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FINDING THE DISTRIBUTION OF BRIDGE LIFETIME LOAD EFFECT BY PREDICTIVE LIKELIHOOD

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ABSTRACT

To assess the safety of an existing bridge, the loads to which it may be subject in its lifetime are required. Statistical analysis is used to extrapolate a sample of load effect values from the simulation period to the required design period. Complex statistical methods are often used and the end result is usually a single value of characteristic load effect. Such a deterministic result is at odds with the underlying stochastic nature of the problem. In this paper, predictive likelihood is shown to be a method by which the distribution of the lifetime extreme load effect may be determined. A basic application to the prediction of lifetime Gross vehicle Weight (GVW) is given. Results are also presented for some cases of bridge loading, compared to a return period approach and important differences are identified. The implications for the assessment of existing bridges are discussed.

NOMENCLATURE

CDF	- Cumulative Density Function;
CDS	- Composite Distribution Statistics;
f_j	- Relative frequency of occurrence of mode *j*;
$F_j(\cdot)$	- Parent CDF of GVW for mode *j*;
$f_{L_p}(z;y)$	- Predictand distribution;
$G(y;\theta)$	- GEV CDF;
$g(y;\theta)$	- GEV PDF;
$G_C(\cdot)$	- Composite CDF for *N* different types of loading event;
$g_C(y)$	- Composite PDF;
GEV	- Generalized Extreme Value distribution;
GVW	- Gross Vehicle Weight;
$L_{MP}(z \mid y)$	- modified profile predictive likelihood;
$l_{MP}(z \mid y)$	- log of $L_{MP}(z \mid y)$;
$L_P(z \mid y;\theta_z)$	- Fisherian predictive likelihood;
$L_y(\theta;y)$	- Likelihood function of the data;
$l_y(\theta;y)$	- Log-likelihood of data;
$L_z(\theta;z)$	- Predictand likelihood function;
n_d	- Expected number of daily events;
PDF	- Probability Density Function;
PL	- Predictive Likelihood;
W	- Daily maximum GVW;
$y,\ y_i$	- Data vector and data point;
θ	- GEV Parameter vector determined from data alone;
θ_z	- GEV parameter vector determined jointly from data and predictand;
μ	- GEV location parameter;
σ	- GEV scale parameter;
ξ	- GEV shape parameter;
$\sqrt{\lvert I\ (\theta_z)\rvert}$	- The square root of the absolute determinant of the Fisher information matrix;
$\lvert\partial\theta_z/\partial\theta\rvert$	- Parameter transformation constant;

1. INTRODUCTION

To assess the safety of structures, it is necessary to have estimates for the load or load effect to which it is subject. Statistical approaches are commonly adopted as the tools through which loads with an acceptably small probability of occurrence are determined. The assessment of existing bridges is a particular case when such analyses are very useful. In general, it is particularly expensive to repair or replace deteriorated bridges due to the cost of the new structure, disruption to traffic and the cost of resulting delays. Large savings may be made by proving that many bridges are safe without intervention and statistical analysis of bridge loading facilitates this.

Loading data is found through measurement or simulation, as is the usual case with site-specific bridge loading. An extreme value theory form of analysis (be it block-maxima or threshold based) is performed on these results and used to estimate the load effect with the acceptably small probability of occurrence. For example, the Eurocode for bridge loading [1] defines this to be 10% probability of exceedance in 100 years, usually expressed as a 1000-year return period.

The idea that a single value of load effect may represent the load effects that can occur at a structure's lifetime is flawed. The inherent variability of traffic loading means that, in general, different samples of load effect would result in different characteristic values. Of course, there must be some particular value of load effect which has a 10% probability of exceedance in 100 years (for example), but such a value needs to be derived from a distribution which takes into account many sources of variability. Various methods exist in the statistical literature for calculating such distributions – the delta method [2] and bootstrapping [3] being two. However predictive likelihood has advantages over these as it accounts for more sources of variability.

In this paper, the authors present an application of predictive likelihood [4] to the problem of estimating a bridge lifetime-maximum load effect distribution. This approach provides more information from the given data as it gives, not just an estimate of the lifetime-maximum effect, but also the nature of its variability.

