Research Repository UCD

Title	Harmonic divisors and rationality of zeros of Jacobi polynomials
Authors(s)	Render, Hermann
Publication date	2013-08
Publication information	Render, Hermann. "Harmonic Divisors and Rationality of Zeros of Jacobi Polynomials" 31, no. 3 (August, 2013).
Publisher	Springer
Item record/more information	http://hdl.handle.net/10197/5488
Publisher's statement	The final publication is available at www.springerlink.com
Publisher's version (DOI)	10.1007/s11139-013-9475-1

Downloaded 2023-04-07T10:52:29Z

The UCD community has made this article openly available. Please share how this access benefits you. Your story matters! (@ucd_oa)

© Some rights reserved. For more information

HARMONIC DIVISORS AND RATIONALITY OF ZEROS OF JACOBI POLYNOMIALS

HERMANN RENDER

ABSTRACT. Let $P_n^{(\alpha, \beta)}(x)$ be the Jacobi polynomial of degree n with parameters α, β . The main result of the paper states the following: If $b \neq 1, 3$ and c are non-zero relatively prime natural numbers then $P_n^{(k+(d-3)/2, k+(d-3)/2)}(\sqrt{b/c}) \neq 0$ for all natural numbers d, n and $k \in \mathbb{N}_0$. Moreover, under the above assumption, the polynomial $Q(x) = \frac{b}{c}(x_1^2 + \dots + x_{d-1}^2) + (\frac{b}{c} - 1)x_d^2$ is not a harmonic divisor, and the Dirichlet problem for the cone $\{Q(x) < 0\}$ has polynomial harmonic solutions for polynomial data functions.

1. INTRODUCTION

A polynomial $Q(x)$ is called a *harmonic divisor* if there exists a polynomial $p(x) \neq 0$ such that the product $Q(x)p(x)$ is harmonic, i.e. that

$$\Delta(Q(x)p(x)) = 0 \text{ for all } x \in \mathbb{R}^d,$$

where $\Delta = \frac{\partial^2}{\partial x_1^2} + \dots + \frac{\partial^2}{\partial x_d^2}$ is the Laplace operator in the euclidean space $\mathbb{R}^d$. The notion of a harmonic divisor arises naturally in the investigation of stationary sets for the wave and heat equation [1],[2], and the injectivity of the spherical Radon transform [3]. In the study of the Cauchy problem in the category of formal power series it is often necessary to assume that a given polynomial $Q(x)$ is *not* a harmonic divisor, see [15], [16], [17], [18].

Let $\gamma \in (0, 1)$. In this paper we are interested in the Dirichlet problem for the closed cone

$$(1) \quad \Omega_\gamma := \{(x_1, \dots, x_d) \in \mathbb{R}^d : x_d \geq 0 \text{ and } \gamma^2(x_1^2 + \dots + x_{d-1}^2) \leq (1 - \gamma^2)x_d^2\}.$$

Using some standard arguments we shall see that the Dirichlet problem for polynomial data functions has unique harmonic polynomial solutions provided that the quadratic homogeneous polynomial

$$(2) \quad Q_\gamma(x_1, \dots, x_d) = \gamma^2(x_1^2 + \dots + x_{d-1}^2) + (\gamma^2 - 1)x_d^2$$

is not a harmonic divisor.

1991 Mathematics Subject Classification: 33C45; 11C08; 31B05.

The author was partially supported by Grant MTM2009-12740-C03-03 of the D.G.I. of Spain.

Throughout the paper $\mathbb{N}$ denotes the set of all natural numbers $n = 1, 2, 3, \dots$ and $\mathbb{N}_0$ denotes the set $\mathbb{N} \cup \{0\}$. D. Armitage has shown in [6] that Q_γ is not a harmonic divisor if and only if

$$(3) \quad C_{m-k}^{k+(d-2)/2}(\gamma) \neq 0$$

for all $m \in \mathbb{N}_0$ and for all $k \in \{0, \dots, m\}$. Here $C_n^\lambda(x)$ is the Gegenbauer polynomial (or ultraspherical polynomial) of degree n and parameter λ . Using the fact that Gegenbauer polynomials are expressible by Jacobi polynomials $P_n^{(\alpha, \beta)}(x)$ (see Section 2) the condition (3) is equivalent to

$$(4) \quad P_n^{(k+(d-3)/2, k+(d-3)/2)}(\gamma) \neq 0 \text{ for all } k, n \in \mathbb{N}_0.$$

Since Jacobi polynomials have rational coefficients it is clear that (4) is satisfied for transcendental numbers γ . The question arises whether one may find rather simple numbers γ , say rational numbers, such that (4) holds. In this paper we shall prove that

$$(5) \quad P_n^{(k+(d-3)/2, k+(d-3)/2)}\left(\sqrt{b/c}\right) \neq 0 \text{ for all } k, n \in \mathbb{N}_0$$

for all relatively prime natural numbers b, c with $b \neq 1, 3$. Our method of proof relies on simple divisibility arguments and an old result of Legendre about the divisibility properties of binomial coefficients.

The paper is organized as follows. In Section 2 we shall recall some standard identities for Jacobi polynomials which will be essential for our arguments. Section 3 contains the main result which will be derived from a more general theorem for Jacobi polynomials $P_n^{(\alpha, \beta)}$ where the parameters α, β are integers or half-integers.

In Section 4 we apply our results to Chebyshev polynomials providing a new proof of the following fact proven by D. H. Lehmer in [27]: Let k be an integer and $m \in \mathbb{N}_0$. If there exist a natural number c and $b \in \mathbb{N}_0$ such that

$$x_{k,m} := \cos \frac{k\pi}{m+1} = \sqrt{b/c}$$

then $x_{k,m}$ is equal to one of the numbers $0, 1, 1/2, 1/\sqrt{2}, 3/\sqrt{2}$.

