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Generalisations of the classical strain-energy function to model biological soft tissue

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ABSTRACT Strain measures consistent with the classical, infinitesimal form of the strain-energy function are obtained within the context of isotropic, homogeneous, compressible, non-linear elasticity. It will be shown that there are two distinct families of such measures. One family has already been much studied in the literature, the most important member being the strains whose principal values are a function only of the corresponding principal stretches. The second family of strains appears new. The motivation for studying such strains is the intuitive expectation that, for at least moderate deformations, a good fit with experimental data from material characterisation tests will be obtained with the corresponding strain-energy functions. In particular, there is the expectation that such models could prove useful for the modelling of biological soft tissue, whose physiological response is characterised by moderate strains. It will be shown that this is indeed the case for simple tension tests on porcine brain tissue.

Keywords: classical strain-energy function, moderate deformations, non-linear elasticity, soft tissue.

1. Introduction

An obvious method of modelling non-linearly elastic deformations is to generalise the structures of the classical linear theory to the non-linear regime. A short history of the earliest of these attempts can be found in Section 94 of Truesdell and Noll [1]. One focus of attention has been on obtaining non-linear equivalents of the classical stress-strain relation. Although considered as early as 1915 by Armanni [2], this problem is still of interest, as can be seen in the recent work of Batra [3,4], Nader [5], Chiskis and Parvis [6] and Murphy [7].

Another related approach has been to seek non-linear equivalents of the classical strain-energy function,

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$$W = \mu(\hat{e}_1^2 + \hat{e}_2^2 + \hat{e}_3^2) + \frac{\lambda}{2}(\hat{e}_1 + \hat{e}_2 + \hat{e}_3)^2, \quad (1)$$

where λ, μ are the Lamé constants and $\hat{e}_i$, $i = 1, 2, 3$, are the principal values of the infinitesimal strain tensor. The usual approach adopted is to replace these principal values with the principal values, denoted by e_i , $i = 1, 2, 3$, of the non-linear strain tensor defined by

$$e_i = f(\lambda_i), \quad i = 1, 2, 3, \quad (2)$$

where $\lambda_1, \lambda_2, \lambda_3$ are the principal stretches and f is some given, well-behaved function (see, for example, Truesdell and Noll [1]). An important choice for f was introduced by Hencky [8] who considered principal strains of the form

$$e_i = \ln(\lambda_i), \quad i = 1, 2, 3. \quad (3)$$

The corresponding strain-energy function is still of much interest, as can be seen, for example, in the work of Bruhns, Xiao and Meyers [9,10]. Much of this interest is motivated by the results of Anand [11], who demonstrated an excellent fit of this strain-energy function with some experimental data from material characterisation tests for moderate deformations of many different types of material.

Necessary and sufficient conditions will be obtained here so that strain-energy functions of the form

$$W = \mu(e_1^2 + e_2^2 + e_3^2) + \frac{\lambda}{2}(e_1 + e_2 + e_3)^2, \quad (4)$$

reduce to the classical linear strain-energy function on restriction to infinitesimal deformations. Here λ, μ are the Lamé constants and the principal strains $e_i = e_i(\lambda_i, \lambda_j, \lambda_k)$ are symmetric with respect to the interchange of the principal stretches λ_j and λ_k , with (2) then being the special case of the principal strains e_i being independent of λ_j and λ_k . There are two complementary sets of conditions. The first set of conditions is satisfied by strains of the form (2) in a natural way. The other set of conditions seem to result in more complicated mathematical models and this complexity is even evident when modelling the standard material characterisation tests for non-linearly elastic materials. The strains that arise from satisfying the first set of conditions therefore are the obvious choice when modelling these tests. It remains to be seen under what circumstances, if any, the second family of strains would be preferable.

The hypothesis adopted here is that, given the excellent agreement of the classical model with experimental data for infinitesimal strains, non-linear generalizations of (1) should yield simple, accurate models of non-linear elastic behaviour for *moderate* deformations. In particular, it is hoped that an excellent fit will be achieved for specific examples of such strain-energies with experimental data for moderate deformations of soft biological tissue, since the physiological range of strains is of the order of 10%. As an example, the theory is applied here to the modelling of simple tension tests on porcine

brain tissue. Finding experimentally validated slightly compressible forms of the strain-energy function is important as slightly compressible strain-energy functions are the basis of most commercial Finite Element codes that simulate soft tissue.

