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Determination of a universal series

A. MOUZE, V. NESTORIDIS, I. PAPADOPERAKIS, N. TSIRIVAS ¹

Abstract

The known proofs for universal Taylor series do not determine a specific universal Taylor series. In the present paper, we isolate a specific universal Taylor series by modifying the proof in [30]. Thus we determine all Taylor coefficients of a specific universal Taylor series on the disc or on a polygonal domain. Furthermore in non simply connected domains, when universal Taylor series exist, we can construct a sequence of specific rational functions converging to a universal function, provided the boundary is good enough. The solution uses an infinite denumerable procedure and a finite number of steps is not sufficient. However we solve a Runge's type problem in a finite number of steps.

1 Introduction

The first universal power series was discovered by Fekete before 1914 ([32]), doing approximations on $[-1, 1]$. In the complex plane $\mathbb{C}$ Seleznev showed the existence of a complex power series with zero radius of convergence satisfying approximations in all compact sets $K \subset \mathbb{C}$ with $0 \notin K$ and K^c connected ([33]). Luh and independently Chui and Parnes ([21], [7]) gave the first example of universal Taylor series with strictly positive radius of convergence $R = 1$ whose partial sums did approximations on every compact set $K \subset \mathbb{C}$, disjoint from the closed unit disc $\bar{D} = \{z \in \mathbb{C} ; |z| \leq 1\}$ with K^c connected. The universal approximation was not valid on the boundary of the domain of definition of the universal Taylor series. The above result was strengthened in [28] to allow approximation on the boundary also and took the following form.

Theorem 1.1 *There exists a Taylor series $\sum_{n=0}^{+\infty} a_n z^n$, $a_n \in \mathbb{C}$, converging for $|z| < 1$, so that for every compact set $K \subset \mathbb{C}$ with $K \cap \{z \in \mathbb{C} ; |z| < 1\} = \emptyset$ and K^c connected, and every function $h : K \rightarrow \mathbb{C}$ continuous on K and holomorphic in $\overset{\circ}{K}$, there exists a strictly increasing sequence $\lambda_n \in \{0, 1, 2, \dots\}$, $n = 1, 2, \dots$ so that $\sum_{j=0}^{\lambda_n} a_j z^j \rightarrow h(z)$, uniformly on K , as $n \rightarrow +\infty$.*

Explicit Taylor series satisfying the requirements of Theorem 1.1 do not seem to be known. Further universal Taylor series as in Theorem 1.1 have wild properties. We refer

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the reader to [28], [2], [4], [6], [8], [9], [10], [12], [16], [18], [19], [20], [23], [24], [25], [26], [29], [17]. For instance they are outside the Nevanlinna class and any classical space of holomorphic functions in the open unit disc ([25]). Maybe this explains the difficulty of finding explicitly a universal Taylor series although the class of universal series is residual in $H(D)$, $D = \{z \in \mathbb{C} ; |z| < 1\}$, where $H(D) = \{f : D \rightarrow \mathbb{C}, f \text{ holomorphic}\}$ is endowed with the topology of uniform convergence on compact sets.

The proofs of theorems such as Theorem 1.1 using Baire's Category Theorem give that the set of universal Taylor series is dense and G_δ , hence non void. In this way, we can not isolate a specific universal Taylor series. The proofs of theorems such as Theorem 1.1 avoiding Baire's Category Theorem show that the set of universal Taylor series is non void (and dense). The construction in these proofs does not lead to a specific universal Taylor series. The main reason for this is that, in the course of the construction, they have to use (infinitely many times) Runge's (or Mergelyan's) theorem and approximate by a polynomial some functions on arbitrary compact sets with connected complement. Thus, one concludes that there exists a polynomial P_n doing the approximation. For each n , the sets of these good polynomials is non-empty, but there does not exist a rule to select a unique good polynomial P_n . For any such choice of P_n 's the series $\sum_{n=1}^{+\infty} z^{k_n} P_n(z)$ is a universal Taylor series for suitably chosen natural numbers k_n . Thus, one concludes that the set of universal Taylor series is non-void, but we do not have determined a specific universal Taylor series.

To give a simpler example, if we have to choose a number in $(0, \delta)$ where $\delta > 0$ is known, there are lots of choices. A rule is to choose $\delta/2$. Another rule would be to find the smallest natural number N so that $\frac{1}{N} < \delta$ and to choose the element $\frac{1}{N} \in (0, \delta)$. In the previously considered Runge's problem there is not an obvious such rule of selection.

In section 2 we show that the proof of Theorem 1.1 in [30] without use of Baire's Theorem can be transformed so that it allows to determine all coefficients $a_n \in \mathbb{C}$, $n = 0, 1, 2, \dots$, of a specific universal Taylor series using an infinite denumerable procedure. Each coefficient can be constructed after a finite number of steps, but in order to compute all the coefficients we need a denumerable infinite number of steps. The corresponding program would be a finite program with orders of repetition.

The solution of the above problem consists in solving a Runge type problem. We have to compute a polynomial approximating simultaneously another polynomial on a compact set K , $K \cap D = \emptyset$, K^c connected and the zero function on a compact set $L \subset D$, $K \cap L = \emptyset$. A careful examination of the proof of Runge's Theorem in [22] shows that an explicit construction is possible if, on the one hand, we know a curve surrounding K and separating L from the boundary ∂D and if, on the other hand, we know a positive lower bound of the distance of this curve to $K \cup L$. Then we follow this curve in moving the poles of the rational functions appearing in the discretization of a Cauchy's integral. The only problem would be to separate L from the boundary of the unit disc. But this can easily be realized using a circumference with center 0 and radius r : $\max_{z \in L} |z| < r < 1$. We also can control the distance of this circumference from L and the unit circle. If we replace the open unit disc D by an arbitrary simply connected domain Ω then we do not know a specific way to separate a compact set $L \subset \Omega$ from $\partial\Omega$. So we can not determine a universal Taylor series for the general simply connected domain Ω , although we know that there exist such functions and their class is residual in $H(\Omega)$ ([26], [29]). In the particular cases where $\partial\Omega$ consists of a finite number of segments or circular arcs, then we can find rules to separate a compact set $L \subset \Omega$ from $\partial\Omega$ and a procedure that constructs universal Taylor series on discs, polygons, half-planes or angles.

The reader can verify that every step in our approach in section 2 can be done following a finite program (with orders of repetition) after a finite number of steps. Each partial result is unique, specific and deterministic. If we repeat our approach another time this result will be the same. In our approach, when we use the fact that there exists an object satisfying some requirements, then one can verify that we can find a rule to choose it in a unique way and have a specific result. The Runge's type problems we solve are solved after a finite number of steps of this procedure; see also [1]. On the contrary, to determine a specific universal Taylor series, we need all the Taylor coefficients and a finite number of steps is not sufficient; that is why we can not say that this procedure is algorithmic, but we call it an infinite denumerable procedure. If we stop after a finite number of steps, we obtain a polynomial P . But any polynomial can be completed to a power series and become universal or non universal. So in that sense it is natural that a finite number of steps is not sufficient to determine a universal Taylor series.

In section 3 we present some new results concerning the existence of universal Taylor series on some unbounded doubly connected domains Ω . The proofs presented here avoid Baire's Theorem and allow a specific determination in the case where $\partial\Omega$ is a Jordan curve consisting of a finite number of segments or circular arcs. We find all members of a sequence of rational functions converging to a universal function. This sequence has simultaneously an additional universal property.

In section 4 we extend the results of section 3 in the case where the unbounded domain Ω has arbitrary connectivity, but has a good geometry in accordance to [34]. In the case of "good" boundaries we can determine a universal series, even if the connectivity is infinite.

In conclusion we can say that if the boundary of a domain Ω is good enough and universal Taylor series exist on Ω , then we can determine by an infinite denumerable procedure a specific universal Taylor series on Ω , while in the general case we do not see any hope of success.

2 The case of simply connected domains

We recall the following well-known theorem ([26], [29]).

Theorem 2.1 *Let Ω be a simply connected domain in $\mathbb{C}$ and $\zeta_0 \in \Omega$. There exists a holomorphic function $F \in H(\Omega)$ such that the sequence of partial sums $S_N(f, \zeta_0)(z) = \sum_{n=0}^N \frac{F^{(n)}(\zeta_0)}{n!} (z - \zeta_0)^n$, $N = 0, 1, 2, \dots$ has the following property. For every compact set $K \subset \mathbb{C}$ with $K \cap \Omega = \emptyset$ and K^c connected, and every function $h : K \rightarrow \mathbb{C}$ continuous on K and holomorphic in $\overset{\circ}{K}$, there exists a sequence $\lambda_n \subset \{0, 1, 2, \dots\}$ with $\lambda_n < \lambda_{n+1}$ ($n = 1, 2, \dots$) such that $S_{\lambda_n}(F, \zeta_0)(z) \rightarrow h(z)$, uniformly on K , as $n \rightarrow +\infty$.*

No explicit function F as before seems to be known. Our aim is to show that, if the boundary of Ω is good enough, then it is possible to construct our infinite denumerable procedure determining in a unique way all the Taylor coefficients of such an F . First we sketch a proof of Theorem 2.1 without use of Baire's Theorem. Then we examine if each step of this proof can be done constructively.