In Section 5 we give applications to the Dirichlet problem as explained above.

2. JACOBI POLYNOMIALS

Let us recall that the Pochhammer symbol $(\alpha)_k$ for a complex number α and $k \in \mathbb{N}_0$ is defined by

$$(\alpha)_k = \alpha(\alpha+1)\dots(\alpha+k-1)$$

with the convention that $(\alpha)_0 = 1$. The Gegenbauer polynomial $C_n^\lambda(x)$ can be expressed through Jacobi polynomials by the formula (see [5, p. 302])

$$C_n^\lambda(x) = \frac{(2\lambda)_n}{(\lambda + (1/2))_n} P_n^{(\lambda-(1/2), \lambda-(1/2))}(x),$$

where the Jacobi polynomial $P^{(\alpha, \beta)}(x)$ for complex parameters α and β is defined by

$$P_n^{(\alpha, \beta)}(x) = (-1)^n \frac{(\alpha + 1)_n}{n!} \sum_{k=0}^n \frac{(-n)_k}{k!} \frac{(n + \alpha + \beta + 1)_k}{(\alpha + 1)_k} \left(\frac{1-x}{2} \right)^k,$$

see [5, p. 99]. For our purposes the following formula

$$P_n^{(\alpha, \beta)}(x) = \frac{(\alpha + 1)_n}{n!} \left(\frac{1+x}{2} \right)^n \sum_{k=0}^n \frac{(-n)_k}{k!} \frac{(-n - \beta)_k}{(\alpha + 1)_k} \left(\frac{x-1}{x+1} \right)^k,$$

is very convenient, see [5, p. 117]. Using that

$$\frac{(-n)_k}{k!} = \frac{(-1)^k}{k!} n(n-1) \dots (n-(k-1)) = (-1)^k \binom{n}{k}$$

and $(-1)^k (-n - \beta)_k = (n + \beta + 1 - k)_k$ one obtains the formula

$$(6) \quad P_n^{(\alpha, \beta)}(x) = \frac{(\alpha + 1)_n}{n!} \left(\frac{1+x}{2} \right)^n Q_n^{(\alpha, \beta)} \left(\frac{x-1}{x+1} \right)$$

where we define the polynomial $Q_n^{(\alpha, \beta)}(y)$ by

$$(7) \quad Q_n^{(\alpha, \beta)}(y) = \sum_{k=0}^n \frac{(n + \beta + 1 - k)_k}{(\alpha + 1)_k} \binom{n}{k} y^k.$$

Clearly (6) implies that

$$(8) \quad P_n^{(\alpha, \beta)}(2x^2 - 1) = \frac{(\alpha + 1)_n}{n!} x^{2n} Q_n^{(\alpha, \beta)} \left(\frac{x^2 - 1}{x^2} \right).$$

We recall from [5, p. 117] that

$$(9) \quad P_{2n}^{(\alpha, \alpha)}(x) = \frac{\Gamma(2n + \alpha + 1) n!}{\Gamma(n + \alpha + 1) (2n + 1)!} P_n^{(\alpha, -1/2)}(2x^2 - 1).$$

Taking the parameter β equal to $-1/2$ in formula (8) one obtains from (9) the formula

$$(10) \quad P_{2n}^{(\alpha, \alpha)}(x) = \frac{\Gamma(2n + \alpha + 1) (\alpha + 1)_n}{\Gamma(n + \alpha + 1) (2n + 1)!} x^{2n} Q_n^{(\alpha, -1/2)} \left(\frac{x^2 - 1}{x^2} \right).$$

For $x = \sqrt{b/c}$ this means that

$$(11) \quad P_{2n}^{(\alpha, \alpha)} \left(\sqrt{b/c} \right) = \frac{\Gamma(2n + \alpha + 1) (\alpha + 1)_n}{\Gamma(n + \alpha + 1) (2n + 1)!} \frac{b^n}{c^n} Q_n^{(\alpha, -1/2)} \left(\frac{b - c}{b} \right).$$

Similarly we have (see [5, p. 117])

$$P_{2n+1}^{(\alpha, \alpha)}(x) = \frac{\Gamma(2n + \alpha + 2) n!}{\Gamma(n + \alpha + 1) (2n + 1)!} \cdot x \cdot P_n^{(\alpha, 1/2)}(2x^2 - 1)$$

and

$$(12) \quad P_{2n+1}^{(\alpha, \alpha)}(x) = \frac{\Gamma(2n + \alpha + 2)(\alpha + 1)_n}{\Gamma(n + \alpha + 1)(2n + 1)!} x^{2n+1} Q_n^{(\alpha, 1/2)}\left(\frac{x^2 - 1}{x^2}\right).$$

Thus

$$(13) \quad P_{2n+1}^{(\alpha, \alpha)}\left(\sqrt{b/c}\right) = \frac{\Gamma(2n + \alpha + 2)(\alpha + 1)_n}{\Gamma(n + \alpha + 1)(2n + 1)!} x^{2n+1} Q_n^{(\alpha, 1/2)}\left(\frac{b - c}{b}\right).$$

In the next section we shall analyse the polynomial $Q_n^{(\alpha, \beta)}(x)$.

3. THE MAIN RESULT

At first let us introduce some definitions and notations: for an integer $a \neq 0$ and a prime number p (so by definition $p \geq 2$) define $v_p(a)$ as the largest number $m \in \mathbb{N}_0$ such that p^m divides a , and define $v_p(0) = \infty$. Thus, $v_p(a)$ is the multiplicity of the prime factor p occurring in the prime decomposition of a . For a rational number $r = \frac{a}{b}$ one defines $v_p(r) := v_p(a) - v_p(b)$.