2. LOAD EFFECT PREDICTION

2.1 CONVENTIONAL PREDICTION

A basic yet practical example is used to illustrate the proposed method. A representative tri-modal distribution of GVW is specified in Table 1. It is taken that there are $n_d = 2000$ occurrences of trucks per day. The distribution of the daily maximum GVW is given by [5]:

$$P[W \le w] = \left(\sum_{j=1}^{3} F_j(w) \cdot f_j \right)^{n_d} \qquad (1)$$

Using the Eurocode definition of design life as 100 years, the distribution of lifetime maximum GVW is given by:

$$P[W \le w] = \left(\sum_{j=1}^{3} F_j(w) \cdot f_j \right)^{25000 n_d} \qquad (2)$$

In which it is taken that there are 250 working days per year. Both distributions (1) and (2) are given figuratively further on.

Table 1. GVW distribution properties

Mode	Weight	Mean	Standard Deviation
1	0.246	14.83	1.722
2	0.485	25.01	6.998
3	0.269	39.07	2.888

Using Monte Carlo simulation 1000 sample observations of daily maximum GVW are obtained. Following conventional statistical analysis, this sample is fit using the Generalized Extreme Value (GEV) distribution [2]:

$$G(y;\theta) = \exp\left\{ -\left[1 - \xi\left(\frac{y-\mu}{\sigma} \right) \right]_+^{1/\xi} \right\} \qquad (3)$$

where $[h]_+ = \max(h, 0)$. The probability density function (PDF) is:

$$g(y;\theta) = G(y;\theta) \cdot \sigma^{-1} \left\{ 1 + \xi\left(\frac{y-\mu}{\sigma} \right) \right\}^{-1/\xi-1} \qquad (4)$$

Maximum likelihood estimation is used, based on the log-likelihood function for the GEV distribution, given by [2].

This model is then used to extrapolate to a return period of 1000 years to obtain the characteristic value. That is, the value that has 10% probability of exceedance in 100 years. This process is shown in Figure 1 and the result obtained is 69.46 tonnes.

The process described for GVW is similar for conventional load effect prediction. Usually, for a particular site, Monte Carlo simulation of statistically modelled traffic is carried out for a bridge(s) and load effect(s) of interest. This data is then analysed similarly to the GVW data.

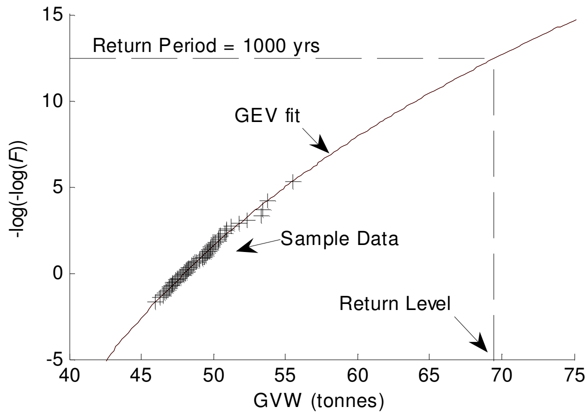

Figure 1. Extrapolation of GVW sample.

2.2 BRIDGE LOAD PREDICTION

Authors have used many different methods to predict the lifetime bridge load effect from measured or simulated load effect data. In many studies by Nowak and others [6]–[9] straight lines are superimposed on the tails of the distributions and extrapolated to determine the characteristic load effect values. In other studies by Nowak [10]–[11], curved lines on normal probability paper are used for the extrapolation. Based on measured traffic samples [12]–[13] consider and compare several methods of extrapolation of the basic histogram of load effect. Grave et al [14] use a weighted least-squares approach to fit Weibull distributions to load effect values. This process is repeated to give an estimate of the distribution of characteristic values. These authors use the upper 2√n data points as recommended by Castillo [15] for data that may not be convergent to an extreme value population. Bailey and Bez [16]–[17] determine that the Weibull distribution is most appropriate to model the tails of the load effect distributions and used maximum likelihood estimation. Cooper [18]–[19] uses measured truck loading events to determine the distribution of load effect. Cooper raises this distribution to a power to establish the distribution of the maximum load effect from 4.5 days of traffic. This is fit with a Gumbel distribution which is used to extrapolate to a 2400 year return period. Crespo-Minguillón and Casas [20] adopt a Peaks-Over-Threshold approach and use the Generalized Pareto Distribution to model the exceedances of weekly maximum traffic effects over a certain threshold. An optimal threshold is selected based on the overall minimum least-squares value and it is the distribution that corresponds to this threshold that is used as the basis for extrapolation.

