(z)} \cdot \frac{q_n(w) - q_n(z)}{w - z} \, dz, \quad w \in C_3.$$
Clearly, $p_n(m)$ is a sum of two polynomials of degree at most $n - 1$. By the global Cauchy Theorem

$$F_m(w) - p_n(m)(w) = \frac{1}{2\pi i} \int_{\gamma_1} \frac{F_m(z)}{z - w} \cdot \frac{q_n(w)}{q_n(z)} \, dz + \frac{1}{2\pi i} \int_{\gamma_2} \frac{F_m(z)}{z - w} \cdot \frac{q_n(w)}{q_n(z)} \, dz, \quad w \in C_3. \quad (1)$$

Using (1) and the fact that the sequence $1/\beta_n$ is bounded we can find a constant $M_0 > 0$, independent of $n, m$, such that

$$\|F_m - p_n(m)\|_{C_3} < M_0 \cdot \inf_{\gamma_1 \cup \gamma_2} |q_n|, \quad n, m \in \mathbb{N}, \quad n \geq 2. \quad (2)$$

Let $G := C_3^c$. The set $\partial G$ is non-polar, so $G$ possesses a Green function $g_G$. Let $r_G$ be the Harnack distance for $G$ [12, Definition 1.3.4], $c(C_3)$ be the logarithmic capacity of $C_3$ and $\delta_n(C_3)$ be the $n$-th diameter of $C_3$ [12, Definition 5.1.1], for $n \geq 2$. By Bernstein’s Lemma [12, Theorem 5.5.7 (b)],

$$\left( \left| \frac{q_n(z)}{\|q_n\|_{C_3}} \right| \right)^{1/n} \geq e^{g_G(z, \infty)} \cdot \left( \frac{c(C_3)}{\delta_n(C_3)} \right)^{r_G(z, \infty)} \quad (3)$$

for $n \geq 2, z \in G$.

By the Fekete-Szegö Theorem [12, Theorem 5.5.2] we have

$$\lim_{n \to \infty} \delta_n(C_3) = c(C_3). \quad (4)$$

Applying (3) to the curves $\gamma_1$ and $\gamma_2$ and using (2) and (4) we can find $\theta_1 \in (0, 1)$ and $\nu_0 \in \mathbb{N}$ such that

$$\|F_m - p_n(m)\|_{C_3}^{1/n} < \theta_1, \quad n, m \in \mathbb{N}, \quad n \geq \nu_0. \quad (5)$$

Using the fact that $L \cup K_{m_0} \subset C_3$, the definition of $F_m$, the condition $\sqrt[n]{|\beta(n)|} \to 1$ and (5), we can find a natural number $N_0$ such that

$$\left| p_{N_0}(N_0) - p_0 \right| < \varepsilon_0 \quad \text{on} \quad L \quad (6)$$

and

$$\left| \beta(N_0)p_{N_0}(N_0) - f_{j_0} \right| < \frac{1}{s_0} \quad \text{on} \quad K_{m_0}. \quad (7)$$

We set $f := p_{N_0}(N_0)$. Then $f \in H(\Omega)$ and $f = S_{N_0}(f)$ because $f$ is a polynomial of degree at most $N_0 - 1$. Thus we have proved the desired inequalities in $(\ast)$. 

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Case b) $\lim_{n \to \infty} \beta_n = 0$

The proof is almost the same as in case a). The only change is to replace the constant $M_0$ in (2) by $(1 + \delta)^m$, where $\delta > 0$ is arbitrarily small and $m = n$ is sufficiently large (depending on $\delta$). ■

Now using Lemmas 2.2, 2.3 and 2.4, Baire’s Category Theorem and the completeness of the metric space $(\mathcal{H}(\Omega), T_u)$, we conclude that $U(\Omega, z_0, \beta)$ is a $G_\delta$ dense subset of $(\mathcal{H}(\Omega), T_u)$.