2. Some restrictions

Consider strain energy-functions of the form (4). To preserve positive definiteness on restriction to infinitesimal deformations, it will be assumed that

$$\mu > 0, 3\kappa = 3\lambda + 2\mu > 0, \quad (5)$$

where κ is the infinitesimal bulk modulus. For technical reasons later, it will also be assumed that $\lambda + \mu \neq 0$. Solid rubbers and biological tissue are the materials of interest here and for such materials it is usually assumed that

$$\mu/\kappa \ll 1. \quad (6)$$

There are a number of additional restrictions that will be imposed here. The first is that the strain energy is zero in the reference configuration. Therefore it is required that

$$e_1(1,1,1) = 0. \quad (7)$$

The second requirement will be that the reference configuration is stress-free. This is identically satisfied here since the principal Cauchy stresses, T_i , are given by

$$T_i = \frac{1}{\lambda_j \lambda_k} \{ 2\mu(e_1 e_{1,i} + e_2 e_{2,i} + e_3 e_{3,i}) + \lambda(e_1 + e_2 + e_3)(e_{1,i} + e_{2,i} + e_{3,i}) \}, \quad (8)$$

which, on noting (7), are identically zero in the reference configuration.

It will finally be required that (4) reduce to (1) on restriction to the infinitesimal deformations. This can be summarized (see, for example, Ogden [12]) as follows:

$$\overline{\frac{\partial^2 W}{\partial \lambda_i \partial \lambda_j}} = 2\mu \delta_{ij} + \lambda, i, j \in \{1, 2, 3\}, \quad (9)$$

where the superposed bar notation denotes evaluation in the reference configuration. First note that for (4),

$$\begin{aligned} \overline{\frac{\partial^2 W}{\partial \lambda_1^2}} &= 2\mu(\overline{e_{1,1}^2} + \overline{e_{2,1}^2} + \overline{e_{3,1}^2}) + \lambda(\overline{e_{1,1}} + \overline{e_{2,1}} + \overline{e_{3,1}})^2, \\ \overline{\frac{\partial^2 W}{\partial \lambda_1 \partial \lambda_2}} &= 2\mu(\overline{e_{1,1} e_{1,2}} + \overline{e_{2,2} e_{2,1}} + \overline{e_{3,1} e_{3,2}}) + \lambda(\overline{e_{1,1}} + \overline{e_{2,1}} + \overline{e_{3,1}})(\overline{e_{1,2}} + \overline{e_{2,2}} + \overline{e_{3,2}}), \end{aligned} \quad (10)$$

By symmetry it follows that

$$\overline{e_{1,1}} = \overline{e_{2,2}} \equiv x, \overline{e_{1,2}} = \overline{e_{2,1}} = \overline{e_{3,1}} = \overline{e_{3,2}} \equiv y. \quad (11)$$

Without loss of generality, it follows that (9) are satisfied if and only if

$$y(2x + y) = 0, x^2 + 2y^2 = 1, (x + 2y)^2 = 1. \quad (12)$$

This over-determined system has the two solutions

$$x = \pm 1, y = 0; x = \pm 1/3, y = -2x. \quad (13)$$

These are then necessary and sufficient conditions to ensure that (4) reduces to the classical form on restriction to infinitesimal deformations, provided the initial condition (7) holds.

3. A general class of strains

The general class of strains (2) are a natural class of strains to associate with the first set of compatibility conditions (13)_{1,2} since to satisfy these, as well as the initial condition (7), it is only required that

$$f(1) = 0, f'(1) = \pm 1, \quad (14)$$

where the prime notation denotes differentiation. Henceforth it will be assumed that these conditions are satisfied if the general family of strains (2) is being considered. The corresponding Cauchy stress response is obtained by substituting (2) into (8) and is given by

$$T_i = \frac{f'(\lambda_i)}{\lambda_j \lambda_k} \left\{ (\lambda + 2\mu) f(\lambda_i) + \lambda (f(\lambda_j) + f(\lambda_k)) \right\}, \quad (15)$$

The most important family of such strains is given by

$$e_i = \begin{cases} \frac{1}{n} (\lambda_i^n - 1), & n \neq 0 \\ \ln \lambda_i, & n = 0 \end{cases}. \quad (16)$$

This family has been widely studied, mainly in connection with conjugate stresses, an excellent discussion of which is given in Ogden [12]. It is worthwhile noting that special cases of the combination of (4) and the family of strains (16) are strain-energy functions that have been studied independently in the literature. For example, setting $n = 0$ yields the well-known Hencky strain-energy function,

$$W = \mu \left((\ln \lambda_1)^2 + (\ln \lambda_2)^2 + (\ln \lambda_3)^2 \right) + \frac{\lambda}{2} (\ln \lambda_1 \lambda_2 \lambda_3)^2, \quad (17)$$

which Anand [11] showed accurately models the moderate stress response in material characterization tests for a number of different materials, including some solid rubbers. Setting $n = 1$ yields

$$W = \mu \left((\lambda_1 - 1)^2 + (\lambda_2 - 1)^2 + (\lambda_3 - 1)^2 \right) + \frac{\lambda}{2} (\lambda_1 + \lambda_2 + \lambda_3 - 3)^2, \quad (18)$$

which is the Standard Harmonic Material introduced by John [13]. This strain-energy is also the Kirchhoff-Saint Venant strain-energy function discussed in Truesdell and Noll [1] and Ciarlet [14].