Let n be a natural number and p be a prime number. Let us write its p -adic decomposition by $n = n_t p^t + n_{t-1} p^{t-1} + \dots + n_1 p + n_0$ where $n_0, \dots, n_t \in \{0, 1, \dots, p-1\}$. The sum of the p -digits of n is defined by $\sigma_p(n) = n_0 + \dots + n_t$. A beautiful result due to Legendre says that

$$v_p(n!) = \frac{n - \sigma_p(n)}{p - 1},$$

see e.g. [40]. Since the sum $n_0 + \dots + n_t$ is positive for $n \geq 1$ we conclude that

Lemma 1. *For any prime number p and any natural number n one has*

$$v_p(n!) \leq \frac{n - 1}{p - 1}.$$

The following simple lemma will be our main tool. For convenience of the reader we include the proof although it might be part of mathematical folklore.

Lemma 2. *Let $Q_n(x) = \sum_{k=0}^n a_k x^k$ be a polynomial with rational coefficients and $a_n \neq 0$ and $a_0 \neq 0$. Let b and c be non-zero integers and let p be a prime number dividing c and not b . Assume that*

$$(14) \quad v_p\left(c^k \frac{a_{n-k}}{a_n}\right) \geq 1$$

for $k = 1, \dots, n$. Then $Q_n\left(\frac{b}{c}\right) \neq 0$.

Proof. We write $Q_n(x) = \sum_{k=0}^n a_{n-k} x^{n-k}$ and obtain

$$(15) \quad \frac{c^n}{a_n} Q_n\left(\frac{b}{c}\right) = b^n + \sum_{k=1}^n b^{n-k} c^k \frac{a_{n-k}}{a_n}.$$

Note that in the sum in (15), each term has p -adic valuation ≥ 1 . On the other hand, b^n is not divisible by p . Hence $Q_n\left(\frac{b}{c}\right)$ can not be zero and we actually have proved that

$$(16) \quad v_p\left(Q_n\left(\frac{b}{c}\right)\right) = v_p\left(\frac{a_n}{c^n}\right).$$

□

Remark 3. Let D_n be the least natural number such that $D_n a_{n-k}/a_n$ is an integer for all $k = 1, \dots, n$. Multiplying (15) with D_n shows that $D_n \frac{c^n}{a_n} Q_n\left(\frac{b}{c}\right)$ is a non-zero integer and therefore the following inequality holds:

$$(17) \quad \left|Q_n\left(\frac{b}{c}\right)\right| \geq \frac{|a_n|}{|c^n|} \cdot \frac{1}{D_n}.$$

We shall need the following elementary lemma. The proof is included for convenience of the reader:

Lemma 4. If m is a natural number and $k \in \mathbb{N}_0$ then

$$(18) \quad 2^{2k-1} \cdot \left(m - \frac{1}{2}\right)_k = \frac{(2m+2k-3)!(m-1)!}{(m+k-2)!(2m-2)!}.$$

Proof. For $k \geq 1$ the term $2^{2k-1} \cdot (m-1/2)_k$ is equal to

$$2^{k-1} (2m-1)(2m+1) \dots (2m+2k-3).$$

Clearly this is equal to

$$2^{k-1} \frac{(2m-1)(2m)(2m+1) \dots (2m+2k-4)(2m+2k-3)}{(2m)(2m+2) \dots (2m+2k-4)}$$

and from this one obtains the right hand side of (18). For $k = 0$ one easily checks that (18) holds as well. □

Now we will state the main result of the paper and it is convenient to recall formula (7):

$$(19) \quad Q_n^{(\alpha, \beta)}(y) = \sum_{k=0}^n \frac{(n+\beta+1-k)_k}{(\alpha+1)_k} \binom{n}{k} y^k.$$

Theorem 5. Let $n \in \mathbb{N}$, and $\alpha, \beta \in \mathbb{N}_0$ and $\delta \in \{0, 1\}$. Then

$$(20) \quad Q_n^{(-\frac{\delta}{2}+\alpha, -\frac{1}{2}+\beta)}\left(\frac{b}{c}\right) \neq 0$$

for all non-zero relatively prime integers b and c if either (i) 2 divides c or (ii) there exists a prime number $p \geq \beta+3$ dividing c and but not $2\beta+1$, or (iii) there exists a prime number $p > (\beta+3)/2$ such that p^2 divides c .

Proof. 1. Replace β in (19) by $-\frac{1}{2} + \beta$. Lemma 4 (put $m := n + \beta - k + 1 \geq 1$) yields

$$(n + 1/2 + \beta - k)_k = \frac{1}{2^{2k-1}} \frac{(2n + 2\beta - 1)! (n + \beta - k)!}{(n + \beta - 1)! (2n + 2\beta - 2k)!}.$$

2. In the first case suppose that $\delta = 0$. Since $\alpha \in \mathbb{N}_0$ we have $(\alpha + 1)_k = (\alpha + k)!/\alpha!$. Thus the k -th coefficient of the polynomial $Q_n^{(\alpha, -1/2+\beta)}(y)$ is given by

$$(21) \quad a_k := \binom{n}{k} \frac{\alpha!}{(\alpha + k)!} \frac{1}{2^{2k-1}} \frac{(2n + 2\beta - 1)! (n + \beta - k)!}{(n + \beta - 1)! (2n + 2\beta - 2k)!}.$$