Lemma 2.2 [26] *Let $\Omega \subset \mathbb{C}$ be a simply connected domain. Then there exists a sequence of compact sets $K_m \subset \mathbb{C}$, $m = 1, 2, \dots$, $K_m \cap \Omega = \emptyset$, K_m^c connected such that every compact set $K \subset \mathbb{C}$, $K \cap \Omega = \emptyset$, K^c connected is contained in some K_m .*

To define the sequence K_m , $m = 1, 2, \dots$ we consider all the polygonal lines Γ in $\mathbb{C}$ starting from ζ_0 and ending to $n + 1$, where n is any natural number, and having all vertices in $\mathbb{Q} + i\mathbb{Q}$, where $\mathbb{Q}$ denotes the set of rational numbers. Let also $s \in \{1, 2, \dots\}$. We consider the compact set

$$L(n, \Gamma, s, \zeta_0, \Omega) = \{z \in \mathbb{C} : z \in \Omega^c, |z| \leq n, \text{dist}(z, \Gamma) \geq \frac{1}{s}\}.$$

The set of all possible such $L(n, \Gamma, s, \zeta_0, \Omega)$ is denumerable and an enumeration of it gives the sequence K_m , $m = 1, 2, \dots$. Using de Morgan's rule we can easily verify that K_m^c is connected and that the sequence K_m , $m = 1, 2, \dots$ absorbs all compact sets $K \subset \mathbb{C}$ with $K \cap \Omega = \emptyset$, K^c connected. What is important for the sequel is that K_m is defined by a polygonal line Γ with all vertices in $\mathbb{Q} + i\mathbb{Q}$ and that we control the distance of K_m to Γ : $\text{dist}(\gamma, K_m) \geq \frac{1}{s}$, $s \in \{1, 2, \dots\}$.

We consider also an enumeration f_j , $j = 1, 2, \dots$ of all polynomials with coefficients in $\mathbb{Q} + i\mathbb{Q}$. We consider an enumeration (K_{m_n}, f_{j_n}) , $n = 1, 2, \dots$, of all couples (K_m, f_j) , $m, j = 1, 2, \dots$. In some papers it is assumed that each couple (K_m, f_j) appears infinitely many times in the list (K_{m_n}, f_{j_n}) , $n = 1, 2, \dots$; but this assumption is not necessary [25]. Let L_n , $n = 1, 2, \dots$, be an exhausting sequence of compact subsets of Ω . We keep this notation till the end of the paper. We remind that we can set $L_n = \{z \in \Omega : |z| \leq n, \text{dist}(z, \Omega^c) \geq \frac{1}{n}\}$. We consider the function

$$F_1(z) = \begin{cases} 0 & \text{on } L_1 \\ f_{j_1}(z) & \text{on } K_{m_1} \end{cases}$$

By Runge's Theorem we find a polynomial $P_1(z) = \sum_{\lambda=0}^{k_1} a_\lambda (z - \zeta_0)^\lambda$, $k_1 > 0$, such that

$$|F_1(z) - P_1(z)| < 1/1^2 = 1 \text{ on } L_1 \cup K_{m_1}.$$

Next we consider the function

$$F_2(z) = \begin{cases} 0 & \text{on } L_2 \\ f_{j_2}(z) - P_1(z) & \text{on } K_{m_2} \end{cases}$$

Using Runge's Theorem we can find a polynomial $P_2(z) = \sum_{\lambda=k_1+1}^{k_2} a_\lambda (z - \zeta_0)^\lambda$, $k_2 > k_1$, such that

$$|F_2(z) - P_2(z)| < 1/2^2 \text{ on } L_2 \cup K_{m_2}.$$

Inductively, assume that we have constructed the polynomials $P_1, P_2, \dots, P_n$ and the integers $k_1 < k_2 < \dots < k_n$. Then we consider the function

$$F_{n+1}(z) = \begin{cases} 0 & \text{on } L_{n+1} \\ f_{j_{n+1}}(z) - \sum_{i=1}^n P_i(z) & \text{on } K_{m_{n+1}} \end{cases}$$

Using Runge's Theorem we can find a polynomial $P_{n+1}(z) = \sum_{\lambda=k_n+1}^{k_{n+1}} a_\lambda (z - \zeta_0)^\lambda$, $k_{n+1} > k_n$, such that

$$|F_{n+1}(z) - P_{n+1}(z)| < 1/(n+1)^2 \text{ on } L_{n+1} \cup K_{m_{n+1}}.$$

It is easy to see that the series $\sum_{n=1}^{+\infty} P_n$ converges uniformly on compact subsets of Ω . We set $F(z) = \sum_{n=1}^{+\infty} P_n(z) = \sum_{\lambda=0}^{+\infty} a_\lambda(z - \zeta_0)^\lambda$. Further one can easily check that F satisfies the conclusion of Theorem 2.1 ([30], [7], [21], [25]).

We start now to examine whether each step of the previous proof can be realized by a finite or infinite procedure uniquely determining what is needed.

First observe that $\mathbb{Q}^+ = \{\frac{p}{q} : p, q \in \{1, 2, \dots\}\}$ can be enumerated in precise way. To do this, it suffices to enumerate $\{1, 2, \dots\}^2$ following lines parallel to $x + y = 0$. Having enumerations of $\mathbb{Q}^+$, $-\mathbb{Q}^+$ we enumerate $\mathbb{Q}$ starting from 0, then taking the first element of $\mathbb{Q}^+$, after that the first element of $-\mathbb{Q}^+$, then the second element of $\mathbb{Q}^+$ and after that the second element of $-\mathbb{Q}^+$ and so on. In a similar way, we can enumerate $\mathbb{Q} \times \mathbb{Q} \times \dots \times \mathbb{Q}$, $\bigcup_{n=1}^{\infty} \mathbb{Q}^n$ and $\bigcup_{n=1}^{\infty} (\mathbb{Q} + i\mathbb{Q})^n$. Thus, we can have precise enumerations of all polynomials with coefficients in $\mathbb{Q} + i\mathbb{Q}$, all polygonal lines Γ with vertices in $\mathbb{Q} + i\mathbb{Q}$ and all sets $L(n, \Gamma, s, \zeta_0, \Omega)$ ($\zeta_0 \in \Omega$). We notice that it is difficult to construct explicitly each set $K_m = L(n, \Gamma, s, \zeta_0, \Omega)$, but it will suffice that we know the polygonal line Γ and the distance $\text{dist}(\Gamma, K_m) = \frac{1}{s}$. Concerning the exhausting sequence L_n , $n = 1, 2, \dots$, it is also difficult to construct explicitly each L_n . As we will see it will suffice to construct explicitly a closed curve δ in Ω such that $L_n \cap \delta = \emptyset$, $\text{Ind}(\delta, z) = -1$ for all $z \in L_n$ and to know a number $\rho > 0$ such that $\text{dist}(\delta, L_n) \geq \rho$, $\text{dist}(\delta, \Omega^c) \geq \rho$ (obviously $\rho = \rho_n \rightarrow 0$, as $n \rightarrow +\infty$). In the case where Ω is the open unit disc we can set δ to be a circle with center 0 and radius $1 - \frac{1}{2n}$ oriented in the negative sense. Then $\rho = \frac{1}{2n}$. In the case where Ω is the interior of a polygon we can construct δ using segments parallel to the sides of the polygon and ending on the bisectors of the angles of the polygon. In the case where Ω is an open half plane, δ may consist of a segment parallel to the axis of the half plane and the half of a circle. In the case where Ω is an open angle, δ may consist of two segments parallel to the sides of the angle and a circular arc. The curve δ should separate the boundary of Ω from a compact subset $L = L_n$ of Ω and we should control the distances of δ to Ω^c and L . If the boundary of Ω is very bad there is no hope to know explicitly δ with control of the distances $\text{dist}(\delta, \Omega^c)$ and $\text{dist}(\delta, L)$ and we are unable to solve constructively our problem for the general simply connected domain.

In the good cases mentioned above we will see that it is possible to determine in a unique way all the Taylor coefficients of a universal Taylor series F . Each coefficient will be constructed after a finite number of steps; but in order to construct all the coefficients we will need an infinite denumerable number of steps. The corresponding program will be a finite program with orders of repetition. It is also important to say that in order to construct the N th coefficient we will not need all the enumerations of the previous mentioned denumerable sets, but an initial finite part of them. The essential step to be done explicitly is the following Runge's type problem.

