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The Khavinson-Shapiro conjecture and polynomial decompositions

Erik Lundberg

Department of Mathematics
University of South Florida
4202 E. Fowler Ave.
Tampa, FL 33617
USA

Hermann Render

School of Mathematical Sciences
University College Dublin
Dublin 4
Ireland

MANUSCRIPT CORRESPONDENCE INFORMATION:
Erik Lundberg
e-mail: elundber@mail.usf.edu
Fax: 813-974-2700
phone: 305-301-6522

Abstract

The main result of the paper states the following: Let ψ be a polynomial in n variables of degree t. Suppose that there exists a constant $C > 0$ such that any polynomial f has a polynomial decomposition $f = \psi q_f + h_f$ with $\Delta^k h_f = 0$ and $\deg q_f \leq \deg f + C$. Then $\deg \psi \leq 2k$. Here Δ^k is the kth iterate of the Laplace operator Δ. As an application, new classes of domains in $\mathbb{R}^n$ are identified for which the Khavinson-Shapiro conjecture holds.

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1. Introduction

A real-valued function h defined on an open set U in $\mathbb{R}^n$ is called *k-harmonic* or *polyharmonic of order k* if h is differentiable up to the order $2k$ and satisfies the equation $\Delta^k h(x) = 0$ for all $x \in U$. Here Δ denotes the Laplacian $\frac{\partial^2}{\partial x_1^2} + ... + \frac{\partial^2}{\partial x_n^2}$ and Δ^k is the kth iterate of the Laplace operator Δ. Polyharmonic functions have been studied extensively in [6], and they are useful in many branches in mathematics, see [22]. For example, in elasticity theory and dynamics of slow, viscous fluids polyharmonic functions of order 2, or more briefly, *biharmonic functions*, are very important.

Before discussing our main results we still need some notation. By $\mathbb{R}[x_1, ..., x_n]$ we denote the space of all polynomials with real coefficients in the variables $x_1, ..., x_n$. Frequently we use the fact that any polynomial ψ of degree m can be expanded into a sum of homogeneous polynomials ψ_j of degree j for $j = 0, ..., m$, and we write shortly $\psi = \psi_0 + ... + \psi_m$; here $\psi_m \neq 0$ is called the *principal part* or *leading part* of the polynomial ψ. The degree of a polynomial ψ is denoted by $\deg \psi$.

In this article we will be concerned with a conjecture (see below) which arises naturally from of the following statement proven in [25, Theorem 3] (for $k = 1$ see also [8]):

Theorem 1.1. *Let $\psi \in \mathbb{R}[x_1, ..., x_n]$ be a polynomial of degree $2k$ such that the leading part ψ_{2k} is non-negative. Then for any polynomial $f \in \mathbb{R}[x_1, ..., x_n]$ there exist unique polynomials q_f and h_f in $\in \mathbb{R}[x_1, ..., x_n]$ such that*

$$f = \psi q_f + h_f \text{ and } \Delta^k (h_f) = 0. \tag{1.1}$$

Moreover, the decomposition is degree preserving, meaning that $\deg h_f \leq \deg f$ and, consequently, $\deg q_f \leq \deg f - 2k$.

Theorem 1.1 is related to the polynomial solvability of Dirichlet-type problems. For example, let us consider the polynomial

$$\psi_0(x) = \sum_{j=1}^{n} \frac{x_j^2}{a_j^2} - 1, \tag{1.2}$$

so $E_0 := \left\{x \in \mathbb{R}^d : \psi_0(x) < 0\right\}$ is an ellipsoid. Then the decomposition (1.1) (where $k = 1$) shows the well known and old fact that for any polynomial f, restricted to the boundary ∂E_0, there exists a harmonic *polynomial* h which coincides with the data function f on ∂E_0. In other words: the solutions for polynomial data functions of the Dirichlet problem for the ellipsoid are again polynomials, see [7], [9], [12], or [20].

In [20] D. Khavinson and H.S. Shapiro formulated the following two conjectures (i) and (ii) for bounded domains Ω for which the Dirichlet problem is solvable:

(KS): Ω is an ellipsoid if for every polynomial f the solution of the Dirichlet problem u_f is (i) a polynomial and, respectively, (ii) entire.