Then

$$\frac{a_{n-k}}{a_n} = 2^{2k} \binom{n}{k} \frac{(\alpha + n)!}{(\alpha + n - k)!} \frac{(\beta + k)!}{\beta!} \frac{(2\beta)!}{(2\beta + 2k)!}.$$

Note that

$$(22) \quad 2^k \frac{(\beta + k)!}{\beta!} \frac{(2\beta)!}{(2\beta + 2k)!} = 2^k \frac{(\beta + 1) \dots (\beta + k)}{(2\beta + 1) \dots (2\beta + 2k)} = \frac{1}{T_k(\beta)}$$

where

$$(23) \quad T_k(\beta) := (2\beta + 1)(2\beta + 3) \dots (2\beta + 2k - 1).$$

Thus

$$\frac{a_{n-k}}{a_n} = 2^k \binom{n}{k} \frac{(\alpha + n)!}{(\alpha + n - k)!} \frac{1}{T_k(\beta)}.$$

3. In the second case we have $\delta = 1$, so the first parameter in (19) is equal to $-1/2 + \alpha$. By formula (18) applied to $m = \alpha + 1$ we obtain

$$(\alpha + 1)_k = \left(m - \frac{1}{2}\right)_k = \frac{1}{2^{2k-1}} \frac{(2\alpha + 2k - 1)! \alpha!}{(\alpha + k - 1)! (2\alpha)!}.$$

Thus the k -th coefficient of $Q_n^{(-1/2+\alpha, -1/2+\beta)}(x)$ is equal to

$$(24) \quad a_k = \binom{n}{k} \frac{(2n + 2\beta - 1)!}{(n + \beta - 1)!} \frac{(2\alpha)! (\alpha + k - 1)! (n + \beta - k)!}{\alpha! (2\alpha + 2k - 1)! (2n + 2\beta - 2k)!}.$$

Hence

$$(25) \quad \frac{a_{n-k}}{a_n} = \binom{n}{k} \frac{(n - k + \alpha - 1)!}{(n + \alpha - 1)!} \frac{(2n + 2\alpha - 1)!}{(2n - 2k + 2\alpha - 1)!} \frac{(\beta + k)! (2\beta)!}{(2\beta + 2k)! \beta!}.$$

Since

$$\begin{aligned} f_k &:= \frac{(n - k + \alpha - 1)!}{(n + \alpha - 1)!} \frac{(2n + 2\alpha - 1)!}{(2n - 2k + 2\alpha - 1)!} \\ &= \frac{(2n - 2k + 2\alpha) (2n - 2k + 2\alpha + 1) \dots (2n + 2\alpha - 1)}{(n - k + \alpha) \dots (n + \alpha - 1)} \end{aligned}$$

it is easy to see that $f_k = 2^k g_k$ with

$$g_k := (2n - 2k + \alpha + 1)(2n - 2k + \alpha + 3) \dots (2n + 2\alpha - 1).$$

Thus using (22) we obtain the following formula for the case $\delta = 1$:

$$\frac{a_{n-k}}{a_n} = \binom{n}{k} g_k \frac{1}{T_k(\beta)}.$$

4. Let now p be a prime number dividing c . In both cases, δ equal to 0 or 1, the natural number $T_k(\beta)$ is a denominator of a_{n-k}/a_k . We shall show that condition (14) in Lemma 2, namely

$$(26) \quad v_p \left(c^k \frac{a_{n-k}}{a_n} \right) \geq v_p \left(\frac{c^k}{T_k(\beta)} \right) \geq 1 \text{ for } k = 1, \dots, n,$$

is satisfied under the assumptions of the theorem, and therefore the proof will be finished.

If $p = 2$ we see that $v_2(T_k(\beta)) = 0$ for $k = 1, \dots, n$ since $T_k(\beta)$ is a product of odd numbers, so (26) is satisfied.

Assume now that $p \geq \beta + 3$. Then it is easy to see that the inequality

$$(27) \quad \frac{2\beta + 2k - 2}{p - 1} \leq k - 1$$

holds for all $k = 3, \dots, n$. Indeed, (27) says that the function $f(k) = (k - 1)(p - 1) - (2\beta + 2k - 2)$ is non-negative for $k = 3, \dots, n$. Since f is a linear map, we have only to check that $f(3) \geq 0$, so $2(p - 1) - 2\beta - 4 \geq 0$, which is obviously true since $p \geq \beta + 3$. By Lemma 1 we have

$$(28) \quad v_p(T_k(\beta)) \leq v_p((2\beta + 2k - 1)!) \leq \frac{2\beta + 2k - 2}{p - 1}$$

and by (27) we infer $v_p(T_k(\beta)) \leq k - 1$ that for $k = 3, \dots, n$, so (26) is satisfied for $k = 3, \dots, n$. We consider now the cases $k = 1, 2$. By assumption we know that

$$(29) \quad v_p(T_1(\beta)) = v_p(2\beta + 1) = 0.$$

Thus (14) holds for $k = 1$. Moreover, (29) implies

$$v_p(T_2(\beta)) = v_p((2\beta + 1)(2\beta + 3)) = v_p(2\beta + 3).$$

Suppose that $v_p(2\beta + 3) \geq 2$: then $2\beta + 3 \geq p^2 \geq (\beta + 3)^2 = \beta^2 + 6\beta + 9$ which is obviously nonsense. Thus $v_p(T_2(\beta)) \leq 1$ and $v_p(c^2/T_2(\beta)) \geq 1$. Hence (26) holds for all $k = 1, \dots, n$ and the result follows.