Suppose Ω is a simply connected domain of the previously mentioned good types and $\zeta_0 \in \Omega$. Let $K \equiv K_{m_{n+1}}$, $L \equiv L_{m_{n+1}}$, $f \equiv f_{j_{n+1}} - \sum_{i=1}^n P_i(z)$ a known polynomial and $\tilde{k} = k_n$. We also know a polygonal line Γ starting at ζ_0 and ending at $N + 1$, where N is a natural number, and having all vertices in $\mathbb{Q} + i\mathbb{Q}$. An upper bound of $|z|$, $z \in \Gamma$, is $N + 1$. Let $\rho = \frac{1}{s} > 0$ be known and such that $\text{dist}(\Gamma, K) \geq \rho$. Since it suffices to consider only large n , we may assume that we know a number $r > 0$ such that $\{z : |z - \zeta_0| \leq r\} \subset L$. Let also $\varepsilon > 0$ be given. We have to find a unique polynomial $P(z) \equiv P_{n+1}(z) \equiv \sum_{\lambda=\tilde{k}+1}^{k_{n+1}} a_\lambda(z - \zeta_0)^\lambda$, $k_{n+1} > \tilde{k}$, with $|P(z) - f(z)| < \varepsilon$ on K and $|P(z)| < \varepsilon$ on L . We restrict our attention to the case where Ω is bounded and a bound is $1 \leq N$. The good unbounded cases require some modifications. We consider

the following explicitly known curve. One piece is the positively oriented circle σ with center 0 and passing through $N + 1$ (we consider that we start at $N + 1$ and we end at $N + 1$ after turning one time in the positive sense). Then we follow the curve Γ starting from $N + 1$ towards ζ_0 . We stop the first time where Γ meets δ at some point ξ . Then we follow δ , which turns in the negative sense, until we return to ξ . Then we follow Γ from ξ until we arrive again to $N + 1$. We denote this closed curve by Δ . Clearly Δ is disjoint from $K \cup L$ and $\text{Ind}(\Delta, z) = 1$ for all $z \in K$ and $\text{Ind}(\Delta, z) = 0$ for all $z \in L$. One could wonder whether the point ξ can be determined. But the sides of Γ are segments and δ is either a circle or it consists of known segments or circular arcs. The intersection of a known segment with another known segment or circle reduces to the solution of a first or second degree equation. So we can find ξ . We remind that $\text{dist}(K, \Gamma) \geq \frac{1}{s}$, $\text{dist}(\delta, \Omega^c) \geq \rho$, $\text{dist}(\delta, L) \geq \rho$ and $\text{dist}(\sigma, K \cup L) \geq 1$. Setting $R = \min(1, \frac{1}{s}, \rho) > 0$ we have $\text{dist}(\Delta, K \cup L) \geq R$ and R, δ are explicitly known. We repeat the proof of Theorem 3.1 of [22]. According to the Cauchy formula applied to the known polynomial f the following holds :

$$\frac{1}{2i\pi} \int_{\Delta} \frac{f(\zeta)}{\zeta - z} d\zeta = \begin{cases} f(z) & \text{for } z \in K \\ 0 & \text{for } z \in L. \end{cases}$$

The derivative f' is easily found; the degrees of f and f' are known as well as the maximum of the absolute values of their coefficients. An upper bound of $|\zeta|$ for $\zeta \in \Delta$ is $N + 1$. So $\max_{\zeta \in \Delta} |f(\zeta)|$ and $\max_{\zeta \in \Delta} |f'(\zeta)|$ are bounded by a known number M which can be easily determined. Thus, for every $\varepsilon' > 0$ we can compute $\delta(\varepsilon') > 0$ so that, if $\zeta', \zeta'' \in \Delta$ and the length of the arc from ζ' to ζ'' is strictly less than $\delta(\varepsilon')$, then $|f(\zeta') - f(\zeta'')| < \varepsilon'$. It suffices to set $\delta(\varepsilon') = \frac{\varepsilon'}{M}$. Next we divide the curve Δ to small curves with length less than $\min(\delta(\varepsilon'), \varepsilon')$. If $\varepsilon' > 0$ and $\delta(\varepsilon') > 0$, are known, the previous division can be done in a unique way because Δ consists of segments and circular arcs. The number of the smaller curves can be found and is unique if it is the minimum good number. Taking into account relation 3.5 in page 83 of [22] and the fact that we know explicitly Δ, R and M , if $\varepsilon'' > 0$ is given we may compute $\varepsilon' > 0$ and find an explicit rational function φ with simple poles on Δ so that $|\varphi(z) - f(z)| < \varepsilon''$ on K and $|\varphi(z)| < \varepsilon''$ on L . Next we repeat the proof of Lemma of [22] (page 83). We move each pole of φ following the curve δ to the point $N + 1$. This can be done in a unique way because Δ is known and consists of segments or circular arcs and $R > 0$ is known. So we can find the number m and the points $\zeta = \zeta_0, \zeta_1, \dots, \zeta_m = \tilde{\zeta} = N + 1$ on Δ , so that the diameter of the subarcs of Δ defined by ζ_l, ζ_{l+1} is strictly less than $R/2$. The maximum on $K \cup L$ of the absolute value of a known rational function with unique pole at $\zeta \in \Delta$ of multiplicity k (known) can be computed because the absolute value of the denominator is greater than or equal to R^k and the numerator is a known polynomial and $|z| \leq N$ on $K \cup L$. Thus the integer n_1 of [22] (page 84) may be determined if $\varepsilon'' > 0$ is given. So, if $\tilde{\varepsilon}$ is given, we find a rational function φ_1 with unique pole at $N + 1$ such that

$$|\varphi_1(z) - f(z)| < \tilde{\varepsilon} \text{ on } K \text{ and } |\varphi_1(z)| < \tilde{\varepsilon} \text{ on } L.$$

So $\varphi_1(z) = \frac{Q(z)}{(z - N - 1)^\tau}$ where τ is known and Q is a known polynomial. The maximum of $|\varphi_1(z)|$ on $|z| \leq N + \frac{1}{2}$ can be controlled by a constant C which can be computed in a unique way. Cauchy estimates imply that the Taylor coefficients a_n of φ_1 satisfy

$$|a_n| \leq \frac{C}{(N + \frac{1}{2})^n}.$$

For $|z| \leq N$, we easily estimate

$$\sum_{n=\Lambda+1}^{+\infty} |a_n| |z|^n \leq \sum_{n=\Lambda+1}^{+\infty} C \frac{N^n}{(N + \frac{1}{2})^n} = C \left(\frac{1}{1 + 1/2N} \right)^{\Lambda+1} (2N+1).$$

If we impose that this last expression is less than $\tilde{\varepsilon}$, we find the minimum good $\Lambda > \tilde{k}$ and we have

$$|\varphi_1(z) - \sum_{n=0}^{\Lambda} a_n z^n| < \tilde{\varepsilon} \text{ for } z \in K \cup L.$$

Moreover the coefficients a_n , $n = 0, 1, \dots, \Lambda$, can be computed, since φ_1 can be easily decomposed into simple functions with pole $N+1$. Therefore we get

$$|f(z) - \sum_{n=0}^{\Lambda} a_n z^n| < 2\tilde{\varepsilon} \text{ on } K \text{ and } |\sum_{n=0}^{\Lambda} a_n z^n| < 2\tilde{\varepsilon} \text{ on } L.$$

We remind that we know an $r > 0$ so that $\{z \in \mathbb{C} : |z| \leq r\} \subset L$. So by Cauchy's estimates we have $|a_n| \leq \frac{2\tilde{\varepsilon}}{r^n}$. Therefore, for $z \in K \cup L$, we have

$$|\sum_{n=0}^{\tilde{k}} a_n z^n| \leq \sum_{n=0}^{\tilde{k}} |a_n| |z|^n \leq \sum_{n=0}^{\tilde{k}} \frac{2\tilde{\varepsilon}}{r^n} N^n.$$

It follows

$$|\sum_{n=\tilde{k}+1}^{\Lambda} a_n z^n| \leq 2\tilde{\varepsilon} (1 + \sum_{n=0}^{\tilde{k}} \left(\frac{N}{r}\right)^n) \text{ on } L$$

and

$$|f(z) - \sum_{n=\tilde{k}+1}^{\Lambda} a_n z^n| < 2\tilde{\varepsilon} (1 + \sum_{n=0}^{\tilde{k}} \left(\frac{N}{r}\right)^n) \text{ on } K.$$

Choosing $\tilde{\varepsilon} > 0$ so that $2\tilde{\varepsilon} (1 + \sum_{n=0}^{\tilde{k}} \left(\frac{N}{r}\right)^n) \leq \varepsilon$ and setting $P(z) = \sum_{n=\tilde{k}+1}^{\Lambda} a_n z^n$ we have find in a unique way the required polynomial and the universal Taylor series F . Therefore, we see that in the cases where Ω is a disc, a polygon, a half plane or an angle it is possible to write an infinite denumerable procedure determining in a unique way the Taylor coefficients of a universal Taylor series F .

Remarks 2.3 • For the solution of the Runge's type problem we can write a finite program with repetitions and the procedure ends always after a finite number of steps. Thus we may talk of an algorithm in the sense of numerical analysis. For the construction of all the coefficients of the universal series we need infinitely many steps thus we can not speak about an algorithm but rather about an infinite denumerable procedure.

- The function F of Theorem 2.1 has the following additional properties:
 - (i) there exists a strictly increasing sequence $\mu_n \in \{0, 1, 2, \dots\}$, $n = 1, 2, \dots$, so that, for every compact set $L \subset \Omega$ we have $\sup_{z \in L} \sup_{\zeta \in L} |S_{\mu_n}(F, \zeta)(z) - F(z)| \rightarrow 0$, as $n \rightarrow +\infty$,
 - (ii) for every compact set $K \subset \mathbb{C}$, $K \cap \Omega = \emptyset$, K^c connected and every function $h : K \rightarrow \mathbb{C}$, continuous on K and holomorphic in $\overset{\circ}{K}$ there exists a subsequence (λ_n) (μ_n) so that for every compact set $L \subset \Omega$ we have $\sup_{z \in K} \sup_{\zeta \in L} |S_{\lambda_n}(F, \zeta)(z) - h(z)| \rightarrow 0$, as $n \rightarrow +\infty$.

We refer the reader to [26], [27] and [36].