It is clear that a wide range of methods are used in the literature, and that the variability of the characteristic load effect is not generally assessed.

3. PREDICTIVE LIKELIHOOD

3.1 DESCRIPTION

Parametric statistical inference on a set of observations requires the selection of a statistical model and estimation of the parameters of that model. For a given model, there are many possible parameter vectors, θ, representing many possible distributions. Using the maximum likelihood estimator, the most likely distribution, $\hat{\theta}$, given the data, *y*, is that which maximizes the likelihood function. From this parameter vector, the maximum likelihood estimate of the characteristic value, *z* (the predictand), is identified for a given probability level. Predictive likelihood, on the other hand, finds the most likely distribution, given both the data and a postulated predictand. It does this by maximising the likelihood functions of the data, L_y, and the predictand, L_z, jointly:

$$L_P\left(z \mid y\right) = \sup_{\theta} L_y\left(\theta; y\right) L_z\left(\theta; z\right) \qquad (5)$$

Equation (5) is termed Fisherian predictive likelihood after [21].

3.2 THEORY

The likelihood function for the data vector, *y* is:

$$L_y\left(\theta; y\right) = \prod_{i=1}^{n} g\left(y_i; \theta\right) \qquad (6)$$

For a postulated value of *z*, and denoting the PDF of the predictand by $g_z(\cdot)$, the likelihood function is:

$$L_z\left(\theta; z\right) = g_z\left(z; \theta\right) \qquad (7)$$

as there is only a single value, z. Similarly to maximum likelihood estimation, it is easier to use the log-likelihoods – maximization of this function is equivalent to maximization of the likelihood function itself. Therefore, equations (5), (6) and (7) are written as:

$$\begin{aligned}&\log\left[L_P\left(z \mid y\right)\right] \\ &\quad=\sup_{\theta}\left\{\log\left[L_y\left(\theta;y\right)\right]+\log\left[L_z\left(\theta;z\right)\right]\right\} \quad (8) \\ &\quad=\sup_{\theta}\left\{\sum_{i=1}^{n}\log\left[g(y_i;\theta)\right]+\log\left[g_z(z;\theta)\right]\right\}\end{aligned}$$

For a given predictand (at a certain probability level), the joint likelihood of both the data and predictand is maximized. By repeating the process for a range of alternative predictands, a range of distributions are found. An example is illustrated in Figure 2. A random data sample from a GEV distribution with parameter vector $\theta=(300,20,0.1)$ is fit using maximum likelihood estimation. This is shown as the solid black line. The predictive likelihood values for ten values of predictand are also shown. For each of the predictive likelihood maximizations, the ten GEV fits to the data are also shown in the figure. It is to be noted that these distributions are not 'forced' to go through the predictand as the distribution results both from the data and the predictand.

