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Abstract
This paper presents the potential of the cross-entropy method to surmise the properties of a simply supported beam using as input the response of the structure to a moving load. The beam model is discretised into a number of elementary beams with assumed initial statistical distributions of stiffness. Then, an optimisation procedure based on cross-entropy is employed to minimise differences between simulated measurements and the results of the theoretical finite element beam model. The procedure consists of generating a large sample of stiffness distributions for each elementary beam, and selecting those fitting the measured response best. Then, the parameters of the statistical distribution of stiffness assumed for each elementary beam (mean and standard deviation) are updated using the stiffness values of those combinations of elementary beams giving a best solution. It is an iterative procedure where the mean value of each distribution tends towards the true stiffness in successive iterations. The level of accuracy is limited by the quantity and quality of the available measurements. Therefore, the standard deviation of the final stiffness for each beam element (once further iterations do not lead to a reduction of the error) provides an estimation of the reliability of the prediction. Here, the method is demonstrated for the characterisation of the stiffness distribution of a beam from the simulated response to a moving load. First, deflections are calculated using a finite element beam model with assumed initial stiffness properties. There will be a record of simulated responses per measurement point that cross-entropy will try to imitate by adjusting and improving estimations of stiffness in successive iterations. The results show cross-entropy can be used as a valuable tool to estimate structural parameters and it has huge scope for applications in model calibration, bridge weigh-in-motion and monitoring.

Keywords: Finite-Element Updating, Cross-Entropy

1. Introduction
The assessment of the material properties of a bridge through site measurement, without dissecting the structure, is a vibrant field of study. These parameters can be used to model and assess/monitor the structure or for direct damage detection. The ability to measure deflections or strains on site and infer the materials flexural rigidity (EI) or density (ρ) would allow an accurate and calibrated finite element model of the structure to be built.

There exist many such non-destructive alternative methods being pursued, particularly in the area of assessing bridge parameters as a method of damage detection. For example, natural frequency based methods (Chen et al. 1995), mode shape based methods (Ewins 1984) and neural network based methods (Pandey and Barai 1995) to
mention a few approaches. For a comprehensive review of damage detection algorithms using dynamic measurements, see Carden and Fanning (2004).

Walsh and González (2009) propose using the Cross-Entropy (CE) method (Rubinstein and Kroese 2004) in conjunction with static measurements to obtain the distribution of flexural stiffness within a discretized Finite Element (FE) beam model. The CE method can simply be described as an iterative procedure composed of two steps (de Boer et al. 2005); firstly a random data sample is generated and secondly the method by which the random data is generated is altered so as to produce a better fitting sample in the next iteration. It is a form of ‘brute-force’ optimization algorithm, which takes advantage of modern computational capacity to generate many possible solutions, gradually converging to the most probable solution. This paper will use the CE method for determining those properties necessary to construct the stiffness and mass matrices of a FE model.

2. Beam Model

A FE model was programmed with Matlab (Mathworks Inc. 2005) to calculate the response of a simply supported beam to a forcing function. The beam contained a number, N, of beam elements; hence N+1 nodes; with each node having 2 Degrees-of-Freedom (DoF): rotation and vertical displacement. The DoFs of an 8m simply supported beam are shown in Figure 1 (The displacement of the vertical DoFs over the supports are set to zero for a simply-supported beam).

![Figure 1 – FE Beam depicting Degrees of Freedom](image)

The 8m beam depicted in Figure 1 is split into eight 1m long beam elements giving \((8+1) \times 2 = 18\) DoFs. This FE model can be used to calculate the static response to the applied load and to derive the stiffness matrix from the simulated measurements, as in the earlier work of Walsh and González (2009). The model can also be used to calculate the dynamic response to a moving load and to derive mass and stiffness properties from this simulated response as proposed in this paper.

3. The Cross-Entropy Algorithm for FE Beam Models

The CE algorithm used to determine the material properties was developed through a number of trial generations, progressing from one to the next with increasing complexity. The first stage was the verification of the core the CE method implemented by studying the static CE algorithm of Walsh and González (2009) to
calculate the stiffness values of the FE elementary beams (i.e., product of modulus of elasticity and second moment of area, EI). Secondly, change the forcing function from static to time-varying, and use dynamic response to calculate the EI values. And finally, include the mass matrix in the solution and attempt to calculate the material density \( \rho \) in addition to the EI values.