3 The doubly connected case

If Ω is a bounded non simply connected domain, then universal Taylor series do not exist ([16], [26]). However, if K is a compact connected set in $\mathbb{C}$ whose complement is also connected, then in $\Omega = \mathbb{C} \setminus K$ there exist universal Taylor series with respect to one center [8], [24], [37]; see also [3], [5], [11], [13], [14], [15], [31], [35], [38]. In this section we present three new propositions in the doubly connected case $\Omega = \mathbb{C} \setminus K$ and the proofs presented here do not use Baire's Theorem and, as in the previous section, if ∂K is good enough they can be transformed to be realized infinite denumerable procedure. The good situations are when K is a closed disc or a polygon and more generally when ∂K is a Jordan curve consisting of a finite number of segments or circular arcs.

Let K' be a compact set. In the sequel, we denote by $A(K')$ the set $\{h : K' \rightarrow \mathbb{C}, \text{ continuous on } K' \text{ and holomorphic in } \overset{\circ}{K'}\}$ endowed with sup-norm on K' .

Proposition 3.1 *Let $K \subset \mathbb{C}$ be a connected compact set such that $\Omega = \mathbb{C} \setminus K$ is also connected. Let $\zeta_0 \in \Omega$ and $w \in \partial K$ such that $|\zeta_0 - w| = \max_{z \in K} |\zeta_0 - z|$. Then there exists a sequence $g_j, j = 1, 2, \dots$ of rational functions with $g_j(\infty) = 0$ with unique pole w with multiplicity k_j , with $k_j < k_{j+1}$, such that the series $\sum_{j=1}^{+\infty} g_j$ converges uniformly on compact subsets of Ω to a function $f \in H(\Omega)$ so that*

- (i) *the sequence $\sum_{j=1}^N g_j, N \in \mathbb{N}$, is dense in $A(K')$ for every compact set $K' \subset K - \{z : |z - \zeta_0| = |\zeta_0 - w|\}$ with K'^c connected,*
- (ii) *the sequence $S_N(f, \zeta_0)(\cdot), N \in \mathbb{N}$, is dense in $A(K')$, where K' is as in (i).*

Proof. Let $L_m, m = 1, 2, 3, \dots$, be an exhausting sequence of compact subsets of $\Omega \cup \{\infty\}$, with $\zeta_0, \infty \in L_1$. We may also assume that L_m^c is connected, contains K and is a bounded subset of $\mathbb{C}$. Let $f_j, j = 1, 2, \dots$, be an enumeration of all polynomials with coefficients in $\mathbb{Q} + i\mathbb{Q}$. We set $R = |\zeta_0 - w|$. Let us consider a strictly decreasing sequence of positive real numbers σ_m converging to zero, with $\sigma_1 < R$. We set $K_m = K \cap \{z \in \mathbb{C} : |\zeta_0 - z| \leq R - \sigma_m\}$. It is easily seen that $K_m \subset \mathbb{C}$ is a compact set with connected complement. We consider an enumeration $(K_{m_i}, f_{j_i}), i = 1, 2, \dots$, of all couples (K_m, f_j) with $K_m \neq \emptyset$. We construct by induction a sequence $(P_l), l = 1, 2, \dots$, of polynomials, $P_l(w) \neq 0$, and a strictly increasing sequence $(k_l), l = 1, 2, \dots$, of positive integers, $k_l > \deg(P_l)$.

First step: Consider two disjoint open subsets $G_1^{(1)}$ and $G_2^{(1)}$ with $K_{m_1} \subset G_2^{(1)}$, $L_1 \subset G_1^{(1)}$ and $w \notin G_1^{(1)} \cup G_2^{(1)}$. We consider the function $F(z) = f_{j_1}(z)$ on $G_2^{(1)}$ and $F(z) = 0$ on $G_1^{(1)}$. Applying Runge's Theorem we find a polynomial $P_1, P_1(w) \neq 0$, and a positive integer $k_1 > \deg(P_1)$ so that $g_1(z) = \frac{P_1(z)}{(z - w)^{k_1}}$ has unique pole w with multiplicity $k_1, g_1(\infty) = 0$ and satisfies

$$\sup_{z \in K_{m_1}} |g_1(z) - f_{j_1}(z)| < \frac{1}{2} \text{ and } \sup_{z \in L_1} |g_1(z)| < 1.$$

Notice that if $P_1(w) = 0$ then we can add a small constant respecting the other inequalities. So $P_1(w) \neq 0$. We also have $S_N(g_1, \zeta_0)(z) \rightarrow g_1(z)$, as $N \rightarrow +\infty$, uniformly on each compact subset of $D(\zeta_0, R)$, in particular on $K_{m_1} \subset D(\zeta_0, R)$. Therefore, we find N_1 so that

$$\sup_{z \in K_{m_1}} |S_{N_1}(g_1, \zeta_0)(z) - g_1(z)| < \frac{1}{2}.$$

It follows that

$$\sup_{z \in K_{m_1}} |g_1(z) - f_{j_1}(z)| < \frac{1}{2}, \quad \sup_{z \in K_{m_1}} |S_{N_1}(g_1, \zeta_0)(z) - f_{j_1}(z)| < 1 \text{ and } \sup_{z \in L_1} |g_1(z)| < 1.$$

Before proceeding in the 2nd step we mention that the use of Runge's Theorem can be constructive because it suffices to use a closed curve disjoint from L_1 which surrounds K_1 .

Second step: Let $\varepsilon_2 > 0$ to be defined later. By Runge's Theorem we find a polynomial P_2 , $P_2(w) \neq 0$, and a natural number $k_2 > \max(k_1, \deg(P_2))$, so that, if we set $g_2(z) = \frac{P_2(z)}{(z-w)^{k_2}}$, then g_2 is a rational function with $g_2(\infty) = 0$ and unique pole w with multiplicity k_2 , satisfying

$$\sup_{z \in K_{m_2}} |g_1(z) + g_2(z) - f_{j_2}(z)| < \frac{1}{2^2} \text{ and } \sup_{z \in L_2} |g_2(z)| < \varepsilon_2.$$

Because $\zeta_0 \in \overset{\circ}{L}_1$, we find $r > 0$ so that $\overline{D(\zeta_0, r)} \subset L_m$, for all $m = 1, 2, \dots$. Obviously we may assume $r < R$. Let $\sum_{j=0}^{+\infty} b_j^{(2)}(\zeta_0)(z - \zeta_0)^j$ be the Taylor development of g_2 with center ζ_0 . Then by Cauchy's estimates we have

$$|b_j^{(2)}(\zeta_0)| \leq \frac{\varepsilon_2}{r^j}, \quad j = 0, 1, 2, \dots$$

Therefore $S_{N_1}(g_1 + g_2, \zeta_0)(z) = S_{N_1}(g_1, \zeta_0)(z) + S_{N_1}(g_2, \zeta_0)(z)$ and for $z \in K_{m_1}$, we have

$$|S_{N_1}(g_2, \zeta_0)(z)| \leq \sum_{j=0}^{N_1} \frac{\varepsilon_2}{r^j} R^j \leq (N_1 + 1)\varepsilon_2 \left(\frac{R}{r}\right)^{N_1}.$$

We get

$$\sup_{z \in K_{m_1}} |S_{N_1}(g_1 + g_2, \zeta_0)(z) - f_{j_1}(z)| \leq 1 + (N_1 + 1)\varepsilon_2 \left(\frac{R}{r}\right)^{N_1}.$$

Because K_{m_2} is a compact subset of $D(\zeta_0, R)$ we find $N_2 > N_1$ so that

$$\sup_{z \in K_{m_2}} |S_{N_2}(g_1 + g_2, \zeta_0)(z) - (g_1 + g_2)(z)| < \frac{1}{2^2}.$$

The triangle inequality implies

$$\sup_{z \in K_{m_2}} |S_{N_2}(g_1 + g_2, \zeta_0)(z) - f_{j_2}(z)| < \frac{1}{2}.$$

By induction on *the step l* we will find a polynomial P_l , $P_l(w) \neq 0$, and a natural number k_l , $k_l > k_{l-1}$, $k_l > \deg(P_l)$, so that if we set $g_l(z) = \frac{P_l(z)}{(z-w)^{k_l}}$, g_l is a rational function with $g_l(\infty) = 0$ and unique pole w with multiplicity k_l , so that

$$\sup_{z \in K_{m_l}} \left| \sum_{i=1}^l g_i(z) - f_{j_l}(z) \right| < \frac{1}{2^l} \quad (1)$$

and

$$\sup_{z \in L_l} |g_l(z)| < \varepsilon_l, \quad (2)$$

where ε_l will be chosen later on. If $\sum_{j=0}^{+\infty} b_j^{(l)}(\zeta_0)(z - \zeta_0)^j$ is the Taylor development of g_l with center ζ_0 , by Cauchy's estimates we have

$$|b_j^{(l)}(\zeta_0)| \leq \frac{\varepsilon_l}{r^j}, \quad j = 0, 1, 2, \dots$$

For $i = 1, \dots, l-1$, we find

$$\sup_{z \in K_{m_i}} |S_{N_i}(g_1 + g_2 + \dots + g_l, \zeta_0)(z) - f_{j_i}(z)| \leq \frac{1}{2^{i-1}} + \left(\sum_{k=i+1}^l \varepsilon_k \right) (N_i + 1) \left(\frac{R}{r} \right)^{N_i}. \quad (3)$$

Next we choose $N_l > N_{l-1}$ so that

$$\sup_{z \in K_{m_l}} |S_{N_l}(g_1 + g_2 + \dots + g_l, \zeta_0)(z) - (g_1 + g_2 + \dots + g_l)(z)| < \frac{1}{2^l}. \quad (4)$$