Now we suppose that \( n \sqrt{|\beta_n|} \to 1 \) and $\beta_n \neq 0 \ \forall \ n \in \mathbb{N}$. The above proof gives us that for every subsequence $(\beta \circ \mu)$ of $(\beta)$ the set $U(\Omega, z_0, \beta \circ \mu)$ is a $G_\delta$ dense subset of $(\mathcal{H}(\Omega), T_u)$. Now as in the implication (v)$\Rightarrow$(vi) of Theorem 4.2 of [7] (see also [2], [3]), we see that the set $U(\Omega, z_0, \beta)$ contains a dense vector subspace of $\mathcal{H}(\Omega)$ except 0. Passing to a subsequence of $\beta$, the same holds when 1 is a limit point of $\left(\sqrt{n} |\beta_n|\right)$. By the above we have completed the positive cases of Theorem 1.2

Now we examine the negative cases of Theorem 1.2.

**Proposition 2.5.** If the number 1 is not a limit point of the sequence $\left(\sqrt{|\beta_n|}\right)$, then $U(\Omega, z_0, \phi) = \emptyset$.

**Proof.** We distinguish three cases.

**First case:** $\limsup_{n \to \infty} \sqrt{|\beta_n|} < 1$.

We fix $a \in \left(1, 1/\limsup_{n \to \infty} \sqrt{|\beta_n|}\right)$ if $\limsup_{n \to \infty} \sqrt{|\beta_n|} \neq 0$, or else choose an arbitrary number $a > 1$. There exists $n_0 \in \mathbb{N}$ such that

$$|\beta_n| < \frac{1}{a^n}, \quad n \geq n_0. \quad (1)$$

Let $d := \text{dist}(z_0, \Omega^c) := \inf\{|z - z_0| : z \in \Omega^c\}$ and $\varepsilon_0 \in \left(0, d \cdot \frac{a - 1}{a + 1}\right)$. Let $K \subset \Omega^c$ be a compact set with connected complement such that

$$\max\{|z - z_0| : z \in K\} \leq d + \varepsilon_0.$$ 

Let $f \in \mathcal{H}(\Omega)$, and $(a_n)$ be the Taylor coefficients of $f$ about $z_0$. If $R$ is the associated radius of convergence, then $R \geq d$. Thus

$$\sum_{k=0}^{\infty} |a_k| (d - \varepsilon_0)^k = A \in [0, +\infty). \quad (2)$$
From (1) and (2) we see easily that
\[ |\beta_n S_n(f, z_0)| \leq \left( \frac{d + \varepsilon_0}{d - \varepsilon_0} \cdot \frac{1}{a} \right)^n \cdot A \quad \text{on } K \quad \text{for all } n \geq n_0, \]
whence
\[ \sup_{z \in K} |\beta(n) S_n(f, z_0)(z)| \to 0 \quad (3) \]
as \( n \to \infty \) because \( \frac{d + \varepsilon_0}{d - \varepsilon_0} \cdot \frac{1}{a} \in (0, 1) \). The convergence in (3) shows that the arbitrary function \( f \in H(\Omega) \) cannot be universal and the result now follows.

**Second case:** \( \lim \inf_{n \to \infty} \sqrt[\nu]{|\beta_n|} > 1 \).

For this proof we use Theorem 1 of [9]. Since \( \Omega \) is a simply connected domain we can find a compact connected set \( \Gamma \) containing more than one point such that \( \Gamma^c \) is connected, \( \Gamma \subset \Omega^c \), and \( \text{dist}(\Gamma, z_0) = \text{dist}(z_0, \Omega^c) \).

If \( \Omega \) is unbounded we consider a sequence \( (K_n)_{n \in \mathbb{N}} \), \( K_n \in \mathcal{M}_{\Omega^c} \) as in Lemma 2.1 such that \( K_n \subseteq K_{n+1} \) for each \( n \) and \( \Gamma \subset K_1 \). In this case we set \( E = \Omega^c \).

If \( \Omega \) is bounded we choose \( N_0 \in \mathbb{N} \) such that \( \Omega \cup \Gamma \subset D(0, N_0) \), and put \( K_n := \Gamma \cup [N_0 + 1, N_0 + 1 + n] \). In this case we set \( E = \bigcup_{n=1}^{\infty} K_n = \Gamma \cup [N_0 + 1, +\infty) \). In each of these cases the set \( E \) is closed and non-thin at \( \infty \). (For the definition of thinness see [1] or [12]).