The implementation of the CE method in conjunction with the FE model involves rewriting the matrices of the equation of motion so the material properties can be input variables. The general equation of motion for structural dynamics is given in Equation (1).

\[
[M][\ddot{u}] + [C][\dot{u}] + [K][u] = F(t)
\]  

(1)

In Equation 1, \([M]\), \([C]\) and \([K]\) are the system’s mass, damping and stiffness matrices respectively, \(\{u\}\) is the vector of displacements and \(F(t)\) is the forcing function. In order to limit the number of unknowns in the CE algorithm, the influence of damping will be neglected for all the simulations described here.

### 3.1 Stiffness and Mass Matrices

The static CE algorithm solves the equation of motion in a simplistic form, considering only the stiffness matrices as in Equation (2).

\[
[K][u] = F(t)
\]  

(2)

The CE algorithm solving for both, EI and the material \(\rho\), must consider the mass matrix in addition to the stiffness matrix. Equation (3) will be employed for this dynamic simulation.

\[
[M][\ddot{u}] + [K][u] = F(t)
\]  

(3)

The (global) mass and stiffness matrices of Equations (1)-(3) are obtained from assembling the elementary stiffness \((k_e)\) and elementary mass \((m_e)\) matrices. \(k_e\) and \(m_e\) for beam elements with 2 DoFs per node are shown in Equations (4) and (5) (Bathe 1996, Logan 2007),

\[
k_e = \begin{bmatrix}
12/\rho & 6/\rho & -12/\rho & 6/\rho \\
6/\rho & 4/1 & -6/\rho & 2/1 \\
-12/\rho & -6/\rho & 12/\rho & -6/\rho \\
6/\rho & 2/1 & -6/\rho & 4/1
\end{bmatrix}
\]  

(4)

\[
m_e = \frac{3}{420} \begin{bmatrix}
156\rho & 22\rho^2 & 54\rho & -13\rho^2 \\
12\rho^2 & 4\rho^2 & 13\rho^2 & -3\rho^2 \\
54\rho & 13\rho^2 & 150\rho & -22\rho^2 \\
-13\rho^2 & -3\rho^2 & -22\rho^2 & 4\rho^2
\end{bmatrix}
\]  

(5)
where $l$ and $A$ are elementary beam length and area respectively. The deflection measurements used as input to the CE algorithm are shown for reference in Figure 2. The deflections shown in Figures 2 (a) and (b) are the input signals to the CE algorithm described by Equations (2) and (3) respectively. The inclusion of the matrix $[M]$ in the solution creates a more complicated deflection signal with the participation of the inertial forces of the beam. The magnitude and velocity of the load used were 200 kN and 10 m/s respectively.

![Figure 2](image_url)

**Figure 2** – Deflection Measurements

### 3.2 EI estimates, TBs and EIPs

The first step in the CE algorithm is to create the initial distribution of values, some initial EI distribution for each element of the beam (these distributions will be defined by mean ($\mu$) and standard deviation ($\sigma$) values). Let’s call this initial set of EI distributions an EI Profile (EIP). Some initial input is required to the system, a best-engineering estimate of the materials EI value. Variations larger than 30% from theoretical values are not expected except for large amounts of damage.

From the EIP a number, $K$, of Trial Beams (TBs) with elementary EI beam values randomly sampled from the EIP are generated. Deflection measurements are then taken at three locations along the beam to test the algorithm: $\frac{1}{4}$-span, mid-span and $\frac{3}{4}$-span for the applied loading. Clearly, the smaller the number of measurement
locations, the larger the number of possible combinations of model parameters satisfying the solution and the more difficult the optimisation problem. An objective function is defined as the sum of the squares of the differences between the deflections measured for each TB and the deflections measured for the actual beam. An ‘elite set’ (Rubinstein and Kroese 2004) is defined as the top 5% TBs giving the lowest objective function values. The mean and standard deviation EI values of this elite set are then used to create the EIP for the next iteration of TBs. A tolerance is then specified to establish when convergence has been reached, i.e., 0.01% difference in successive objective functions.