Combining the triangle inequality with (1) and (4), we get

$$\sup_{z \in K_{m_l}} |S_{N_l}(g_1 + g_2 + \dots + g_l, \zeta_0)(z) - f_{j_l}(z)| < \frac{1}{2^l} + \frac{1}{2^l} = \frac{1}{2^{l-1}}. \quad (5)$$

That is (3) is also valid for $i = l$. We set

$$\varepsilon_i = \frac{1}{2^i} \left(\frac{r}{R} \right)^{N_{i-1}} \frac{1}{N_{i-1} + 1}, \quad i = 2, 3, \dots \quad (6)$$

We note that the choice of ε_i at the i^{th} step depends on r , R , which are fixed, and N_{i-1} which has been defined in the step of order $i-1$. Because $0 < \frac{r}{R} < 1$ we find

$$\sum_{i=m+1}^{+\infty} \varepsilon_i \leq \left(\frac{1}{2^{m+1}} + \frac{1}{2^{m+2}} + \dots \right) \left(\frac{r}{R} \right)^{N_m} \frac{1}{N_m + 1} \leq \frac{1}{2^m} \left(\frac{r}{R} \right)^{N_m} \frac{1}{N_m + 1}. \quad (7)$$

According to (2) and (6) the series $\sum_{j=1}^{+\infty} g_j$ converges uniformly on compact subsets of Ω and defines a function $f = \sum_{j=1}^{+\infty} g_j \in H(\Omega)$. Relation (1) gives (i) of Proposition 3.1 and (3) and (7) imply

$$\sup_{z \in K_{m_k}} |S_{N_k}(f, \zeta_0)(z) - f_{j_k}(z)| \leq \frac{1}{2^{k-1}} + \frac{1}{2^{k-1}} = \frac{1}{2^{k-2}}, \quad (8)$$

which gives the desired conclusion (ii) of Proposition 3.1. $\square$

Certainly the above construction can become an infinite denumerable procedure if ∂K is a circle, a polygon or a Jordan curve consisting of a finite number of segments or circular arcs. In the same cases we may give a constructive proof of the existence of universal Taylor series in Ω in the sense of Melas [24]; that is, we construct $f = \sum_{j=1}^{+\infty} \tilde{g}_j \in H(\Omega)$, where $\tilde{g}_j$ are rational functions with unique pole w , so that $S_N(f, \zeta_0)(\cdot)$, $N = 1, 2, \dots$, is dense in $A(K)$. But we do not obtain that $\sum_{j=1}^N \tilde{g}_j$, $N = 1, 2, \dots$, is dense in $A(K)$. The modification required in the previous proof is that at the j^{th} step the pole is not w but

$w_j \in \Omega$, where $w \in [\zeta_0, w_j]$ and $|w_j - w| > 0$ and $\text{dist}(w_j, L_j) > 0$ are known. This allows the knowledge of a partial sum $S_{N_j}(g_j, \zeta_0)(\cdot)$ making an approximation to the whole K . Now N_j is fixed and known. By Runge's Theorem we move the pole w_j following $[w_j, w]$ to w and in the same time $\tilde{g}_j$ approximate g_j on L_j . Since N_j is known, Cauchy's estimates suffice so that $S_{N_j}(\tilde{g}_j, \zeta_0)(\cdot)$ is close to $S_{N_j}(g_j, \zeta_0)(\cdot)$ on K . Moreover this can be done in a unique way, because the segment $[w_j, w]$ is known, as well its distance from L_j .

If we do not use Runge's Theorem a last time and we do not move w_j to w , then the series $\sum_{i=1}^{+\infty} g_i$ converges in the sense of meromorphic functions in Ω and $f = \sum_{i=1}^{+\infty} g_i$ is meromorphic. Thus, we obtain the following result.

Proposition 3.2 *Let $K \subset \mathbb{C}$ be a connected compact set such that $\Omega = \mathbb{C} \setminus K$ is also connected. Let $\zeta_0 \in \Omega$ and $w \in \partial K$ such that $|\zeta_0 - w| = \max_{z \in K} |\zeta_0 - z|$. Then there exists a sequence $w_j \in \Omega$, $j = 1, 2, \dots$, with $w_j \rightarrow w$, as $j \rightarrow +\infty$, and a sequence of rational functions g_j , $g_j(\infty) = 0$, with unique pole at w_j , such that the series $\sum_{j=1}^{+\infty} g_j$ converges in Ω to a meromorphic function $f = \sum_{j=1}^{+\infty} g_j$ in Ω and the following hold:*

- (i) *the sequence $\sum_{j=1}^N g_j$, $N = 1, 2, \dots$, is dense in $A(K)$,*
- (ii) *the sequence $S_N(f, \zeta_0)(\cdot)$, $N = 1, 2, \dots$, is dense in $A(K)$.*

Finally we can obtain density of the partial sums in $A(K)$ with f holomorphic in Ω instead of meromorphic in Ω .

Proposition 3.3 *Let $K \subset \mathbb{C}$ be a connected compact set such that $\Omega = \mathbb{C} \setminus K$ is also connected. Let $\zeta_0 \in \Omega$ and $w \in \partial K$ be such that $|\zeta_0 - w| = \max_{z \in K} |\zeta_0 - z|$. There exists a sequence $a_j \in \Omega$, $j = 1, 2, \dots$, with $a_j \rightarrow w$, as $j \rightarrow +\infty$, and rational functions g_j , $j = 1, 2, \dots$, $g_j(\infty) = 0$, with unique pole at a_j , such that the series $g_1 + \sum_{j=1}^{+\infty} (g_{j+1} - g_j)$ converges uniformly on each compact subset of Ω to a function $f \in H(\Omega)$ and the following hold:*

- (i) *the sequence (g_N) , $N = 1, 2, \dots$, is dense in $A(K)$,*
- (ii) *the sequence $S_N(f, \zeta_0)(\cdot)$, $N = 1, 2, \dots$, is dense in $A(K)$.*

Proof. Let f_j , $j = 1, 2, \dots$, be an enumeration of all polynomials with coefficients in $\mathbb{Q} + i\mathbb{Q}$. Let L_m , $m = 1, 2, 3, \dots$, be an exhausting sequence of compact subsets of $\Omega \cup \{\infty\}$, with $\zeta_0, \infty \in \overset{\circ}{L}_1$.

First step: Let $G_1^{(1)}, G_2^{(1)}$ be two disjoint open sets with $K \subset G_2^{(1)}$, $L_1 \subset G_1^{(1)}$. We choose $a_1 \notin G_1^{(1)} \cup G_2^{(1)}$ such that $|a_1 - \zeta_0| > |\zeta_0 - w|$. We consider the function $F_1 = f_1$ on $G_2^{(1)}$ and $F_1 = 0$ on $G_1^{(1)}$. By Runge's Theorem we find a rational function g_1 with unique pole a_1 so that

$$\sup_{z \in K} |g_1(z) - f_1(z)| < \frac{1}{2^2}, \quad \sup_{z \in L_1} |g_1(z)| < 1 \text{ and } g_1(\infty) = 0.$$

Since $S_N(g_1, \zeta_0)(z) \rightarrow g_1(z)$, as $N \rightarrow +\infty$, uniformly on $K \subset \overline{D(\zeta_0, |\zeta_0 - w|)}$, we find $N_1 \in \mathbb{N}$ such that

$$\sup_{z \in K} |S_{N_1}(g_1, \zeta_0)(z) - g_1(z)| < \frac{1}{2^2}.$$

It follows

$$\sup_{z \in K} |S_{N_1}(g_1, \zeta_0)(z) - f_1(z)| < \frac{1}{2}.$$

Second step: Let $\varepsilon_2 > 0$ to be defined later. By Runge's Theorem we find a rational function g_2 with unique pole $a_2 \in \Omega - L_2$, $|a_2 - w| < \frac{1}{2}|a_1 - w|$, $|a_2 - \zeta_0| > |\zeta_0 - w|$, so that

$$\sup_{z \in K} |g_2(z) - f_2(z)| < \frac{1}{2^3}, \quad \sup_{z \in L_1} |g_2(z) - g_1(z)| < \varepsilon_2 \quad \text{and} \quad g_2(\infty) = 0.$$

Since $\zeta_0 \in \overset{\circ}{L}_1$, we find $r > 0$ such that $\overline{D(\zeta_0, r)} \subset L_m$, $m = 1, 2, \dots$. Let also $R = |\zeta_0 - w|$, $r < R$. We choose $N_2 > N_1$ so that

$$\sup_{z \in \overline{D(\zeta_0, R)}} |S_{N_2}(g_2, \zeta_0)(z) - g_2(z)| < \frac{1}{2^3}.$$

The triangle inequality implies

$$\sup_{z \in K} |S_{N_2}(g_2, \zeta_0)(z) - f_2(z)| < \frac{1}{2^2}.$$

We also have

$$\begin{aligned} \sup_{z \in K} |S_{N_1}(g_2, \zeta_0)(z) - f_1(z)| &\leq \sup_{z \in K} |S_{N_1}(g_2, \zeta_0)(z) - S_{N_1}(g_1, \zeta_0)(z)| \\ &\quad + \sup_{z \in K} |S_{N_1}(g_1, \zeta_0)(z) - f_1(z)| \\ &\leq (N_1 + 1) \left(\frac{R}{r} \right)^{N_1} \varepsilon_2 + \frac{1}{2}, \end{aligned}$$

where we have used Cauchy's inequalities $|b_j^{(2)}(\zeta_0)| \leq \frac{\varepsilon_2}{r^j}$, $j = 0, 1, 2, \dots$, for the Taylor coefficients $b_j^{(2)}(\zeta_0)$ of the Taylor development $\sum_{j=0}^{+\infty} b_j^{(2)}(\zeta_0)(z - \zeta_0)^j$ of $g_2 - g_1$ with center ζ_0 .