5. Now assume that p^2 divides c . If p is an integer $> (\beta + 3)/2$ then clearly

$$p \geq \frac{2\beta + 7}{4} = \frac{\beta + 3}{2} + \frac{1}{4}.$$

We have to show that (26) holds for all $k = 1, \dots, n$. Note that by Lemma 1

$$v_p \left(\frac{c^k}{T_k(\beta)} \right) \geq 2k - v_p(T_k(\beta)) \geq 2k - \frac{2\beta + 2k - 2}{p - 1}.$$

We conclude that $v_p(c^k/T_k(\beta)) \geq 1$ for $k = 3, \dots, n$ since $h(k) := (2k - 1)(p - 1) - 2\beta - 2k + 2 \geq 0$ for $k = 3, \dots, n$. The latter is true since $h(k) \geq h(3) = 5(p - 1) - 2\beta - 4$ and by our assumption $p \geq (2\beta + 7)/4$. Now we check that $v_p(c^k/T_k(\beta)) \geq 1$ for $k = 1, 2$. Suppose that $v_p(2\beta + 1) \geq 2$ or $v_p(2\beta + 3) \geq 2$: then $p^2 \leq 2\beta + 3$ and our assumption $(2\beta + 7)/4 \leq p$ yields

$$4\beta^2 + 28\beta + 49 = (2\beta + 7)^2 \leq 16p^2 \leq 32\beta + 48.$$

Hence $(2\beta - 1)^2 \leq 0$, a contradiction since β is an integer. Thus $v_p(2\beta + 1) \leq 1$ and $v_p(2\beta + 3) \leq 1$ and therefore

$$v_p \left(\frac{c}{T_1(\beta)} \right) \geq 2 - 1 \geq 1 \text{ and } v_p \left(\frac{c^2}{T_2(\beta)} \right) \geq 4 - 2 \geq 2 \geq 1.$$

The proof is complete. $\square$

Let us consider the case $n = 1$. From (19) we infer that $Q_1^{(\alpha, \beta)}(x) = 1 + \frac{\beta+1}{\alpha+1}x$, and specializing to our case of half-integers we obtain

$$Q_1^{(-\delta/2+\alpha, -\frac{1}{2}+\beta)}(x) = 1 + \frac{2\beta + 1}{2\alpha + 2 - \delta}x.$$

Thus $x_{1, \alpha, \beta, \delta} := -(2\alpha + 2 - \delta) / (2\beta + 1)$ is a rational zero. This already shows that the assumption that the prime number p does not divide $2\beta + 1$ in (ii) of Theorem 5 can not be omitted. In Section 4 we shall see similar examples where the degree n may be arbitrarily high.

Note that Theorem 5 does not give any information if the denominator c is equal to 1. Indeed, in this case we may have integer zeros, e.g. for $\beta = 1$ and $\delta = 0$ and $\alpha = 5$ we have

$$Q_4^{(5, \frac{1}{2})}(x) = \frac{1}{256} (x + 4) (5x^3 + 100x^2 + 176x + 64).$$

Now we are going to prove the main result announced in the introduction:

Theorem 6. *Let d be a natural number and let b and c be relatively prime natural numbers. If m is even and $b \neq 1$ then*

$$P_m^{(k+(d-3)/2, k+(d-3)/2)} \left(\sqrt{\frac{b}{c}} \right) \neq 0 \text{ for all } k, m \in \mathbb{N}_0.$$

If m is odd and $b \neq 1, 3$ then the same conclusion holds.

Proof. Assume that m is even, say $m = 2n$. For $x = \sqrt{b/c}$ use the identity (11), namely

$$P_{2n}^{(\alpha, \alpha)} \left(\sqrt{b/c} \right) = \frac{\Gamma(2n + \alpha + 1) (\alpha + 1)_n}{\Gamma(n + \alpha + 1) (2n + 1)!} \frac{b^n}{c^n} Q_n^{(\alpha, -1/2)} \left(\frac{b - c}{b} \right).$$

Clearly $b - c$ and b are relatively prime. Since $b \neq 1$ there exists a prime number $p \geq 2$ dividing b . Theorem 5 for the case $\beta = 0$ shows that $Q_n^{(\alpha, -1/2)} \left(\frac{b-c}{b} \right) \neq 0$. For $m = 2n + 1$ we use (13). Since $b \neq 1, 3$ there exists either a prime number $p \neq 3$ dividing b , or 3^2 divides b . Theorem 5 for the case $\beta = 1$ finishes the proof. $\square$

In Theorem 5 it is assumed that the prime number p divides the denominator c . We are now turning to a criterion where the prime number p divides the nominator. In the case $\delta = 1$ we may deduce a result by using a symmetry property of the polynomials $Q_n^{(\alpha, \beta)}(y)$:

Proposition 7. *Let α, β be complex numbers. Then for any $y \neq 0$*

$$Q_n^{(\alpha, \beta)}(y) = \frac{(\beta + 1)_n}{(\alpha + 1)_n} \cdot y^n Q_n^{(\beta, \alpha)} \left(\frac{1}{y} \right).$$

Proof. One may derive this result directly from the definition. Alternatively, one may use the well known fact that $P_n^{(\alpha, \beta)}(x) = (-1)^n P_n^{(\beta, \alpha)}(-x)$ and use formula (6). Then the substitution $y = (x - 1) / (x + 1)$ finishes the proof. $\square$

Theorem 8. *Let $n \in \mathbb{N}$ and $\alpha, \beta \in \mathbb{N}_0$. Then*

$$Q_n^{(-1/2+\alpha, -1/2+\beta)} \left(\frac{b}{c} \right) \neq 0$$

for all non-zero relatively prime integers b and c if either (i) 2 divides b or (ii) there exists a prime number $p \geq \alpha + 3$ dividing b but not $2\alpha + 1$, or (iii) there exists a prime number $p > (\beta + 3) / 2$ such that p^2 divides b .