Conjectures (i) and (ii) are still open, but important contributions have been made by several authors. Most of the results are proven for the two-dimensional case, see e.g. [12], [13], [23] and [17]. M. Putinar and N. Stylianopoulos have shown recently in [24] that the conjecture (i) for a simply connected bounded domain Ω in the complex plane is true if and only if the Bergman orthogonal polynomials satisfy a finite recurrence relation. D. Khavinson and N. Stylianopoulos proved among other things that the Bergman orthogonal polynomials satisfy a recurrence relation of order $N+1$ if and only if conjecture (i) holds and a degree condition for the solution u_f is satisfied, for details and further discussion see [21]. In [25] the second author has given a solution for (i) and (ii) for arbitrary dimension and for a large but not exhaustive class of domains.

The authors believe that the validation of the following conjecture for the case $k=1$ would be an important step for proving the Khavinson-Shapiro conjecture (e.g. confer the proof of Theorem 27 in [25]):

Conjecture 1.2. *Suppose $\psi \in \mathbb{R}[x_1, ..., x_n]$ is a polynomial, such that every polynomial $f \in \mathbb{R}[x_1, ..., x_n]$ has a decomposition $f = \psi q_f + h_f$, where h_f is polyharmonic of order k. Then $\deg \psi \leq 2k$.*

We are able to prove the conjecture if we add a degree condition on the involved polynomials which is in the spirit of the above-mentioned work [21]. More precisely, the main result of the present paper is the following:

Theorem 1.3. *Let $\psi \in \mathbb{R}[x_1, ..., x_n]$ be a polynomial. Suppose that there exists a constant $C > 0$ such that for any polynomial $f \in \mathbb{R}[x_1, ..., x_n]$ there exists a decomposition $f = \psi q_f + h_f$ with $\Delta^k h_f = 0$ and*

$$\deg q_f \leq \deg f + C. \tag{1.3}$$

Then $\deg \psi \leq 2k$.

Theorem 1.3 will be a consequence of a somewhat stronger result proved in Section 3 after a short discussion of harmonic divisors in Section 2. In passing we note that the conjecture 1.2 does not hold for polynomials ψ with complex coefficients, see [18].

It is a natural question under which conditions at the given polynomial $\psi(x)$ the degree condition in Theorem 1.3 is automatically satisfied. In other words, can we conclude from the equation

$$f = \psi q_f + h_f \text{ with } \Delta^k h_f = 0$$

a restriction on the degree of q_f or h_f in terms of the degree of f? For the case $k = 1$ we shall prove in Section 4 that the degree condition (1.3) is satisfied if (i) the leading part ψ_t of ψ contains a non-negative non-constant factor or (ii) ψ has a homogeneous expansion of the form $\psi = \psi_t + \psi_s + ... + \psi_0$ where $\psi_s \neq 0$ contains a non-negative non-constant factor. An extension of these results for arbitrary k is also given. These results allow to identify new types of domains in $\mathbb{R}^n$ for which the Khavinson-Shapiro conjecture is true.

2. Fischer operators and harmonic divisors

For $Q \in \mathbb{R}[x_1, ..., x_n]$ let us define $Q(D)$ as the differential operator replacing a monomial x^α appearing in Q by the differential operator $\partial^\alpha / \partial x^\alpha$, where α is a multi-index. For two polynomials Q and ψ we call the operator $F_\psi^Q : \mathbb{R}[x_1, ..., x_n] \to \mathbb{R}[x_1, ..., x_n]$ defined by

$$F_\psi^Q(q) := Q(D)(\psi q) \qquad q \in \mathbb{R}[x_1, ..., x_n] \tag{2.1}$$

the *Fischer operator*; for the significance of this notion we refer to the excellent exposition [26], or [8], [25]. We shall need the following result due to E. Fischer [16] which is in a slightly modified form valid for polynomials with complex coefficients, see [26]:

Theorem 2.1. *Let $Q \in \mathbb{R}[x_1, ..., x_n]$ be a homogeneous polynomial. Then the operator $q \longmapsto Q(D)(Qq)$ is bijective.*

At first we observe that the conjecture 1.2 is equivalent to the surjectivity of the Fischer operator with $Q = (\sum_{i=1}^n x_i^2)^k$; this fact is well known but for convenience of the reader we include the short proof.

Proposition 2.2. *Suppose $k \in \mathbb{N}$ and ψ is a polynomial. The operator*

$$F_{\psi}^{k}(q) := \Delta^{k}(\psi q)$$

is surjective if and only if every polynomial f can be decomposed as $f = \psi q_f + h_f$, where h is polyharmonic of order k.