4. A new general class of strains

Another class of strains will now be introduced as the counterpart of (2) for the *second* set of compatibility conditions $(13)_{3,4}$. A natural general class of strains for these conditions is defined by

$$e_i = g(s_i), s_i \equiv \frac{\lambda_i}{\lambda_j^2 \lambda_k^2}, \text{ arbitrary } g, i \neq j \neq k \neq i, \quad (19)$$

since $(13)_{3,4}$ are satisfied provided

$$g(1) = 0, g'(1) = \pm 1/3. \quad (20)$$

A power-law family of strains for (19), corresponding to (16) for the general family (2), is given by

$$e_i = \begin{cases} \frac{1}{3n} \left(\left(\frac{\lambda_i}{\lambda_j^2 \lambda_k^2} \right)^n - 1 \right), & n \neq 0 \\ \frac{1}{3} \ln \left(\frac{\lambda_i}{\lambda_j^2 \lambda_k^2} \right), & n = 0 \end{cases}, \quad (21)$$

Surprisingly, choosing $n = 0$ again yields the Hencky strain-energy function (17).

The corresponding principal stress response follows from (8) and is given by

$$T_i = \frac{1}{i_3} \left\{ 2\mu \left(s_i g(s_i) g'(s_i) - 2s_j g(s_j) g'(s_j) - 2s_k g(s_k) g'(s_k) \right) + \lambda \left(g(s_i) + g(s_j) + g(s_k) \right) \left(s_i g'(s_i) - 2s_j g'(s_j) - 2s_k g'(s_k) \right) \right\}, \quad (22)$$

where $i_3 \equiv \lambda_1 \lambda_2 \lambda_3$. The contrast of the algebraic simplicity between this stress response and that given by (15) is immediate: the general family of strains (2), associated with the first set of compatibility conditions given in (13), leads to a much simpler stress response. As will be shown in the next section, this is even evident when considering the stress response for typical material characterisation tests used to determine the mechanical response of non-linear solids. The modelling of experimental data from these tests is therefore much more easily achieved using models discussed in the previous section than those developed here.

It is worthwhile noting that the family of strains based on (16) with the stretches replaced by the normalised stretches $\lambda_i^* \equiv \lambda_i / i_3^{1/3}$, i.e.,

$$e_i = \frac{1}{n} \left((\lambda_i^*)^n - 1 \right), \quad (23)$$

as discussed, for example, by Simo and Taylor [15] in the context of the Finite Element simulation of hyperelastic materials, satisfies neither of the two sets of conditions given in (13) and therefore a linear form based on these strains is *not* possible.

5. Simple tension/compression tests

In simple tension/compression tests, an axial force is applied to two parallel faces of a rectangular specimen, with the other faces remaining traction-free. Assuming that the principal *engineering* stresses, P_i , are specified, this experiment is described mathematically by

$$P_2 = P_3 = 0; \quad \lambda_2 = \lambda_3, \quad (24)$$

with the principal stress in the direction of applied force assumed prescribed at discrete values of the axial stretch.

For the general class of strains defined by (2), the stress-free boundary conditions and (15) yield

$$f(\lambda_2) = -\frac{\lambda}{2(\lambda + \mu)} f(\lambda_1). \quad (25)$$

In contrast, for the general family of strains defined by (19), the stress-free boundary conditions do not allow determination of $g(s_2)$ in terms of $g(s_1)$ and the modelling process requires therefore specification of the arbitrary function in order to determine the transverse stretch in terms of the axial stretch. Therefore, at least for simple tension experiments, the strain family (2) seem much more appropriate and will only be considered here.