The proof of this case is similar to that in Proposition 3.7, so it is omitted.

**Third case:** \( \lim \inf_{n \to \infty} \sqrt[\nu]{|\beta(n)|} < 1 < \lim \sup_{n \to \infty} \sqrt[\nu]{|\beta(n)|} \).

We consider the same sets \( \Gamma, E \) and the same sequence of compact sets \( (K_n)_{n \in \mathbb{N}} \) as in the previous case, where \( \Gamma \subset K_1 \), and suppose that \( \mathcal{U}(\Omega, z_0, \beta) \neq \emptyset \) for the sake of contradiction.

Let \( f \in \mathcal{U}(\Omega, z_0, \beta) \). Then we can find a strictly increasing sequence \( \lambda = (\lambda_n)_{n \in \mathbb{N}} \) of natural numbers such that
\[ \sup_{z \in K_n} \left| \beta_{\lambda_n} S_{\lambda_n}(f)(z) - 1 \right| < \frac{1}{n}, \quad n \in \mathbb{N}. \quad (4) \]
It is easy to see that the sequence \( (|\beta_{\lambda_n}|), n = 1, 2, \ldots \) has only two possible limit points, namely 0 and \( +\infty \).

Suppose that 0 is a limit point. Let \( w_0 \in \Gamma \) be such that \( |w_0 - z_0| = \text{dist}(z_0, \Omega^c) \). Then, as in the proof of the second case, we see that
\[ |\beta_{\mu_n} S_{\mu_n}(f, z_0)(w_0)| \to 0 \quad \text{as } n \to \infty, \]
which is a contradiction. Thus the only limit
point of $|β_n|$ is $+∞$. By our assumptions on $β$ we have $\liminf_{n→∞} \lambda \sqrt{|β_n|} > 1$. There exists some $θ_0 ∈ (0, 1)$ and $ν_0 ∈ N$ such that

$$\|S_{λ_n}(f)\|^{1/λ_n}_{K_n} < θ_0, \quad n ≥ ν_0. \quad (5)$$

We can now use the argument in the first case, following (5), to obtain again a contradiction. ■

3. A Theorem of Seleznev

A result of Seleznev [14] gives the first example of a universal Taylor series in the complex plane with radius of convergence zero. A recent extension of it (Theorem 6.2 of [7]) corresponds, roughly speaking, to our Theorem 1.2 in the case where the universal Taylor series have radius of convergence zero. In this paragraph we preserve the original terminology of [7]. Thus we consider a sequence of non-zero complex numbers $(φ_n)$, where $1/φ_n$ will play the same role as $β_n$ did earlier. However, [7] dealt only with the case where $\limsup_{n→∞} |φ_n| > 0$. In this section we will address the case where $\lim_{n→∞} φ_n = 0$, to complete the result.

Of course, the condition $\lim_{n→∞} φ_n = 0$ implies that $\limsup_{n→∞} n \sqrt{|φ_n|} ≤ 1$. We will show that, if $\lim_{n→∞} φ_n = 0$, then the conclusion of Theorem 6.2 of [7] holds when $\limsup_{n→∞} n \sqrt{|φ_n|} = 1$ but fails when $\limsup_{n→∞} n \sqrt{|φ_n|} < 1$. We write $M_{[0]e}$ for the collection of compact subsets of $C\setminus\{0\}$ with connected complement, and consider the set $U(φ)$ of sequences of $C^{N_0}$ such that for all $(K, f) ∈ M_{[0]e} × A(K)$

$$∃ λ = (λ_n)_{n∈N} a sequence of natural numbers so that \frac{1}{φ(λ_n)} \sum_{j=0}^{λ_n} a_j z^j → f \text{ uniformly on } K \text{ as } n→∞.$$  

Now we consider the space $C^{N_0}$ endowed with the Cartesian topology that is induced by the metric $ρ : C^{N_0} × C^{N_0} → \mathbb{R}^+ \text{ with } ρ(a, b) = \sum_{i=0}^{∞} \frac{1}{2^i} \frac{|a_i - b_i|}{1 + |a_i - b_i|}, (a, b) ∈ (C^{N_0})^2$. We write $(C^{N_0}, T_c)$ for the above space.