Proof. Taking Δ^k of both sides of $f = \psi q + h$ gives $\Delta^k f = F_{\psi}^{k}(q)$. Given g we can find f such that $g = \Delta^k f$, showing F_{ψ}^{k} is surjective. Conversely, if F_{ψ}^{k} is surjective, then given f there is a q such that $\Delta^k f = F_{\psi}^{k}(q)$, showing that $h = f - \psi q$ is polyharmonic of order k. □

A polynomial f_m is called *homogeneous* of degree m if $f_m(rx) = r^m f_m(x)$ for all $r > 0$ and for all $x \in \mathbb{R}^n$. We will use $\mathbb{P}^N$ to denote the space of polynomials of degree at most N, and $\mathbb{P}_{\text{hom}}^{N}$ the space of homogeneous polynomials of degree N. For a homogeneous polynomial ψ we define the space of all homogeneous k-harmonic divisors of degree m of ψ by

$$D_k^m(\psi) = \left\{ q \in \mathbb{P}_{\text{hom}}^{m} : \Delta^k(\psi q) = 0 \right\}.$$

For $k = 1$ we obtain the definition of a harmonic divisor (of degree m) which arises in the investigation of stationary sets for the wave and heat equation, see [2], [3], and the injectivity of the spherical Radon transform, see [4], [1].

It is an interesting but difficult problem to compute the dimension of the space $D_k^m(\psi)$ in dependence of the polynomial ψ. In the proof of our main result Theorem 1.3 we shall use the rough upper estimate provided in the next proposition and the remarks following:

Proposition 2.3. *Let $\psi \in \mathbb{R}[x_1, ..., x_n]$ be a homogeneous polynomial. Then*

$$\dim D_k^m(\psi) \leq \dim \left\{ f \in \mathbb{P}_{\text{hom}}^{m} : \Delta^k f = 0 \right\}.$$

Proof. Let $q \in D_k^m(\psi)$. Then $q \in \mathbb{P}_{\text{hom}}^{m}$ and $q\psi = h$ for some $h \in \mathbb{P}_{\text{hom}}^{m+t}$ with $\Delta^k h = 0$, where t is the degree of ψ. Clearly we have $\psi(D)(\psi q) = \psi(D) h$ and

$$0 = \psi(D)\left(\Delta^k h\right) = \Delta^k(\psi(D) h). \tag{2.2}$$

By Theorem 2.1 the operator F defined by $F(q) = \psi(D)(\psi q)$ is bijective, and from $\psi q = h$ we infer that $q = F^{-1}(w)$ with $w := \psi(D) h$. Equation (2.2) shows that $w \in \left\{ f \in \mathbb{P}_{\text{hom}}^{m} : \Delta^k f = 0 \right\}$. Thus

$$D_k^m(\psi) \subset F^{-1}\left(\left\{ f \in \mathbb{P}_{\text{hom}}^{m} : \Delta^k f = 0 \right\}\right).$$

Since F^{-1} is a bijective operator the claim is now obvious. □

Let us define $H_k^m := \left\{ f \in \mathbb{P}_{\text{hom}}^m : \Delta^k f = 0 \right\}$. By Theorem 2.1 for $Q(x) = |x|^{2k}$ it follows that any polynomial f has a Fischer decomposition $f = |x|^{2k} q + h$ where h is k-harmonic. Moreover, h and q are homogeneous iff f is. So we have

$$\mathbb{P}_{\text{hom}}^m = |x|^{2k} \mathbb{P}_{\text{hom}}^{m-2k} \oplus H_k^m.$$

Thus we obtain

$$\dim D_k^m(\psi) \leq \dim H_k^m = \dim \mathbb{P}_{\text{hom}}^m - \dim \mathbb{P}_{\text{hom}}^{m-2k}. \tag{2.3}$$

The following question was posed by M. Agranovsky for the case $k = 1$ in [1], where it was also answered in the case that ψ factors completely into linear factors.

Question 2.4 (Agranovksy)**.** *What is the asymptotic behavior of* $\dim D_k^m(\psi)$, *as $m \to \infty$?*

We expect that a full answer to this question would allow to relax the asumption on degree appearing in Theorem 1.3.