By induction in the step l we will have constructed $g_1, \dots, g_{l-1}$, $N_1 < N_2 < \dots < N_{l-1}$ and $\varepsilon_2, \dots, \varepsilon_{l-1}$ have been used. Let $\varepsilon_l > 0$ to be defined later. By Runge's Theorem we find a rational function g_l , $g_l(\infty) = 0$, with unique pole $a_l \in \Omega - L_l$, $|a_l - w| < \frac{1}{2}|a_{l-1} - w|$, $|a_l - \zeta_0| > |w - \zeta_0|$, such that

$$\sup_{z \in K} |g_l(z) - f_l(z)| < \frac{1}{2^{l+1}}, \quad (9)$$

$$\sup_{z \in L_{l-1}} |g_l(z) - g_{l-1}(z)| < \varepsilon_l. \quad (10)$$

If we write $\sum_{j=0}^{+\infty} b_j^{(l)}(\zeta_0)(z - \zeta_0)^j$ the Taylor development of $g_l - g_{l-1}$ with center ζ_0 , then Cauchy estimates imply

$$|b_j^{(l)}(\zeta_0)| \leq \frac{\varepsilon_l}{r^j}, \quad j = 0, 1, 2, \dots \quad (11)$$

For $j = 1, \dots, l-1$, we have

$$\begin{aligned} \sup_{z \in K} |S_{N_j}(g_l, \zeta_0)(z) - f_j(z)| &\leq \sup_{z \in K} |S_{N_j}(g_l, \zeta_0)(z) - S_{N_j}(g_{l-1}, \zeta_0)(z)| \\ &\quad + \sup_{z \in K} |S_{N_j}(g_{l-1}, \zeta_0)(z) - f_j(z)| \\ &\leq \sum_{k=j+1}^l \sup_{z \in K} |S_{N_j}(g_k, \zeta_0)(z) - S_{N_j}(g_{k-1}, \zeta_0)(z)| \\ &\quad + \sup_{z \in K} |S_{N_j}(g_j, \zeta_0)(z) - f_j(z)| \\ &\leq (N_j + 1)(\varepsilon_l + \varepsilon_{l-1} + \dots + \varepsilon_{j+1}) \left(\frac{R}{r} \right)^{N_j} + \frac{1}{2^j}. \end{aligned} \quad (12)$$

For $j = l$, we choose $N_l > N_{l-1}$ so that

$$\sup_{z \in \overline{D(\zeta_0, R)}} |S_{N_l}(g_l, \zeta_0)(z) - g_l(z)| < \frac{1}{2^{l+1}}. \quad (13)$$

The triangle inequality with (9) imply

$$\sup_{z \in K} |S_{N_l}(g_l, \zeta_0)(z) - f_l(z)| < \frac{1}{2^{l+1}} + \frac{1}{2^{l+1}} = \frac{1}{2^l}. \quad (14)$$

We set

$$\varepsilon_j = \frac{1}{2^j} \left(\frac{r}{R} \right)^{N_{j-1}} \frac{1}{N_{j-1} + 1} < \frac{1}{2^j}, \quad (15)$$

depending on r , R , which are known, and on N_{j-1} constructed in the $(j-1)^{th}$ step. Combining the inequality (10) with (14) and (15) we derive that $f = g_1 + \sum_{j=1}^{+\infty} (g_{j+1} - g_j) = g_l + \sum_{j=l}^{+\infty} (g_{j+1} - g_j)$, for all $l \geq 1$, converges uniformly on each compact subset of Ω and $f \in H(\Omega)$. We also have

$$\sum_{i=m+1}^{+\infty} \varepsilon_i \leq \left(\frac{1}{2^{m+1}} + \frac{1}{2^{m+2}} + \dots \right) \left(\frac{r}{R} \right)^{N_m} \frac{1}{N_m + 1} \leq \frac{1}{2^m} \left(\frac{r}{R} \right)^{N_m} \frac{1}{N_m + 1}. \quad (16)$$

Combining (16) with (12) we obtain

$$\begin{aligned} \sup_{z \in K} |S_{N_j}(f, \zeta_0)(z) - f_j(z)| &\leq \left(\frac{R}{r} \right)^{N_j} (N_j + 1) \left(\sum_{i=j+1}^{+\infty} \varepsilon_i \right) + \frac{1}{2^j} \\ &\leq \frac{1}{2^j} + \frac{1}{2^j} = \frac{1}{2^{j-1}}. \end{aligned} \quad (17)$$

This gives (ii). Finally (i) follows from (9) because $g_1 + \sum_{i=1}^N (g_{i+1} - g_i) = g_{N+1}$. $\square$

Remark 3.4 We can compare the last construction with [5, page 883].

4 The case of arbitrary connectivity

In the case where the compact set K is not connected but it has a finite number of components K_i , $i = 1, \dots, s$, so that $\max_{z \in K_i} |\zeta_0 - z|$ is the same for all $i = 1, \dots, s$, then under the natural condition $\Omega = \mathbb{C} \setminus K$ is connected, we can extend the results of section 3. Instead of one point w , we will have $w^{(1)}, \dots, w^{(s)}$ so that $w^{(i)} \in K_i$ and $|\zeta_0 - w^{(i)}| = \max_{z \in K_i} |\zeta_0 - z|$ and the rational functions g_j will have poles inside $\{w^{(1)}, \dots, w^{(s)}\}$ or points close to them. The proofs require minor modifications. It is also important to say that if each ∂K_i is a Jordan curve consisting of a finite number of segments or circular arcs, then the construction can be done in a unique way. If we do not pay attention to infinite denumerable procedures, then we can generalize the previous results in the case of arbitrary (even non-denumerable) connectivity according to a recent result of Tsirivas [34].

Let K be a compact set, $K \subset \mathbb{C}$, $K \neq \emptyset$, whose complement $\Omega = \mathbb{C} - K$ is connected. Let $K = \cup_{i \in I} K_i$, where K_i , $i \in I$, are the connected components of K and I is an arbitrary (even non-denumerable) set of indices. We assume that K has some specific position in $\mathbb{C}$. Fix $\zeta_0 \in \Omega$ and suppose that there exists R , $0 < R < +\infty$, so that $K \subset \overline{D(\zeta_0, R)}$ and $K_i \cap C(\zeta_0, R) \neq \emptyset$ for all $i \in I$. Obviously such an R is unique.

Proposition 4.1 *Let $K = \cup_{i \in I} K_i$ be a compact set as above, $\Omega = \mathbb{C} - K$. We choose $w_i \in K_i$ such that $|\zeta_0 - w_i| = R$ for all $i \in I$. Then there exists a sequence of rational functions g_j , $j = 1, 2, \dots$, with poles in $W = \{w_i, i \in I\}$, $g_j(\infty) = 0$ such that the series $\sum_{j=1}^{+\infty} g_j$ converges uniformly on compact subsets of Ω and defines a holomorphic function $f \in H(\Omega)$, $f = \sum_{j=1}^{+\infty} g_j$ so that*

- (i) *the sequence $\sum_{j=1}^N g_j$, $N \in \mathbb{N}$, is dense in $A(K')$ for every compact set $K' \subset K - \{z : |z - \zeta_0| = R\}$ with K'^c connected,*
- (ii) *the sequence $S_N(f, \zeta_0)(\cdot)$, $N \in \mathbb{N}$, is dense in $A(K')$, where K' is as in (i).*

Proof. Let L_m , $m = 1, 2, 3, \dots$, be an exhausting sequence of compact subsets of Ω , with $\zeta_0 \in \overset{\circ}{L}_1$. Let f_j , $j = 1, 2, \dots$, be an enumeration of all polynomials of one complex variable with coefficients in $\mathbb{Q} + i\mathbb{Q}$. Let us consider a strictly decreasing sequence of positive real numbers σ_m converging to zero. We set $K_m = \{z \in K : |\zeta_0 - z| \leq R - \sigma_m\}$ for $m = 1, 2, \dots$, where $\sigma_1 < R$. It is easily seen that $K_m \subset \mathbb{C}$, $m = 1, 2, \dots$, is a compact set with connected complement. Now we consider two sequences of open and bounded sets $G_1^{(m)}, G_2^{(m)}$, $m = 1, 2, \dots$, with some good properties. More specifically we assume the following: $G_1^{(m)} \cap G_2^{(m)} = \emptyset$, $m = 1, 2, \dots$. The sets $G_1^{(m)}, G_2^{(m)}, (G_1^{(m)})^c, (G_2^{(m)})^c$ have finite connected components, $m = 1, 2, \dots$. We also have $(G_1^{(m)} \cup G_2^{(m)}) \cap W = \emptyset$, $m = 1, 2, \dots$. The boundary of the sets $G_1^{(m)}, G_2^{(m)}$, $m = 1, 2, \dots$, consists of a finite number of simple closed polygonal lines with segments parallel to axes, whose vertices are in $\mathbb{Q} + i\mathbb{Q}$. Finally we assume that $K_m \subset G_2^{(m)}$, $L_m \subset G_1^{(m)}$, $m = 1, 2, \dots$. We can construct the above sets $G_1^{(m)}, G_2^{(m)}$, $m = 1, 2, \dots$, using the construction of the open sets $\mathcal{O}_n$, $n = 1, 2, \dots$, that is described in the pages 80 and 81 of volume III of [22]. Now we begin the proof of the theorem.