**Theorem 3.1.** Suppose that $n→∞ \lim φ_n = 0$. Then the set $U(φ)$ is non-empty if and only if the number 1 is a limit point of $n→∞ \sqrt{|φ(n)|}$ or, equivalently $\limsup_{n→∞} n \sqrt{|φ(n)|} = 1$. In the case where $U(φ)$ is non-empty it is also $G_δ$ dense in $(C^{N_0}, T_c)$ and contains a dense vector subspace of $(C^{N_0}, T_c)$ except 0.
Firstly, we prove the following:

**Proposition 3.2.** Let $\phi$ be a sequence such that $\lim_{n \to \infty} \phi(n) = 0$ and $\lim_{n \to \infty} \sqrt[n]{|\phi(n)|} = 1$. Then the set $\mathcal{U}(\phi)$ is a $G_\delta$-dense subset of $(\mathbb{C}^{\mathbb{N}_0}, \mathcal{T}_c)$.

**Proof.** Let $(K_n)_{n \in \mathbb{N}}$ be a sequence of compact subsets of $\mathbb{C} \setminus \{0\}$ as in Lemma 2.1. Let $f_j$, $j = 1, 2, \ldots$ be an enumeration of all polynomials of one complex variable with coefficients in $\mathbb{Q} + i\mathbb{Q}$. For every $m, j, s \in \mathbb{N}$ and $n \in \mathbb{N}_0$ let

$$\tilde{E}(m, j, s, n) := \left\{ a = (a_0, a_1, \ldots) \in \mathbb{C}^{\mathbb{N}_0} \middle| \sup_{z \in K_m} \left| \frac{1}{\phi(n)} \sum_{i=0}^{n} a_i z^i - f_j(z) \right| < \frac{1}{s} \right\}.$$

We will need the following results.

**Lemma 3.3.** For every $m, j, s \in \mathbb{N}$ and $n \in \mathbb{N}_0$ the set $\tilde{E}(m, j, s, n)$ is open in $(\mathbb{C}^{\mathbb{N}_0}, \mathcal{T}_c)$.

**Lemma 3.4.** With the above notation,

$$\mathcal{U}(\phi) = \bigcap_{m=1}^{\infty} \bigcap_{j=1}^{\infty} \bigcap_{s=1}^{\infty} \bigcup_{n=0}^{\infty} \tilde{E}(m, j, s, n).$$

The proofs of the above two lemmas are similar to those of Lemma 2.4 and Proposition 2.5 of [7] and are omitted.

**Proposition 3.5.** For every $m, j, s \in \mathbb{N}$ and $n \in \mathbb{N}_0$ the set $\bigcup_{n=0}^{\infty} \tilde{E}(m, j, s, n)$ is dense in $(\mathbb{C}^{\mathbb{N}_0}, \mathcal{T}_c)$.

**Proof.** We fix $m_0, j_0, s_0 \in \mathbb{N}$, and prove that the set $A = \bigcup_{n=0}^{\infty} \tilde{E}(m_0, j_0, s_0, n)$ is dense in $(\mathbb{C}^{\mathbb{N}_0}, \mathcal{T}_c)$. We know that the set $c_{00}$ is dense in $(\mathbb{C}^{\mathbb{N}_0}, \mathcal{T}_c)$. It suffices to prove that $S(a, \varepsilon) \cap A \neq \emptyset$ for every $a \in c_{00}$ and $\varepsilon > 0$ where $S(a, \varepsilon) := \{ x \in \mathbb{C}^{\mathbb{N}_0} | \rho(a, x) < \varepsilon \}$.

So let $a = (a_0, a_1, a_2, \ldots, a_\nu, 0, 0, \ldots) \in c_{00}$ where $a_\nu = 0$ for some fixed $\nu \geq \nu_0 + 1$. Also let $\varepsilon_0 > 0$. We will prove that $S(a, \varepsilon_0) \cap A \neq \emptyset$.