Starting with the initial normal distribution of EI values for each of the N beam elements, the algorithm progress; updating the $\mu$ and $\sigma$ values, informed by the elite set from the previous iteration until convergence has occurred. At this point mean values are adopted as the predictions for the EI values and the standard deviations are a measure of the degree of confidence in these predictions (typically the more available measurements the smaller the standard deviation).

The inverse problem of estimating structural parameters given a structure’s response to a loading event has no unique solution. There are multiple sets of EI and density values that may produce the desired structural response. Hence a common problem encountered along the route to the solution of the algorithm is that it may converge to a false-solution. This is a well documented issue with the CE method and the technique used to prevent early convergence is one penned the ‘Noisy’ CE Method by Szita and Lörincz (2006), which was first proposed as the method of ‘injecting’ extra variance into the samples by Botev and Kroese (2004). This technique simply involves adding noise to the updated standard deviation between iterations. This ‘widens’ distributions at the start of the iterations reducing false convergence. The magnitude of the added variance reduces as the number of iterations progresses, until at some simulation the variance is no longer artificially inflated.

4. Moving-Load Algorithm, Damage Simulation

This section presents the results of the CE algorithm in predicting stiffness distribution taking as input the simulated responses from Equation (2) (Figure 2(a)). Figure 3 presents the results of four CE algorithm simulations. Each simulation attempted to model different locations and severities of damage to the beam in the form of a local reduction in stiffness. The 8m beam used in the simulation was split into 8 metre long elements. The dotted line in Figure 3 shows the initial input to the CE algorithm; the light solid line shows the actual values; the heaviest solid line with empty circular markers are the final predicted EI values from the algorithm.
Figure 3 – EI Predictions for Four Simulations (dotted line: initial input; light solid line: actual values; heaviest solid line with empty circular markers: final predicted EI)

It has been noted in a previous study by Walsh and González (2009) that the predictions of the elements near the supports suffer due to the small magnitudes of the deflection values at these locations. This problem can be overcome with the use of inclinometers. With the exception of these end elements the predictions are very good; the location and magnitude of the damaged elements being identified correctly in every case.

5. Inclusion of Mass Matrix [M] in the Solution

The matrix [M] (Equation (3)) is included here as an unknown input to be predicted by the algorithm. This addition adds the material density to the list of unknowns solved in the CE algorithm. It was decided to solve for a single density value, that is, assume that the density of all beam elements being constant in a preliminary investigation. This increases the number of unknowns merely from the previous eight stiffness values to nine. The simulated input has been shown in Figure 2(b).

In programming the ‘Noisy’ CE algorithm it is accepted that sometimes the objective function of the optimisation algorithm may enlarge, that is, some trial solutions may be worse than in previous iterations. Figure 4, shows the objective function for such a simulation. In this simulation ‘injection’ occurred for the first 49 convergences, with the 50th convergence the simulation ended (at approximately iteration 4,000). The peaks in Figure 4 are at the iteration number where injection occurred, with the objective function values at convergence typically of the order of $10^{-4}$. 
Figure 4 – Objective Function Values

The magnitude of the ‘injection’ was reduced as the iterations progressed. The final algorithm predictions are taken from the TB which gives the minimum objective function value, which may not necessarily be the final iteration. The predictions for this simulation are shown in Figure 5.

Figure 5 – Simulation Predictions, including [M]

The elementary beam EI values being sought in this simulation are distributed randomly about some mean, attempting to model healthy elements subject to usual material variation. This is imitating a scenario where the CE algorithm is being used for model calibration. The simulation prediction of material density was 24,040 kg/m$^3$, overestimating the actual value of 24,000 kg/m$^3$ by only 0.2%. There was a mean error
of 2.7% in the prediction of the EI values (or a mean error of 1.5% when not considering the end elements.)

6. Conclusion

A CE algorithm has been shown to predict the flexural stiffness and density values of a FE beam with great accuracy. There is scope for incorporating other inputs such as strain measurements or additional unknowns, i.e., allowing the material density of each element to vary independently, or increasing the complexity of the structural model to 2-D or 3-D. It is a relatively easy-to-implement method compared to other statistical FE updating procedures, that has shown to be able to infer the stiffness and mass matrices of a FE model from measurements, which can be subsequently used for calibration of Bridge Weigh-in-Motion systems or applications in Structural Health Monitoring.

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References