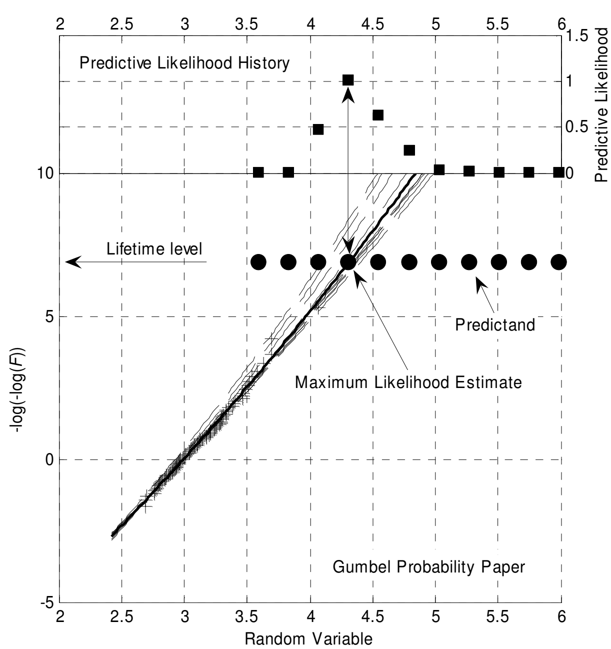

Figure 2. Sample predictive likelihood analysis.

The value of this approach is that additional information is available: for each predictand, the maximized predictive likelihood value is available from equation (8). The distribution of these values for each predictand represents a distribution of predictand given the data; the curve $\{L_p, z\}$, denoted $f^*_{L_P}(z;y)$. The area under this curve is normalized to unity to obtain the predictand distribution – $f_{L_P}(z;y)$ – shown in Figure 2. It can be seen from this figure that the most likely value of the predictand from the predictive likelihood distribution (its mode) coincides, as may be expected, with the maximum likelihood estimate of the predictand.

3.2 MODIFIED PREDICTIVE LIKELIHOOD

Mathiasen [22] notes some problems with Fisherian predictive likelihood. Of particular relevance to this work is that each function maximization does not account for the variability of the derived parameter vector, θ.

Many forms of predictive likelihood have been proposed in the literature to overcome the problems associated with the Fisherian formulation. In this work, the predictive likelihood method proposed by Butler [23], based on that of Fisher [21] and Mathiasen [22] and also considered by Bjørnstad [4], is used. Lindsey [24] describes the reasoning behind its development.

Two modifications are required to the Fisherian formulation for general applicability. The first accounts for the confidence in each parameter vector for each predictand; the second is a constant required to transform the problem into the correct domain. In particular, the square root of determinant of the Fisher information matrix, $\sqrt{\left|I\ \left(\theta_z\right)\right|}$, (the Hessian matrix of the likelihood function) represents the confidence (information) about the parameter values. It is an inverse relationship: larger determinants represent less information and vice versa. The parameter transform modification is required so that the problem is in the domain of the 'free' parameter vector, θ, which is reliant only upon the data. Thomasian [25] provides further information on parameter transformations. That which is relevant here is $\left|\partial\theta_z/\partial\theta\right|$.

Allowing for these modifications to the Fisherian predictive likelihood, the modified profile predictive likelihood (L_{MP}) is given as:

$$L_{MP}\left(z \mid y\right)=\frac{L_P\left(z \mid y;\theta_z\right)}{\left|\frac{\partial\theta_z}{\partial\theta}\right|\sqrt{\left|I\ \ (\theta_z)\right|}} \quad (9)$$

Butler [23] points out that the parameter transform $\left|\partial\theta_z/\partial\theta\right|$ is constant. Therefore normalization of the area under $L_P\left(z \mid y;\theta_z\right)\Big/\sqrt{\left|I\ \ (\theta_z)\right|}$ amounts to evaluation of $\left|\partial\theta_z/\partial\theta\right|$ and hence $L_{MP}\left(z \mid y\right)$ yields the predictive density of the predictand, $f_{L_P}\left(z;y\right)$.

3.3 BRIDGE TRAFFIC LOAD EFFECT FORMULATION

Caprani et al [26] have shown that bridge load effects are caused by a mixture of different types of loading event such as 1-truck and 2-truck loading events. For N different types of loading event, the composite distribution, $G_C\left(\cdot\right)$, of daily maximum load effect is given by:

$$\begin{aligned}G_C\left(y\right)&=\prod_{j=1}^{N}G_j\left(y\right)\\&=\exp\left\{-\sum_{j=1}^{N}\left[1-\xi_j\left(\frac{y-\mu_j}{\sigma_j}\right)\right]^{1/\xi_j}\right\}\end{aligned} \quad (10)$$

where $G_j\left(\cdot\right)$ is the distribution of load effect caused by loading event type j. The composite probability density function, $g_C\left(y\right)$, is evaluated numerically in this work.