This means that we need to find some sequence $b = (b_0, b_1, \ldots, b_n, \ldots) \in \mathbb{C}^{\mathbb{N}_0}$ and a natural number $N_0 \geq 1$ such that:

$$\rho(a, b) < \varepsilon_0 \quad (1)$$

and

$$\sup_{z \in K_{m_0}} \left| \frac{1}{\phi(N_0)} \sum_{i=0}^{N_0} b_i z^i - f_{j_0}(z) \right| < \frac{1}{s_0} \quad (2)$$
Let \( k_0 \in \mathbb{N} \) such that \( k_0 > \max\{2, \nu_0\} \) and
\[
\sum_{i=k_0+1}^{\infty} \frac{1}{2^i} < \varepsilon_0. \tag{3}
\]
We can arrange that the set \( K_{m_0} \) is also connected and has a rectifiable curve as its boundary.

Now we can find a bounded simply connected domain \( W \subseteq \mathbb{C} \setminus \{0\} \) and a smooth simple loop \( \gamma \) such that \( K_{m_0} \subset W, \gamma \subset W, \gamma \cap K_{m_0} = \emptyset \) and \( \text{Ind}\gamma(K_{m_0}) = 1 \). So we have \( K_{m_0} \subset \text{Int}(\gamma) \subset W \subset \mathbb{C} \setminus \{0\} \).

Let \( p(z) = a_0 + a_1 z + \cdots + a_{\nu_0} z^{\nu_0} \), and let \( m \in \mathbb{N} \). We consider the holomorphic function \( F_m : W \to \mathbb{C} \), defined by
\[
F_m(z) = \frac{1}{z^{k_0+1}}(\phi(m)f_{j_0}(z) - p(z)), \quad z \in W.
\]

Applying a similar proof as Lemma 2.4 previously we have that
\[
\|F_m - P_n(m)\|_{K_{m_0}} < \theta_0^n \forall n, m \in \mathbb{N}, \ n \geq \lambda_0 \tag{4}
\]
where \( \theta_0 \in (0, 1) \) and \( \lambda_0 \in \mathbb{N} \) are fixed numbers.

Then (4) holds for every \( n, m \in \mathbb{N}, n \geq \lambda_0 \).

We apply (4) for every \( m \in \mathbb{N}, m > \lambda_0 + k_0, n = m - k_0 \). We see easily from (4) that there exists a constant \( C_1 \) such that
\[
\sup_{z \in K_{m_0}} \left| \frac{1}{\phi(m)}(p(z) + z^{k_0+1}p_n(m)(z)) - f_{j_0}(z) \right| < C_1 \cdot \frac{\theta_0^n}{|\phi(m)|}
\]
for every \( n, m \in \mathbb{N}, n = m - k_0, m > \lambda_0 + k_0 \). \( \tag{5} \)

Let \( \delta_0 \in \left( 0, \frac{1}{\theta_0} - 1 \right) \). Because \( \sqrt[4]{|\phi(n)|} \to 1 \) we can find \( n_1 \in \mathbb{N} \) such that
\[
\frac{1}{|\phi(n)|} < (1 + \delta_0)^n, \quad n \geq n_1. \tag{6}
\]

By (5) and (6) we have that:
\[
\sup_{z \in K_{m_0}} \left| \frac{1}{\phi(m)}(p(z) + z^{k_0+1}p_n(m)(z)) - f_{j_0}(z) \right| < C_1 \cdot (\theta_0(1 + \delta_0))^m
\]
\forall n, m \in \mathbb{N}, \ m > \max\{\lambda_0 + k_0, n_1\}, \ n = m - k_0. \tag{7}
Now because $\theta(1 + \delta) \in (0, 1)$ by (7) we can find a natural number $N_0 > \max\{\lambda_0 + k_0, n_1\}$ such that
\[
\sup_{z \in K_m} \left| \frac{1}{\phi(N_0)} (p(z) + z^{k_0 + 1}p_{N_1}(N_0)(z)) - f_{j_0}(z) \right| < \frac{1}{s_0}, \text{ where } N_1 = N_0 - k_0. \tag{8}
\]