Proof. By Proposition 7 there exists a non-zero rational number $r_n(\alpha, \beta)$ such that

$$(30) \quad Q_n^{(-1/2+\alpha, -1/2+\beta)}(b/c) = r_n(\alpha, \beta) \frac{b^n}{c^n} Q_n^{(-1/2+\beta, -1/2+\alpha)} \left(\frac{c}{b} \right).$$

Now apply Theorem 5 for the case $\delta = 1$ to the right hand side of (30). $\square$

Let us recall that the Jacobi polynomials $P_n^{(0,0)}(x)$ coincide with the Legendre polynomials. It is still an unsolved question whether the Legendre polynomials are irreducible over the rationals, see [23], [24], [30], [40] and [41]. H. Ille has shown that $P_n^{(0,0)}(x)$ has no quadratic factors which implies that $P_n^{(0,0)} \left(\sqrt{b/c} \right) \neq 0$ for all $n, b, c \in \mathbb{N}$ (even for the case $b = 1, 3$). In passing we note that recent research is devoted to the study of irreducibility of the Laguerre polynomials $L_n^\alpha(x)$ initiated by I. Schur, see [20], [22], [36], and for a family of Jacobi polynomials see [12]. For general questions about irreducibility of polynomial with rational coefficients we refer to [28], [31] and [38].

4. APPLICATIONS TO CHEBYSHEV POLYNOMIALS

Note that $Q_n^{(\alpha, \beta)}(x) > 0$ for all $x > 0$ whenever α, β are real numbers $\geq -1/2$. Let us take in Theorem 5 and 8 the parameters α and β equal to zero. Then we infer that

$$(31) \quad Q_n^{(-1/2, -1/2)}\left(\frac{b}{c}\right) \neq 0 \text{ for all } \frac{b}{c} \neq -1.$$

Taking α and β equal to 1 we infer that

$$(32) \quad Q_n^{(1/2, 1/2)}(b/c) \neq 0 \text{ for all } \frac{b}{c} \neq -1, -3, -1/3.$$

Next we shall show that indeed

$$(33) \quad Q_{3m-1}^{(1/2, 1/2)}(-1/3) = 0 \text{ and } Q_{3m-1}^{(1/2, 1/2)}(-3) = 0 \text{ and } Q_{2m-1}^{(1/2, 1/2)}(-1) = 0$$

for all natural numbers m ; in particular one can not omit in Theorem 5 the condition that the prime number p does not divide $3 = 2\beta + 1$ (with $\beta = 1$). For the proof of (33) we use that the relationship of the polynomial $P_n^{(1/2, 1/2)}(x)$ to the Chebyshev polynomial $U_n(x)$ of the second kind, namely

$$(34) \quad P_n^{(1/2, 1/2)}(x) = \frac{(2n+2)!}{2^{n+1}[(n+1)!]^2} U_n(x) = \frac{(2n+2)!}{2^{n+1}[(n+1)!]^2} \frac{\sin(n+1)\theta}{\sin\theta},$$

where $x = \cos\theta$, cf. [5, p. 241], and

$$\cos\left(\frac{\pi}{3}\right) = \frac{1}{2} \text{ and } \cos\left(\frac{2\pi}{3}\right) = -\frac{1}{2} \text{ and } \cos\left(\frac{\pi}{2}\right) = 0.$$

If we set $\theta = \pi/3$ then $x = \cos\theta = 1/2$ and $P_{3m-1}^{(1/2, 1/2)}(1/2) = 0$ by (34). Using (6) we infer that $Q_{3m-1}^{(1/2, 1/2)}(-1/3) = 0$. The cases $\theta = 2\pi/3$ and $\theta = \pi/2$ are similar.

Now we use Theorem 5 to derive the following result (see [27] and [39]):

Theorem 9. *The number $x := \cos \frac{k\pi}{m+1}$ is rational if and only if x is equal to one of the numbers $0, \pm 1, \pm 1/2$.*

Proof. We may assume that $m > 0$ and we put $\theta = k\pi/(m+1)$. By (34), $P_m^{(1/2, 1/2)}(x) = 0$ for $x = \cos\theta$. Assume that $x \neq 0, \pm 1$ and $x = b/c$. Then $b - c \neq 0$ and $b \neq 0$ and by (6)

$$0 = P_m^{(1/2, 1/2)}(b/c) = d_m Q_m^{(1/2, 1/2)}\left(\frac{b-c}{b+c}\right)$$

for some non-zero constant d_m . By (32) we conclude that $\frac{b-c}{b+c} \in \{-1, -3, -1/3\}$. It follows that either $b - c = -(b + c)$ which implies $b = 0$, or $b - c = -3b - 3c$, so $4b = -2c$, so $b/c = -1/2$, or $3(b - c) = -b - c$ which implies that $4b = 2c$, so $b/c = 1/2$. $\square$

Theorem 9 is a special case of the following result due to D.H. Lehmer [27]: Let $n > 2$ and k and n relatively prime. Then $2 \cos(2\pi k/n)$ is an algebraic integer of degree $\varphi(n)/2$ where φ is Euler's φ -function, see also [32, Theorem 3.9]. For example, we have

$$\cos(\pi/4) = 1/\sqrt{2} \text{ and } \cos(\pi/6) = \sqrt{3}/2.$$

The question when $\cos(2\pi k/d)$ is the square root of a positive rational number was discussed by J. L. Varona in [39] using recurrence relations, see also [4, Chapter I]. We shall give here an alternative proof based on Theorem 9.