Since $P_1 = \partial W / \partial \lambda_1$, it now follows from (15) and (25) that

$$P(\lambda_1) = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu} f'(\lambda_1) f(\lambda_1). \quad (26)$$

Let s denote the initial slope of the experimental stress-strain response under tensile loading. Then it follows from (14) and (26) that

$$s = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu}. \quad (27)$$

For the power-law family (16), the relation (26) therefore takes the form

$$P(\lambda_1) = \begin{cases} s \frac{\ln \lambda_1}{\lambda_1}, & n = 0 \\ s \frac{\lambda_1^{n-1} (\lambda_1^n - 1)}{n}, & n \neq 0 \end{cases}. \quad (28)$$

Thus n is now the only parameter that is available to model experimental data. Although a sophisticated non-linear least squares approach could be adopted in order to determine an optimal value of this parameter to fit a given set of experimental data, a more naïve, and probably more efficient, approach is chosen here. Since integer values of n are appealing intuitively, beginning with $n = 0$, alternate positive and negative integer values are substituted into (28) and a decision is made visually as to the best choice, with the hope that at least one choice will result in an excellent fit. The method is illustrated in the next section.

6. An example

Fresh porcine brains were collected from a local slaughter house and preserved in physiological saline solution at 4 to 6°C prior to being tested under uniaxial tension within six hours of post-mortem. Cylindrical samples of nominal diameter 15.0 ± 0.2 mm were cut using a circular steel die cutter of 15.5 mm diameter. The samples were then inserted in a cylindrical metal disk with 15.2 mm internal diameter and 4.1 mm thickness. The excessive brain portion was then removed with a surgical scalpel to maintain an approximate specimen height of 4.0 ± 0.2 mm. A contraction of the cylindrical samples occurred immediately after they were removed from the brains, revealing the presence of residual stresses *in-vivo*. Numerous preliminary experiments and measurements indicated that the nominal dimensions were reached after a few minutes; it was at this stage that testing commenced. All samples were prepared and tested at 22°C.

Uniaxial tensile tests were performed using a Tinius Olsen Universal Tensile Testing machine as illustrated in Figure 1. The surfaces of the top and lower platens were first covered with a masking tape substrate to which a thin layer of surgical glue was applied. The prepared cylindrical specimen of tissue was then placed on the lower platen. The top platen, which was attached to the 10 N load cell on the test machine, was then lowered slowly so as to just touch the top surface of the specimen. Four minutes settling

time was sufficient to ensure proper adhesion of the specimen to the top and lower platens. The distance between the top and lower platens was measured with a Vernier caliper before the start of experimentation. The cylindrical brain specimens of mixed white and gray matter were stretched up to some 60% strain at a quasi-static velocity of 500 mm/min, which corresponded to a strain rate of 2 s^{-1} . Each specimen was tested only once. A more comprehensive modelling of the resulting data obtained will be given elsewhere.

Figure 1. Experimental setup (overview and detail) for quasi-static tensile tests of porcine brain tissue.

For our purposes, a representative set of data from these experiments will suffice. In the table below are engineering stress and stretch data for porcine brain tissue:

stretch	engineering stress (Pa)	stretch	engineering stress (Pa)
1	0	1.060	196.17
1.010	41.82	1.070	218.06
1.020	79.59	1.080	237.55
1.030	113.63	1.090	254.88
1.040	144.22	1.100	270.26
1.050	171.65		

Table 1. Simple tension of porcine brain specimens

First note that this data determines the ratio of material constants (27) to be

$$4182 = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu}. \quad (29)$$

In Figure 2, this data is plotted together with some predictive curves from (28).

Figure 2. Experimental data and predicted response.

It is immediately obvious that an excellent fit is achieved for the family of strains (16) with $n = -2$, i.e.,

$$e_i = \frac{1}{2} \left(\frac{\lambda_i^2 - 1}{\lambda_i^2} \right). \quad (30)$$

These are the principal values of the Almansi-Hamel strain tensor (see, for example, Batra [4]). The corresponding strain-energy function is given by

$$W = \frac{\mu}{4} \left(3 - \frac{2}{I_3} (I_1 + I_2) + \frac{I_2^2}{I_3^2} \right) + \frac{\lambda}{8} \left(3 - \frac{I_2}{I_3} \right)^2, \quad (31)$$

where the material constants satisfy (29) and I_1 , I_2 , I_3 are the Cauchy-Green strain invariants defined by

$$I_1 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2, \quad I_2 = \lambda_1^2 \lambda_2^2 + \lambda_1^2 \lambda_3^2 + \lambda_2^2 \lambda_3^2, \quad I_3 = \lambda_1^2 \lambda_2^2 \lambda_3^2. \quad (32)$$

Conclusions

The physiological range of strains of biological soft tissue is of the order of 10%. It is intuitively appealing therefore to model such strains using strain-energy functions of the same form as the classical, linear strain-energy. Our intuitive expectation that models of this form efficiently capture the mechanical response of biological soft tissue is validated with the excellent fit of the Almansi-Hamel strain tensor with moderate strains of porcine brain tissue in simple tension.

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