Now the polynomial $R(z) = p(z) + z^{k_0 + 1}p_{N_1}(N_0)(z)$ has degree at most $N_0$.
We write $R(z)$ as $\sum_{i=0}^{N_0} b_i z^i$. Then $b_i = a_i$ for $i = 0, 1, \ldots, \nu_0$, and $b_i = 0$ for $i = \nu_0 + 1, \nu_0 + 2, \ldots, k_0$. Let $b := (b_0, b_1, \ldots, b_{N_0}, 0, 0, \ldots) \in c_{00}$. Then $b \in \mathbb{C}^{N_0}$, $ho(a, b) < \varepsilon_0$ by (3) and
\[
\sup_{z \in K_m} \left| \frac{1}{\phi(N_0)} \sum_{i=0}^{N_0} b_i z^i - f_{j_0}(z) \right| < \frac{1}{s_0}
\]
and (1) and (2) are satisfied now and our result follows. ■

Now by Lemmas 3.3, 3.4, Proposition 3.5, Baire’s Category Theorem and the fact that $(\mathbb{C}^{N_0}, T_c)$ is a complete metric space the proof of Proposition 3.2 is complete. ■

**Remark 3.6.** The above argument also yields the classical theorem of Selznev.

**Proposition 3.7.** If $\limsup_{n \to \infty} \sqrt[n]{|\phi(n)|} < 1$ then $U(\phi) = \emptyset$.

**Proof.** Let $\limsup_{n \to \infty} \sqrt[n]{|\phi(n)|} = \theta_0 \in [0, 1)$. We suppose, to obtain a contradiction, that $U(\phi) \neq \emptyset$. Let $a = (a_0, a_1, a_2, \ldots) \in U(\phi)$. We consider the compact subsets $K_n := [1, n]$ for $n = 2, 3, \ldots$ of $\mathbb{C}$. We set $E = \bigcup_{n=2}^{\infty} K_n = [1, +\infty)$ which is closed and non-thin at infinity. Let $\theta_1 \in (\theta_0, 1)$. We can find a natural number $n_0$ such that
\[
|\phi(n)| < \theta_1^n \quad n \geq n_0. \tag{1}
\]

We apply now the definition of the set $U(\phi)$ for the compact set $K_2$ and the constant function $\mathbb{I} : K_2 \to \mathbb{C}$, with formula $\mathbb{I}(z) = 1$ for all $z \in K_2$. It follows that there exists a subsequence of natural numbers $\lambda^2 = (\lambda_n^2)_{n \in \mathbb{N}}$ such that
\[
\sup_{z \in K_2} \left| \frac{1}{\phi(\lambda^2_n)} \sum_{i=0}^{\lambda^2_n} a_i \cdot z^i - 1 \right| \to 0 \quad \text{as } n \to \infty.
\]
By the above convergence and (1) there exists some natural number \( \mu_2 > n_0 \) such that
\[
\sup_{z \in K_2} \left| \frac{1}{\phi(\mu_2)} \sum_{i=0}^{\mu_2} a_i \cdot z^i - 1 \right| < \frac{1}{2} \tag{2}
\]
and
\[
\sup_{z \in K_2} \left| \sum_{i=0}^{\mu_2} a_i \cdot z^i \right| < \frac{3}{2} \theta_1^{\mu_2} \tag{3}
\]
Inductively we see that there exists a sequence \( (\mu_n)_{n \in \mathbb{N}} \) of natural numbers such that
\[
\sup_{z \in K_n} \left| \frac{1}{\phi(\mu_n)} \sum_{i=0}^{\mu_n} a_i \cdot z^i - 1 \right| < \frac{1}{n} \tag{4}
\]
and
\[
\sup_{z \in K_n} \left| \sum_{i=0}^{\mu_n} a_i \cdot z^i \right| < \frac{3}{2} \theta_1^{\mu_n} \quad \forall \ n \in \mathbb{N}, \ n = 2, 3, \ldots \tag{5}
\]
Now we consider the polynomials \( p_n = \sum_{i=0}^{\mu_n} a_i z^i \) for \( n = 2, 3, \ldots \). We have
\[
\|p_n\|_{K_n} < \frac{3}{2} \theta_1^{\mu_n}, \quad n = 2, 3, \ldots \tag{6}
\]
For the polynomials \( p_n, n = 2, 3, \ldots, \Gamma = K_2, E = [1, +\infty) \) and \( d_n = \mu_n, n = 2, 3, \ldots \) we see that the two conditions (i) and (ii) of Theorem 1 of [9] are satisfied (or Theorem 10 of [15]), and so \( \limsup_{n \to \infty} \|p_n\|_E^{1/\mu_n} < 1 \) for every compact subset \( K \) of \( \mathbb{C} \).