Theorem 10. *Let k be an integer and $m \in \mathbb{N}_0$. Suppose that there exist natural numbers b, c such that*

$$\cos \frac{k\pi}{m+1} = \sqrt{b/c}.$$

Then $\cos(k\pi/(m+1))$ is equal to one of the numbers $0, 1, 1/2, 1/\sqrt{2}, \sqrt{3}/2$.

Proof. This is a simple consequence of Theorem 9 using that $2 \cos^2 \alpha - 1 = \cos(2\alpha)$. Thus, if $\cos(k\pi/(m+1))$ is a square root of a rational number, then $\cos(2k\pi/(m+1))$ is a rational number and by Theorem 9 is one of $0, \pm 1, \pm 1/2$. $\square$

5. APPLICATIONS TO THE DIRICHLET PROBLEM

Let $G \subset \mathbb{R}^d$ be a domain and ∂G the boundary of G . We say that the Dirichlet problem is solvable if for each continuous function f on ∂G there exists a continuous function u defined on the closure of G such that u is harmonic in G and $f(\xi) = u(\xi)$ for all $\xi \in \partial G$.

It is well known that the Dirichlet problem can be solved explicitly if G is a ball or an ellipsoid, see [7]. An elegant proof of this fact was presented in [25] (see also [8] and [9]), which can be extended to domains defined by quadratic polynomials in the following way:

Theorem 11. *Let $Q(x)$ be a polynomial of degree ≤ 2 . If Q is not a harmonic divisor then for each polynomial $f(x)$ of degree $\leq m$ there exists a harmonic polynomial u of degree $\leq m$ such that*

$$(35) \quad u(\xi) = f(\xi) \text{ for all } \xi \in Q^{-1}\{0\} := \{x \in \mathbb{R}^d : Q(x) = 0\}$$

Proof. Let $\mathcal{P}(\mathbb{R}^d)$ be the set of all polynomials in the variables $x_1, \dots, x_d$. The so-called Fischer operator $F_Q : \mathcal{P}(\mathbb{R}^d) \rightarrow \mathcal{P}(\mathbb{R}^d)$ is defined by

$$F_Q(p) := \Delta(Pq) \text{ for all } q \in \mathcal{P}(\mathbb{R}^d).$$

The fact that $Q(x)$ is *not* a harmonic divisor is equivalent to the injectivity of F_Q . Since $Q(x)$ is a polynomial of degree ≤ 2 the Fischer operator F_Q maps the space of all polynomials of degree $\leq m$ into itself. Therefore injectivity of F_Q implies the bijectivity of F_Q . To find the solution u of the Dirichlet problem one defines

$$u = f - Q \cdot F_Q^{-1}(\Delta(f)).$$

Then u obviously satisfies (35) and u is harmonic since $\Delta u = \Delta f - F_Q \circ F_Q^{-1}(\Delta f) = 0$. $\square$

Theorem 12. *Let $\gamma := \sqrt{b/c} < 1$ with relatively prime natural numbers b, c with $b \neq 1, 3$. Let Ω_γ be the cone defined in (1). Then for each polynomial f of degree $\leq m$ there exists a harmonic polynomial u of degree $\leq m$ such that $f(\xi) = u(\xi)$ for all $\xi \in \partial\Omega_\gamma$.*

Proof. The assumptions of Theorem 12 imply that Q_γ is not a harmonic divisor. By Theorem 11 there exists a harmonic polynomial u such that $u(\xi) = f(\xi)$ for all $\xi \in Q_\gamma^{-1}(0)$. Since $\partial\Omega_\gamma \subset Q_\gamma^{-1}(0)$ the proof is complete. $\square$

For more applications of the Fischer operator we refer to [35] and [37]. For a discussion of polynomial solutions in the Dirichlet problem (Khavinson-Shapiro conjecture) we refer to [10], [11], [13], [14], [19], [26], [29], [34].

Acknowledgements: The author wishes to thank Prof. Dr. G. Skordev for valuable discussions, and an unknown referee for improving condition (iii) in Theorem 5 and for providing elegant proofs of Lemma 2 and Theorem 10.

REFERENCES

- [1] M.L. Agranovsky, Y. Krasnov, *Quadratic Divisors of Harmonic polynomials in $\mathbb{R}^n$* , Journal D'Analyse Mathématique, 82 (2000), 379–395.
- [2] M.L. Agranovsky, E.T. Quinto, *Geometry of stationary sets for the wave equation in $\mathbb{R}^n$. The case of finitely supported initial data*, Duke Math. J. 107 (2001), 57–84.
- [3] M.L. Agranovsky, V.V. Volchkov, L.A. Zalcman, *Conical Uniqueness sets for the spherical Radon Transform*, Bull. London Math. Soc. 31 (1999), 231–236.
- [4] M. Aigner, G.M. Ziegler, *Proofs from THE BOOK*, 3rd ed., Springer 2004.
- [5] G.E. Andrews, R. Askey, R. Roy, *Special Functions*, Encyclopedia of Math. Appl. 71, Cambridge Univ. Press, 1999.
- [6] D. Armitage, *Cones on which entire harmonic functions can vanish*, Proc. Roy. Irish Acad. Sect. A 92 (1992), 107–110.
- [7] S. J. Axler, P. Bourdon, W. Ramey, *Harmonic Function Theory*, 2nd Edition, Springer, 2001.
- [8] S. Axler, P. Gorkin, K. Voss, *The Dirichlet problem on quadratic surfaces*, Math. Comp. 73 (2003), 637–651.
- [9] J. A. Baker, *The Dirichlet problem for ellipsoids*, Amer. Math. Monthly, Vol. 106, No. 9 (Nov., 1999), 829–834.
- [10] S.R. Bell, P. Ebenfelt, D. Khavinson, H.S. Shapiro, *On the classical Dirichlet problem in the plane with rational data*. J. d'Analyse Math., 100 (2006), 157–190.
- [11] M. Chamberland, D. Siegel, *Polynomial solutions to Dirichlet problems*, Proc. Amer. Math. Soc., 129 (2001), 211–217.
- [12] J. Cullinan, F. Hajir, E. Sell, *Algebraic properties of a family of Jacobi polynomials*, J. Théor. Nombres Bordeaux 21 (2009), 97–108.
- [13] P. Ebenfelt, *Singularities encountered by the analytic continuation of solutions to Dirichlet's problem*, Complex Variables, 20 (1992), 75–91.
- [14] P. Ebenfelt, D. Khavinson, H.S. Shapiro, *Algebraic Aspects of the Dirichlet problem*, Operator Theory: Advances and Applications, Vol 156., (2005), 151–172.
- [15] P. Ebenfelt, H. Render, *The mixed Cauchy problem with data on singular conics*, J. London Math. Soc. 78 (2008), 248–266.