First step: We consider the function $F_1 : G_1^{(1)} \cup G_2^{(1)} \rightarrow \mathbb{C}$ defined by

$$F_1(z) = \begin{cases} f_1(z) & \text{if } z \in G_2^{(1)}, \\ 0 & \text{if } z \in G_1^{(1)}. \end{cases} \quad (18)$$

By Runge's Theorem we can find a rational function g_1 with poles only in W , with $g_1(\infty) = 0$, such that

$$\sup_{z \in K_1} |g_1(z) - f_1(z)| < \frac{1}{2}, \quad (19)$$

and

$$\sup_{z \in L_1} |g_1(z)| < 1. \quad (20)$$

We can easily compute a natural number $N_1 \in \mathbb{N}$ such that

$$\sup_{z \in K_1} |S_{N_1}(g_1, \zeta_0)(z) - g_1(z)| < \frac{1}{2}. \quad (21)$$

By (19), (21) and the triangle inequality we take

$$\sup_{z \in K_1} |S_{N_1}(g_1, \zeta_0)(z) - f_1(z)| < 1. \quad (22)$$

Second step: We consider the function $F_2 : G_1^{(2)} \cup G_2^{(2)} \rightarrow \mathbb{C}$ defined by

$$F_2(z) = \begin{cases} f_2(z) - g_2(z) & \text{if } z \in G_2^{(2)}, \\ 0 & \text{if } z \in G_1^{(2)}. \end{cases} \quad (23)$$

Let $\varepsilon_2 > 0$ to be defined later. By Runge's Theorem we find a rational function g_2 with poles in W and $g_2(\infty) = 0$ such that

$$\sup_{z \in K_2} |g_1(z) + g_2(z) - f_2(z)| < \frac{1}{2^2}, \quad (24)$$

and

$$\sup_{z \in L_2} |g_2(z)| < \varepsilon_2. \quad (25)$$

Because of $\zeta_0 \in \overset{\circ}{L}_1$ we can find a fixed $\tau > 0$, $\tau < R$, such that $\{z : |z - \zeta_0| \leq \tau\} \subset L_m$ for $m = 1, 2, \dots$. Let $\sum_{j=0}^{+\infty} b_j^{(2)}(\zeta_0)(z - \zeta_0)^j$ to be the Taylor development of g_2 with center ζ_0 . By (25) and Cauchy's estimates we have

$$|b_j^{(2)}(\zeta_0)| \leq \frac{\varepsilon_2}{\tau^j}, \quad j = 0, 1, 2, \dots \quad (26)$$

By (26) we have

$$\sup_{z \in K_1} |S_{N_1}(g_2, \zeta_0)(z)| \leq \varepsilon_2(N_1 + 1) \left(\frac{R}{\tau}\right)^{N_1}. \quad (27)$$

By (22), (27) and the triangle inequality we have

$$\sup_{z \in K_1} |S_{N_1}(g_1 + g_2, \zeta_0)(z) - f_1(z)| \leq 1 + \varepsilon_2(N_1 + 1) \left(\frac{R}{\tau}\right)^{N_1}. \quad (28)$$

Because $K_2 \subset D(\zeta_0, R)$ we find $N_2 > N_1$ so that

$$\sup_{z \in K_2} |S_{N_2}(g_1 + g_2, \zeta_0)(z) - (g_1 + g_2)(z)| < \frac{1}{2^2}. \quad (29)$$

By (24), (29) and the triangle inequality we get

$$\sup_{z \in K_2} |S_{N_2}(g_1 + g_2, \zeta_0)(z) - f_2(z)| < \frac{1}{2}. \quad (30)$$

By induction in the step l we will have constructed $g_1, \dots, g_{l-1}$, $N_1 < N_2 < \dots < N_{l-1}$ and $\varepsilon_2, \dots, \varepsilon_{l-1}$ positive. Let ε_l to be defined later. By Runge's Theorem we find a rational function g_l , $g_l(\infty) = 0$ with poles in W so that

$$\sup_{z \in K_l} \left| \sum_{i=1}^l g_i(z) - f_l(z) \right| < \frac{1}{2^l}, \quad (31)$$

and

$$\sup_{z \in L_l} |g_l(z)| < \varepsilon_l, \quad (32)$$

where ε_l will be chosen later on. If $\sum_{j=0}^{+\infty} b_j^{(l)}(\zeta_0)(z - \zeta_0)^j$ is the Taylor development of g_l with center ζ_0 , by Cauchy's estimates we have

$$|b_j^{(l)}(\zeta_0)| \leq \frac{\varepsilon_l}{\tau^j}, \quad j = 0, 1, 2, \dots \quad (33)$$

For $i = 1, \dots, l-1$, we find

$$\sup_{z \in K_i} |S_{N_i}(\sum_{k=1}^l g_k, \zeta_0)(z) - f_i(z)| \leq \frac{1}{2^{i-1}} + (\sum_{k=i+1}^l \varepsilon_k)(N_i + 1) \left(\frac{R}{\tau}\right)^{N_i}. \quad (34)$$

Next we choose $N_l > N_{l-1}$ so that

$$\sup_{z \in K_l} |S_{N_l}(\sum_{k=1}^l g_k, \zeta_0)(z) - \sum_{k=1}^l g_k(z)| < \frac{1}{2^l}. \quad (35)$$

Combining the triangle inequality with (31) and (35), we get

$$\sup_{z \in K_l} |S_{N_l}(\sum_{k=1}^l g_k, \zeta_0)(z) - f_l(z)| < \frac{1}{2^{l-1}}. \quad (36)$$

That is (34) is also valid for $i = l$. Finally we set

$$\varepsilon_i = \frac{1}{2^i} \left(\frac{\tau}{R} \right)^{N_{i-1}} \frac{1}{N_{i-1} + 1}, \quad i = 2, 3, \dots \quad (37)$$

We note that the choice of ε_i at the i^{th} step depends on τ, R which are fixed and N_{i-1} which has been defined in the step of order $i - 1$. Because $0 < \frac{\tau}{R} < 1$ we find

$$\sum_{i=m+1}^{+\infty} \varepsilon_i \leq \frac{1}{2^m} \left(\frac{\tau}{R} \right)^{N_m} \frac{1}{N_m + 1}, \quad (38)$$

for $m = 1, 2, \dots$. According to (32), (37) and (38) the series $\sum_{j=1}^{+\infty} g_j$ converges uniformly on compact subsets of Ω and defines a function $f \in H(\Omega)$. Relation (31) gives (i) of Proposition 4.1. Now let $K' \subset K - \{z : |z - \zeta_0| = R\}$ be a compact set with K'^c connected. Using now the fact that the sequence $K_m, m = 1, 2, \dots$ is increasing, the sequence $f_j, j = 1, 2, \dots$, is dense in $A(K')$, the relations (34), (38) and that the linear operators of partial sums S_N are continuous, we take the desired conclusion (ii) of Proposition 4.1. $\square$

Remark 4.2 We note that we use the proof of Runge's Theorem of [22] in the above proposition in the cases where the set K has a "good" boundary. More precisely we use Theorem 3.1 (page 81) and the lemma in page 83 of Volume III [22]. The same remark holds for Proposition 4.4 below.

In a similar way, we can prove the following proposition.

Proposition 4.3 *Let $K = \cup_{i \in \mathbb{N}} K_i, \Omega = \mathbb{C} - K$ be as in Proposition 4.1. Then there exist points $(a_j^{(i)})_{j=1,2,\dots}$ with $i = 1, \dots, \mu_j$, where $\mu_j \in \mathbb{N}$, for all j , such that $a_j^{(i)} \in \Omega$ and $|a_j^{(i)} - \zeta_0| \rightarrow R$ as $j \rightarrow +\infty$ for all i and rational functions $g_j, j = 1, 2, \dots$, with poles $(a_j^{(i)})_{i=1,\dots,\mu_j}, g_j(\infty) = 0$, so that the series $\sum_{j=1}^{+\infty} g_j$ defines a meromorphic function f in Ω and the following hold:*

- (i) *the sequence $\sum_{j=1}^N g_j, N \in \mathbb{N}$, is dense in $A(K)$,*
- (ii) *the sequence $S_N(f, \zeta_0)(\cdot), N \in \mathbb{N}$, is dense in $A(K)$.*

Our last statement is the following.

Proposition 4.4 *Let $K = \cup_{i \in \mathbb{N}} K_i, \Omega = \mathbb{C} - K$ be as above. Then there exist points $(a_j^{(i)})_{j=1,2,\dots} \in \Omega$ with $i = 1, \dots, \mu_j$, where $\mu_j \in \mathbb{N}$, for all j , such that $a_j^{(i)} \notin \{z \in \mathbb{C} : |z - \zeta_0| \leq R\}$ and $\text{dist}(a_j^{(i)}, K) < \frac{1}{j}$ for $j = 1, 2, \dots$ and $i = 1, \dots, \mu_j$ and rational*

functions g_j with poles in the set $A = \{(a_j^{(i)}), j = 1, 2, \dots, i = 1, \dots, \mu_j\}$ and $g_j(\infty) = 0$, so that the sequence g_N , $N = 1, 2, \dots$, converges to a holomorphic function $f \in H(\Omega)$ uniformly on compact subsets of Ω and the following hold:

- (i) the sequence g_N , $N \in \mathbb{N}$, is dense in $A(K)$,
- (ii) the sequence $S_N(f, \zeta_0)(\cdot)$, $N \in \mathbb{N}$, is dense in $A(K)$.