We apply this conclusion for
\[
K = \bar{D}(0, 1) = \bar{D} = \{ z \in \mathbb{C} | |z| \leq 1 \}.
\]
Let \( \theta_2 := \limsup_{n \to \infty} \|p_n\|_{\bar{D}}^{1/\mu_n} \). Then \( \theta_2 \in (0, 1) \). Let \( \theta_3 \in (\theta_2, 1) \). Then there exists \( m_0 \in \mathbb{N} \) such that \( \|p_n\|_{\bar{D}} < \theta_3^{\mu_n} \) for all \( n \geq m_0 \). Thus \( \|p_n\|_{\bar{D}} \to 0 \) as \( n \to \infty \). By the maximum principle we see that \( 0 \leq |p_n(0)| \leq \|p_n\|_{\bar{D}} \to 0 \) for all \( n \geq 2 \). Thus we have \( |p_n(0)| \to 0 \), whence \( a_0 = 0 \).

So, we have
\[
p_n = \sum_{i=1}^{\mu_n} a_i z^i = z \cdot \sum_{i=0}^{\mu_n-1} a_{i+1} z^i = z \cdot p_n^1
\]
where
\[ p_n^1 = \sum_{i=0}^{\mu_n-1} a_{i+1} z^i \quad n \geq 2. \]

We have \( \|p_n\|_{\tilde{B}} \to 0 \) so \( \|p_n^1\|_{\tilde{B}} \to 0 \). Thus by the maximum principle we conclude that
\[ 0 \leq |p_n^1(0)| \leq \|p_n^1\|_{\tilde{B}} \to 0, \]
which implies that \( a_1 = 0 \). Inductively we see that \( a_n = 0 \) for all \( n = 0, 1, 2, \ldots \).

So \( p_n = 0 \) for all \( n \in \mathbb{N}_0 \).

This contradicts (4) and the result now follows. \( \blacksquare \)

**Proof of Theorem 3.1.** Suppose that \( \lim_{n \to \infty} \phi(n) = 0 \). Then \( \limsup_{n \to \infty} \sqrt[n]{|\phi(n)|} \leq 1 \).

If \( \limsup_{n \to \infty} \sqrt[n]{|\phi(n)|} < 1 \) then by Proposition 3.7 we have that \( U(\phi) = \emptyset \).

Now let \( \limsup_{n \to \infty} \sqrt[n]{|\phi(n)|} = 1 \). Then there exists a sequence of natural numbers \( \lambda = (\lambda_n)_{n \in \mathbb{N}} \) such that
\[ \lim_{n \to \infty} \sqrt[n]{|\phi(\lambda_n)|} = 1. \]

Since \( \lim_{n \to \infty} \phi(n) = 0 \) we see that
\[ \lim_{n \to \infty} \phi(\lambda_n) = 0. \]

Then making a proof similar to that in Proposition 3.5 for the sequence \( \phi \circ \lambda \) instead of \( \phi \) we can take that \( U(\phi) \) is a \( G_\delta \) dense subset of \( (\mathbb{C}^{\mathbb{N}_0}, T_c) \). The previous proof holds for every subsequence \( \mu \) of \( \phi \circ \lambda \). Then we argue as at the end of the proof of Lemma 2.4 to complete the proof of Theorem 3.1. \( \blacksquare \)

**Remark 3.8.** *Theorem 3.1 tells us that the condition for the function \( \phi \) of Theorem 5.1 of [7] cannot be removed but it is not sharp.*

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