3. Proof of the main result

Assume that $2k \leq t$ and let ψ be a polynomial of degree $\leq t$ and let F_ψ^k be the Fischer operator defined in Proposition 2.2. The following technical notion is a crucial tool for proving our main result Theorem 1.3: For a natural number M define $S_i \subset \mathbb{P}^i$ as the subspace whose image under F_ψ^k is contained in $\mathbb{P}^{M+t-2k}$, i.e.,

$$S_i := \{ q \in \mathbb{P}^i : F_\psi^k(q) \in \mathbb{P}^{M+t-2k} \}$$

for $i \in \mathbb{N}_0$. Since ψ has degree $\leq t$ it follows that

$$\mathbb{P}^M = S_M \subset S_{M+1} \subset ... \subset S_{M+j}$$

for all $j \geq 1$.

Proposition 3.1. *Let $\psi = \psi_t + ... + \psi_0$ be a polynomial of degree $\leq t$ and let M be a natural number. Then for all $j \in \mathbb{N}$*

$$\dim S_{M+j} \leq \dim S_{M+j-1} + \dim D_k^{M+j}(\psi_t).$$

Proof. For given $j \in \mathbb{N}$ we will construct a space Q_j such that $S_{M+j} = S_{M+j-1} \oplus Q_j$, and $\dim Q_j \leq \dim D_k^{M+j}(\psi_t)$. First define $Q_{H,j} := \{q_{M+j} : q_{M+j}$ is the degree-$(M+j)$ homogeneous term of some $q \in S_{M+j}\}$. Choose (finitely many) polynomials in S_{M+j} whose leading terms form a basis for $Q_{H,j}$, and define Q_j to be the subspace of S_{M+j} spanned by these polynomials. Suppose $\hat{q} \in S_{M+j}$. The degree-$(M+j)$ homogeneous term $\hat{q}_{M+j}$ (possibly zero) can be matched by the leading homogeneous term of some $q \in Q_j$ so that $\hat{q} - q \in S_{M+j-1}$. This shows that $S_{M+j} = S_{M+j-1} \oplus Q_j$.

Now, we will establish $\dim Q_j \leq \dim D_k^{M+j}(\psi_t)$. It suffices to show that $Q_{H,j} \subset D_k^{M+j}(\psi_t)$, since $\dim Q_j = \dim Q_{H,j}$ by construction. Suppose $q_{M+j} \in Q_{H,j}$ is nonzero, i.e., there is a $q \in S_{M+j}$ and $\deg q = M+j$ such that q_{M+j} is the leading homogeneous term of q. Since $F_\psi^k(q) \in \mathbb{P}^{M+t-2k}$, we have $\deg(\Delta^k(\psi q)) \leq M+t-2k$. This implies that the leading term, $\Delta^k(\psi_t q_{M+j})$, of $\Delta^k(\psi q)$ is zero (since it has degree $M+j+t-2k$). i.e., $\psi_t q_{M+j}$ is k-harmonic. Therefore, $Q_{H,j} \subset D_k^{M+j}$. □

The main result of this paper, Theorem 1.3, follows now from the following more general result by taking $\alpha = 1$:

Theorem 3.2. *Let ψ be a polynomial of degree t. Suppose that there exist constants $\alpha \geq 1$, $C > 0$ such that for any polynomial f there exists a decomposition $f = \psi q_f + h_f$ with $\Delta^k h_f = 0$ and*

$$\deg q_f \leq \alpha \deg f + C.$$

Then $t \leq 2k \cdot \alpha^{n-1}$.

Proof. Suppose $t \geq 2k$. (If $t < 2k$, there is nothing to prove.) Let $f \in \mathbb{P}^{M+t-2k}$ and suppose that $M > 2k$. Choose a polynomial $g \in \mathbb{P}^{M+t}$ with $\Delta^k g = f$. By assumption there exists q_f and h_f with $g = \psi q_f + h_f$ and $\Delta^k h_f = 0$ and $\deg q_f \leq \alpha(M+t) + C$. Then $f = \Delta^k g = F_\psi^k(q_f)$ and we infer the inclusion

$$\mathbb{P}^{M+t-2k} \subset F_\psi^k\left(\mathbb{P}^{B_M}\right) \tag{3.1}$$

with $B_M := \alpha M + \alpha t + C \geq M$. Using the above notation $S_{B_M} = \{q \in \mathbb{P}^{B_M} : F_\psi^k(q) \in \mathbb{P}^{M+t-2k}\}$ we see that (3.1) implies that $\mathbb{P}^{M+t-2k} \subset F_\psi^k(S_{B_M})$. Since F_ψ^k is a linear operator, we have

$$\dim \mathbb{P}^{M+t-2k} \leq \dim F_\psi^k(S_{B_M}) \leq \dim S_{B_M}. \tag{3.2}$$