Proof. Let L_m , $m = 1, 2, 3, \dots$, be an exhausting sequence of compact subsets of Ω , with $\zeta_0 \in \overset{\circ}{L}_1$. Let f_j , $j = 1, 2, \dots$, be an enumeration of all polynomials of one complex variable with coefficients in $\mathbb{Q} + i\mathbb{Q}$. We can find two sequences of open and bounded sets $G_1^{(m)}, G_2^{(m)}$, $m = 1, 2, \dots$, such that $G_1^{(m)} \cap G_2^{(m)} = \emptyset$, for $m = 1, 2, \dots$, with some good properties. More specifically we assume that $K \subset G_2^{(m)}$ and $L_m \subset G_1^{(m)}$, $m = 1, 2, \dots$, the sets $G_1^{(m)}, G_2^{(m)}, (G_1^{(m)})^c, (G_2^{(m)})^c$ contain a finite number of connected components, as well as the sets $(G_1^{(m)} \cup G_2^{(m)})^c$, $m = 1, 2, \dots$ ([22], page 80, Vol. III). We can take the sets $G_1^{(m)}, G_2^{(m)}$, $m = 1, 2, \dots$, such that there exist points $(a_j^{(i)})_{j=1,2,\dots}$ with $i = 1, \dots, \mu_j$ where $\mu_j \in \mathbb{N}$, for all j , such that $a_j^{(i)} \notin \{z \in \mathbb{C} : |z - \zeta_0| \leq R\}$ and $\text{dist}(a_j^{(i)}, K) < \frac{1}{j}$ for $j = 1, 2, \dots$ and $i = 1, \dots, \mu_j$ and for every $m = 1, 2, \dots$ each one of the complex numbers $a_m^{(i)}$, $i = 1, \dots, \mu_m$, belong to each connected component of the set $(G_1^{(m)} \cup G_2^{(m)})^c$. That is, we assume that the set $(G_1^{(m)} \cup G_2^{(m)})^c$ has μ_m bounded connected components for $m = 1, 2, \dots$.

First step: We consider the function $F_1 : G_1^{(1)} \cup G_2^{(1)} \rightarrow \mathbb{C}$ defined by

$$F_1(z) = \begin{cases} f_1(z) & \text{if } z \in G_2^{(1)}, \\ 0 & \text{if } z \in G_1^{(1)}. \end{cases}$$

By Runge's Theorem we find a rational function with poles in the set $\{a_1^{(1)}, a_1^{(2)}, \dots, a_1^{(\mu_1)}\}$ such that

$$\sup_{z \in K} |g_1(z) - f_1(z)| < \frac{1}{2^2}, \quad (39)$$

$$\sup_{z \in L_1} |g_1(z)| < 1 \quad (40)$$

and $g_1(\infty) = 0$. Let $N_1 \in \mathbb{N}$ be such that

$$\sup_{z \in K} |S_{N_1}(g_1, \zeta_0)(z) - g_1(z)| < \frac{1}{2^2}. \quad (41)$$

By (39) and (41) we have

$$\sup_{z \in K} |S_{N_1}(g_1, \zeta_0)(z) - f_1(z)| < \frac{1}{2}. \quad (42)$$

Second step: Let ε_2 to be defined later. We consider the function $F_1 : G_1^{(2)} \cup G_2^{(2)} \rightarrow \mathbb{C}$ defined by

$$F_2(z) = \begin{cases} f_2(z) & \text{if } z \in G_2^{(2)}, \\ g_1(z) & \text{if } z \in G_1^{(2)}. \end{cases}$$

By Runge's Theorem we find a rational function with poles only in $\{a_2^{(1)}, a_2^{(2)}, \dots, a_2^{(\mu_2)}\}$ such that

$$\sup_{z \in K} |g_2(z) - f_2(z)| < \frac{1}{2^3}, \quad (43)$$

$$\sup_{z \in L_1} |g_2(z) - g_1(z)| < \varepsilon_2 \quad (44)$$

and $g_2(\infty) = 0$. We find a natural number $N_2 > N_1$ such that

$$\sup_{z \in \overline{D}(\zeta_0, R)} |S_{N_2}(g_2, \zeta_0)(z) - g_2(z)| < \frac{1}{2^3}. \quad (45)$$

Since $K \subset \overline{D}(\zeta_0, R)$, by (43) and (45) we obtain

$$\sup_{z \in K} |S_{N_2}(g_2, \zeta_0)(z) - f_2(z)| < \frac{1}{2^2}. \quad (46)$$

Since $\zeta_0 \in \overset{\circ}{L}_1$ we choose a positive number $\tau_0 \in (0, R)$ such that $\overline{D}(\zeta_0, \tau_0) \subset L_1 \subset L_m$ for $m = 1, 2, \dots$. Let $b_j^{(2)}(\zeta_0)$, $j = 0, 1, 2, \dots$, be the coefficients of the Taylor development of $g_2 - g_1$ about ζ_0 . By Cauchy's estimates we get

$$|b_j^{(2)}(\zeta_0)| \leq \frac{\varepsilon_2}{\tau_0^j}, \quad j = 0, 1, 2, \dots \quad (47)$$

By (47) we have

$$\sup_{z \in K} |S_{N_1}(g_2, \zeta_0)(z) - S_{N_1}(g_1, \zeta_0)(z)| < \varepsilon_2(N_1 + 1) \left(\frac{R}{\tau_0} \right)^{N_1}. \quad (48)$$

By (42) and (48) we have

$$\sup_{z \in K} |S_{N_1}(g_2, \zeta_0)(z) - f_1(z)| < \varepsilon_2(N_1 + 1) \left(\frac{R}{\tau_0} \right)^{N_1} + \frac{1}{2}. \quad (49)$$

By induction in the step l we will have constructed rational functions $g_1, \dots, g_{l-1}$ and natural numbers $N_1 < N_2 < \dots < N_{l-1}$, and positive numbers $\varepsilon_2, \dots, \varepsilon_{l-1}$ have been used. Let $\varepsilon > 0$ to be defined later. By Runge's Theorem we find a rational function g_l , $g_l(\infty) = 0$, with poles in $\{a_l^{(1)}, a_l^{(2)}, \dots, a_l^{(\mu_l)}\}$ such that

$$\sup_{z \in K} |g_l(z) - f_l(z)| < \frac{1}{2^{l+1}}, \quad (50)$$

$$\sup_{z \in L_{l-1}} |g_l(z) - g_{l-1}(z)| < \varepsilon_l. \quad (51)$$

If we write $\sum_{j=0}^{+\infty} b_j^{(l)}(\zeta_0)(z - \zeta_0)^j$ the Taylor development of $g_l - g_{l-1}$ with center ζ_0 , then Cauchy's estimates imply

$$|b_j^{(l)}(\zeta_0)| \leq \frac{\varepsilon_l}{\tau_0^j}, \quad j = 0, 1, 2, \dots \quad (52)$$

For $j = 1, \dots, l-1$, we have

$$\sup_{z \in K} |S_{N_j}(g_l, \zeta_0)(z) - f_j(z)| < \left(\sum_{k=j+1}^l \varepsilon_k \right) (N_j + 1) \left(\frac{R}{\tau_0} \right)^{N_j} + \frac{1}{2^j}. \quad (53)$$

For $j = l$, we choose $N_l > N_{l-1}$ so that

$$\sup_{z \in \overline{D}(\zeta_0, R)} |S_{N_l}(g_l, \zeta_0)(z) - g_l(z)| < \frac{1}{2^{l+1}}. \quad (54)$$

The triangle inequality with (50) imply

$$\sup_{z \in K} |S_{N_l}(g_l, \zeta_0)(z) - f_l(z)| < \frac{1}{2^l}. \quad (55)$$

We set

$$\varepsilon_j = \frac{1}{2^j} \left(\frac{\tau}{R} \right)^{N_{j-1}} \frac{1}{N_{j-1} + 1} < \frac{1}{2^j}. \quad (56)$$

Combining the inequalities (51), (55) and (56) we derive that the series $g_1 + \sum_{j=1}^{+\infty} (g_{j+1} - g_j) = g_l + \sum_{j=l}^{+\infty} (g_{j+1} - g_j)$, for all $l \geq 1$, converge uniformly on each compact subset of Ω towards a function $f \in H(\Omega)$. We also have

$$\sum_{i=m+1}^{+\infty} \varepsilon_i \leq \frac{1}{2^m} \left(\frac{\tau}{R} \right)^{N_m} \frac{1}{N_m + 1}. \quad (57)$$

By (50) we get the first conclusion (i) of this Proposition 4.4. Combining (53) and (57) we obtain

$$\sup_{z \in K} |S_{N_j}(f, \zeta_0)(z) - f_j(z)| \leq \frac{1}{2^{j-1}}. \quad (58)$$

This gives (ii). $\square$

Remark 4.5 The way of proving Propositions 4.1, 4.3, 4.4 presented here allows the proof of the main result of [34] without the use of the topological Lemma in [34].

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