The likelihood of the data for the CDS distribution is defined in this work to be the combined likelihood of each of the mechanisms of the CDS distribution:

$$\begin{aligned}&\log\left[L_y\left(\theta;y\right)\right]\\&\quad=l_y\left(\theta;y\right)\\&\quad=\sum_{j=1}^{N}\left\{\sum_{i=1}^{n_j}\log\left[g_j\left(\theta_j;y_{j,i}\right)\right]\right\}\end{aligned} \quad (11)$$

where n_j is the number of data points for each event type; $y_{j,i}$ is the ith data point of event type j, and; $\theta_j=\left(\mu_j,\sigma_j,\xi_j\right)$ is the parameter vector for each $G_j\left(\cdot\right)$. The distribution of a maximum of m sample repetitions, $G_{Z,C}\left(\cdot\right)$, is defined as [27]:

$$\begin{aligned}G_{Z,C}\left(z\right)&=\left[G_C\left(z\right)\right]^m\\g_{Z,C}\left(z\right)&=m\cdot g_C\left(z\right)\cdot\left[G_C\left(z\right)\right]^{m-1}\end{aligned} \quad (12)$$

Therefore, the likelihood of the predictand, given the initial distribution is:

$$\begin{aligned}\log\left[L_z\left(\theta;z\right)\right]&=\log\left[g_{Z,C}\left(z\right)\right]\\&=\log\left\{m\cdot g_C\left(z\right)\cdot\left[G_C\left(z\right)\right]^{m-1}\right\}\end{aligned} \quad (13)$$

Thus the distributions required for use in the predictive likelihood approach have been defined with consideration to the underlying stochastic process.

3.4 ESTABLISHING THE PREDICTIVE DISTRIBUTION

Curves of log predictive likelihood are used to determine the predictive distribution, $f_{L_P}\left(z;y\right)$. Firstly, the log predictive likelihoods are defined:

$$l_{MP}\left(z \mid y\right)=\log\left[L_{MP}\left(z \mid y\right)\right] \quad (14)$$

and its maximum value is defined as:

$$\hat{l}_{MP}\left(z \mid y\right)=\sup_z\left\{\log\left[L_{MP}\left(z \mid y\right)\right]\right\} \quad (15)$$

The curve of likelihood ratios is determined as:

$$f_{L_P}^{*}\left(z;y\right)=\exp\left\{l_{MP}\left(z \mid y\right)-\hat{l}_{MP}\left(z \mid y\right)\right\} \quad (16)$$

This curve is then normalized to the predictive distribution:

$$f_{L_P}\left(z;y\right)=\frac{f_{L_P}^{*}\left(z;y\right)}{\int f_{L_P}^{*}\left(z;y\right)} \tag{17}$$

Save for Davison [28], the statistical literature on predictive likelihood does not generally consider its implementation. Numerical instability is a feature of predictive likelihood function maximization; the details of the algorithm used to address these problems is given elsewhere [5].

4. APPLICATION

4.1 GVW EXAMPLE

The GVW example is based upon the daily maximum observed GVWs. Following conventional procedure, it is considered that this distribution is not a mixture distribution and this simplifies the application significantly. Figure shows the results of the application of predictive likelihood to the generated data set. Also shown is the result of the exact analysis of equation (2).

It is interesting to note that Mode 2 governs the lifetime maximum GVW distribution. This is because of its larger variance. However, it is usually assumed that it is the trucks of Mode 3 that govern. Of significance is that even though the data is mixed, predictive likelihood has determined the governing mode from a single sample of data, and has approximated it quite well. In judging the quality of the match it is important to note that the conventional extrapolation approach only returns a single number.

The 90-percentile of the predictive likelihood distribution, gives the characteristic value (by the Eurocode) as 69.13 tonnes. This is similar to the GEV extrapolated value (69.46 tonnes) obtained earlier. This similarity is not general, however. The exact characteristic value, determined from equation (3) is 65.20. The overestimation of both the conventional and predictive likelihood approaches – Figure 4 – is due to their neglect of the mixture in the underlying distribution.