- [16] P. Ebenfelt, H. Render, *The Goursat Problem for a Generalized Helmholtz Operator in $\mathbb{R}^2$* , Journal Analyse Math. 105 (2008), 149–168.
- [17] P. Ebenfelt, H.S. Shapiro, *The Cauchy–Kowaleskaya theorem and Generalizations*, Commun. Partial Differential Equations, 20 (1995), 939–960.
- [18] P. Ebenfelt, H.S. Shapiro, *The mixed Cauchy problem for holomorphic partial differential equations*, J. D’Analyse Math. 65 (1996) 237–295.
- [19] P. Ebenfelt, M. Viscardi, *On the Solution of the Dirichlet Problem with Rational Holomorphic Boundary Data*. Computational Methods and Function Theory, 5 (2005), 445–457.
- [20] M. Filaseta, T.Y. Lam, *On the irreducibility of the generalized Laguerre polynomials*, Acta Arith. 105 (2002), 177–182.
- [21] M. Filaseta, C. Finch, J. Russel Leidy, *T.N. Shorey’s influence in the theory of irreducible polynomials*, Diophantine equations, 77–102, Tata Inst. Fund. Res. Stud. Math., 20, Tata Inst. Fund. Res., Mumbai, 2008.
- [22] F. Hajir, *Algebraic properties of a family of generalized Laguerre polynomials*, Canad. J. Math. 61 (2009), no. 3, 583–603.
- [23] J. B. Holt, *On the irreducibility of Legendre polynomials II*, Proc. London Math. Soc. (2), 12 (1913), 126–132.
- [24] H. Ille, *Zur Irreduzibilität der Kugelfunktionen*, 1924, Jahrbuch der Dissertationen der Univ. Berlin.
- [25] D. Khavinson, H. S. Shapiro, *Dirichlet’s Problem when the data is an entire function*, Bull. London Math. Soc. 24 (1992), 456–468.
- [26] D. Khavinson, N. Stylianopoulos, *Recurrence relations for orthogonal polynomials and the Khavinson-Shapiro conjecture* (in preparation)
- [27] D.H. Lehmer, *A note on trigonometric algebraic numbers*, Amer. Math. Monthly, 40 (1933), 165–166.
- [28] A.K. Lenstra, H.W. Lenstra, Jr., L. Lovász, *Factoring polynomials with rational coefficients*, Math. Ann. 261 (1982), 515–534.
- [29] E. Lundberg, *Dirichlet’s problem and complex lightning bolts*, Comput. Methods and Function Theory, 9 (2009), No. 1, 111–125.
- [30] R.F. McCoart, *Irreducibility of certain classes of Legendre polynomials*, Duke Math. J. 28 (1961), 239–246.
- [31] J. Mott, *Eisenstein-type irreducibility criteria, Zero-dimensional commutative rings* (Knoxville, TN, 1994), 307–329, Lecture Notes in Pure and Appl. Math., 171, Dekker, New York, 1995.
- [32] I. Niven, *Irrational numbers*, Carus Monographs, Vol. 11, John Wiley and Sons, 1965.
- [33] T.D. Noe, *On the divisibility of generalized central trinomial coefficients*, J. Integer Sequences 9 (2006), 1–12.
- [34] M. Putinar, N. Stylianopoulos, *Finite-term relations for planar orthogonal polynomials*, Complex Anal. Oper. Theory 1 (2007), no. 3, 447–456.
- [35] H. Render, *Real Bargmann spaces, Fischer decompositions and Sets of Uniqueness for Polyharmonic Functions*, Duke Math. J. 142 (2008), 313–351.
- [36] E. Sell, *On a family of generalized Laguerre polynomials*, J. Number Theory 107 (2004), 266–281.
- [37] H.S. Shapiro, *An algebraic theorem of E. Fischer and the Holomorphic Goursat Problem*, Bull. London Math. Soc. 21 (1989), 513–537.
- [38] R. Thangadurai, *Irreducibility of polynomials whose coefficients are integers*, Math. Newsletter 17 (2007), 29–37.
- [39] J. L. Varona, *Rational values of the arccosine function*, Central European J. Math., 4 (2006), 319–322.
- [40] J.H. Wahab, *New cases of irreducibility for Legendre polynomials*, Duke Math. J. 19 (1952), 165–176.

- [41] J.H. Wahab, *New cases of irreducibility for Legendre polynomials II*, Duke Math. J. 27 (1960), 481–482.

UNIVERSITY COLLEGE DUBLIN, BELFIELD 4, DUBLIN, IRELAND.
E-mail address: `hermann.render@ucd.ie`