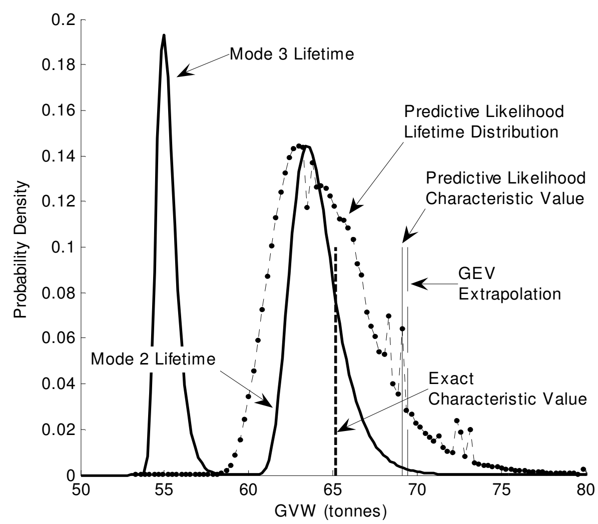

Figure 4. Enlarged upper mode of Figure 3.

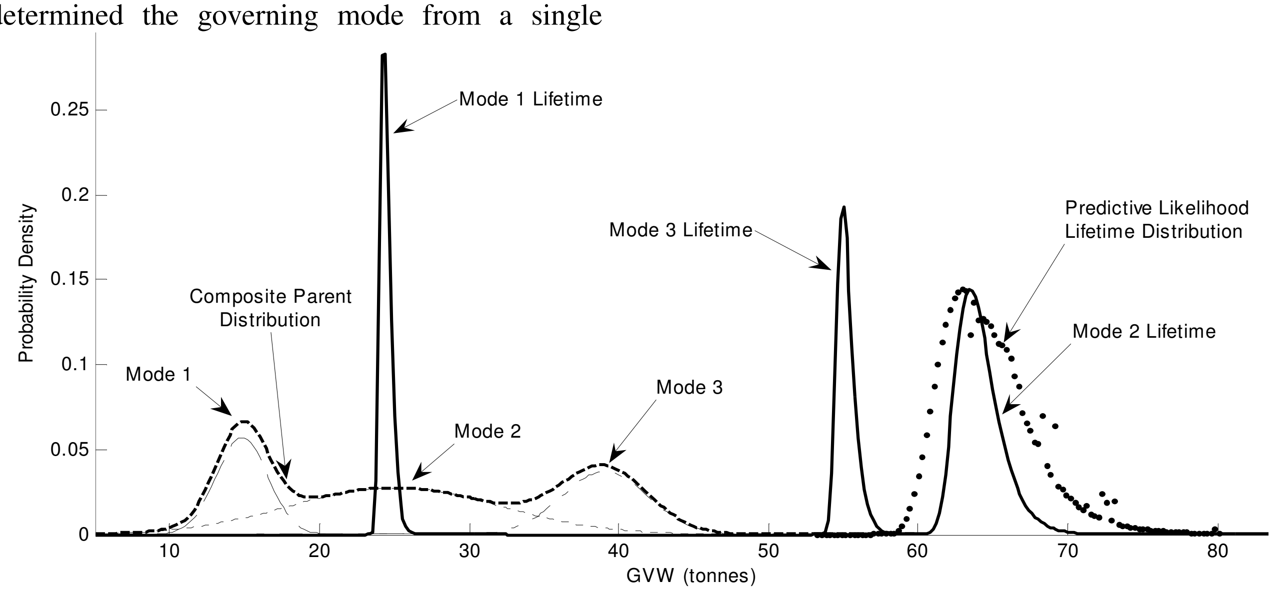

Figure 3. GVW example predictive likelihood and exact analysis.

4.2 PRACTICAL APPLICATION

Weigh-In-Motion data, taken from the A6 motorway near Auxerre, France, is used to assess the implications of predictive likelihood on the estimation of characteristic bridge traffic loading. Weight and dimensional data were collected for 36 373 trucks travelling in the two slow lanes of the 4-lane motorway. The statistical models of the traffic characteristics were used in Monte-Carlo simulations of traffic at the measured site. The distribution for headways, in particular, is known to be important and is modelled as described by OBrien and Caprani [29].

A 1000-day sample period of two-lane bi-directional truck traffic is generated and the resulting load effects are determined for bridge lengths in the range 20 m to 50 m. The particular load effects considered are:

- Load Effect 1: Bending moment at the mid-span of a simply supported bridge;
- Load Effect 2: Left support shear in a simply-supported bridge;
- Load Effect 3: Bending moment at central support of a two-span continuous bridge.

To minimize computing requirements only significant crossing events were processed and are defined as multiple-truck presence events and single truck events with Gross Vehicle Weight (GVW) in excess of 40 tonnes. When a significant crossing event is identified, the comprising truck(s) are moved in 0.02 second intervals across the bridge and the maximum load effects of interest for the event identified.

The load effects resulting from the 1000-day simulation of Auxerre traffic are analysed using predictive likelihood and the results are given in Table 2. In general the information matrices exhibited considerable numerical instability and so the modification for parameter variability is not made to the results presented. In any case, this modification is found to be generally slight [5].

Table 2. Table of predictive likelihood and conventional results

Load Effect	Bridge Length (m)	Characteristic Load Effect			Percentage difference[a]	
		PL[b]	GEV[c]	CDS[d]	GEV	CDS
1 (kNm)	20	4074	4073	4067	0.0	-0.2
	30	7830	7827	7852	0.0	0.3
	40	10814	10801	10701	-0.1	-1.0
	50	14150	14173	13893	0.2	-1.8
2 (kNm)	20	1074	1074	1067	0.0	-0.6
	30	1636	1641	1643	0.3	0.4
	40	2841	2854	2921	0.5	2.8
	50	3825	3839	3785	0.4	-1.1
3 (kN)	20	927	926	922	-0.1	-0.6
	30	969	969	963	0.0	-0.6
	40	1153	1187	1079	2.9	-6.5
	50	1235	1253	1185	1.4	-4.0

[a] Relative to numerical PL results;

[b] 90-percentile of 100-year distribution based on predictive likelihood points;

[c] 90-percentile of 100-year distribution GEV fit to predictive likelihood points;

[d] 1000-year return level based on CDS extrapolation.

Sample predictive distributions of 100-year lifetime-maximum load effect are presented in Figures 5 and 6. Also shown is a GEV fit to the predictive distribution. The GEV distribution is reasonable as it is sufficiently flexible and by virtue of the stability postulate [27] is the exact form of distribution of the return level. Further, the load effect with 10% probability of exceedance in 100 years is indicated, both for the predictive likelihood points (PL RL) and the GEV fit to these points (GEV PL fit). Also given in each figure is the 1000-year maximum likelihood estimate of the return level (CDS RL), derived from the CDS distribution.

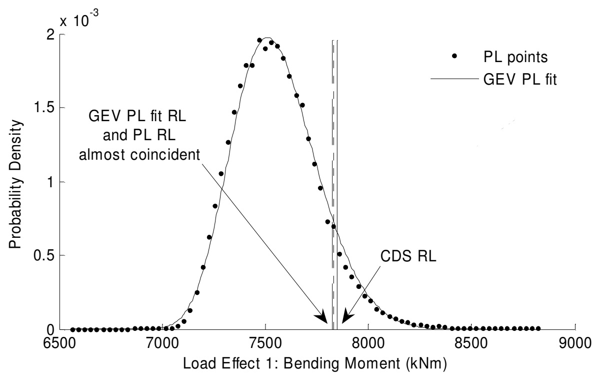

Figure 5: Characteristic load effect prediction for Load Effect 1, 30 m bridge length (see text for details).

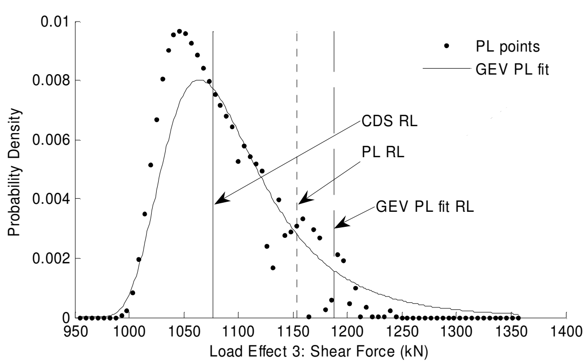

Figure 6: Characteristic load effect prediction for Load Effect 3, 40 m bridge length (see text for details).

Some of the GEV fits to the raw predictive likelihood points are not obtained through fully objective means. In such cases, the approach is to fit the upper tail more closely than either the lower tail or the mode. Due to the numerical nature of the predictive distributions themselves, such GEV fits may be considered as a smoothing process. In any case, the results have been derived from both the fits and the raw distributions and may be seen to be comparable from Table 2 – the maximum difference is about 3% for Load Effect 2, 40 m bridge length.

Comparison of the predictive likelihood results with the 1000-year CDS results are given in Figure 7. Of significance is the fact that the usual method of extrapolation to a 1000-year return period results in general non-conservative results (with the exception of Load Effect 2, 40 m bridge length), compared with either of the predictive likelihood-based results. However, the differences are not substantial. It may be surmised that the predictive likelihood results are closer to the actual lifetime load effect as more information is obtained from the sample. Thus more confidence in the characteristic value results from the use of predictive likelihood, compared with the usual extrapolation procedure.

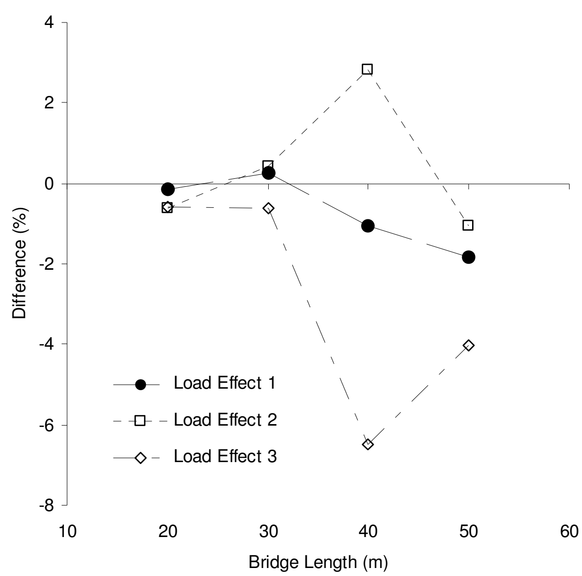

Figure 7: Differences in the CDS return period characteristic load effect prediction relative to the GEV fit to the predictive likelihood results.

Given the differences between the predictive likelihood result (100-year with 10% probability of exceedance) and the conventional CDS result (1000-year return period), it is apparent that these two definitions of probability level are no longer equivalent. This has implications for the specification of acceptable probabilities and the manner in which practitioners estimate the associated design levels.

5. CONCLUSIONS

The method of predictive likelihood is presented and applied to the bridge loading problem. An extension of predictive likelihood is presented which caters for composite distribution statistics problems. This method is applied to problems for which the results are known and the result found to be good. The method is then applied to the results of bridge load simulations. Predictive likelihood generally gives larger lifetime load effect values than the usual return period approach. This is as a result of inclusion of sources of variability within the predictive likelihood distribution. The differences in lifetime load effects are considerable, yet within reason, and are also dependent on the influence line and bridge length. This is to be expected from the physical nature of the problem.

The application of predictive likelihood is shown to require strict definition of acceptable safety levels, as the more usual return period definition does not yield the same results in general. This will have implications for practitioners and code definitions. Also, it is shown that in comparison to the return period approach, which generates a single predictand, the predictive likelihood distribution represents a considerable increase in the information gained from a sample. This increase in information represents more confidence about the result in comparison with the return period approach. Therefore predictive likelihood is a valuable tool in estimating distributions of extremes of stochastic processes.

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