Applying Proposition 3.1 inductively we obtain

$$\dim S_{B_M} \leq \dim(\mathbb{P}^M) + \sum_{j=M+1}^{B_M} \dim D_k^j(\psi_t) \tag{3.3}$$

Since $\mathbb{P}^{M+t-2k} = \mathbb{P}^M \oplus \mathbb{P}^{M+1}_{\text{hom}} \oplus ... \oplus \mathbb{P}^{M+t-2k}_{\text{hom}}$ and $\dim \mathbb{P}^{M+1}_{\text{hom}} \leq \dim \mathbb{P}^{M+j}_{\text{hom}}$ for $j \geq 1$ we infer from (3.2) and (3.3) the interesting formula

$$(t-2k)\dim \mathbb{P}^{M+1}_{\text{hom}} \leq \sum_{j=M+1}^{B_M} \dim D_k^j(\psi_t)\,. \tag{3.4}$$

Further we know from (2.3) that $\dim D_k^j(\psi_t) \leq \dim \mathbb{P}^j_{\text{hom}} - \dim \mathbb{P}^{j-2k}_{\text{hom}}$. Thus the right hand side in (3.4) is a telescoping sum. Using that $\dim \mathbb{P}^j_{\text{hom}} \leq \dim \mathbb{P}^{B_M}_{\text{hom}}$ for $j = B_M - 2k + 1, ..., B_M$ and $\dim \mathbb{P}^{M+1-2k}_{\text{hom}} \leq \dim \mathbb{P}^j_{\text{hom}}$ for the lower indices we can estimate

$$\sum_{j=M+1}^{B_M} \dim D_k^j(\psi_t) \leq 2k \dim \mathbb{P}^{B_M}_{\text{hom}} - 2k \dim \mathbb{P}^{M+1-2k}_{\text{hom}}.$$

Thus we infer from (3.4) and the well known fact

$$\dim \mathbb{P}^{M+1}_{\text{hom}} = \binom{n+M}{n-1} = \binom{n+M}{M+1},$$

proven in [7] that

$$\begin{aligned} & (t-2k)\,\frac{(M+2)\,...\,(M+n)}{(n-1)!} \\ \leq \;\; & 2k\frac{(B_M+1)\,...\,(B_M+n-1) - (M+2-2k)\,....\,(M+n-2k)}{(n-1)!} \end{aligned}$$

Clearly the term $(n-1)!$ can be canceled in the inequality. Divide the inequality by M^{n-1} on both sides and recall that $B_M = \alpha M + \alpha t + C$. Now take the limit $M \to \infty$ and we obtain

$$t - 2k \leq 2k\left(\alpha^{n-1} - 1\right).$$

This implies $t \leq 2k\alpha^{n-1}$ and the proof is complete. □

4. Criteria for degree-related decompositions

We are now turning to the question under which conditions the degree condition is automatically satisfied. The first criterion is simple to prove:

Proposition 4.1. *Suppose that ψ is a polynomial of degree $t > 2$ and $\psi = \psi_t + ... + \psi_0$ is the decomposition into a sum of homogeneous polynomials. Assume the polynomial ψ_t contains a non-negative non-constant factor. Let f be a polynomial and assume that there exists a decomposition*

$$f = \psi q + h$$

where h is harmonic and q is a polynomial. Then $\deg q \leq \deg f - t$ and $\deg h \leq \deg f$.

Proof. Write $q = q_M + ... + q_0$ with homogeneous polynomials q_j of degree $j = 0, ..., M$. Expand the product ψq into a sum of homogeneous polynomials, so $\psi q = \psi_t q_M + R(x)$ where $R(x)$ is a polynomial of degree $< M + t$. Suppose that $M + t > \deg f$. Since $\Delta f = \Delta(\psi q)$ we conclude that $\Delta(\psi_t q_M) = 0$, so $\psi_t q_M$ is harmonic. By the Brelot-Choquet theorem, a harmonic polynomial cannot have non-negative factors, see [11]. Thus $\psi_t q_M = 0$, and we obtain a contradiction. □

The next criterion is more difficult to prove and uses again ideas from the proof of the Brelot-Choquet theorem:

Theorem 4.2. *Suppose that ψ is a polynomial of degree $t > 2$ and $\psi = \psi_t + \psi_s + \psi_{s-1} + ... + \psi_0$ is the decomposition into a sum of homogeneous polynomials. Assume the polynomial ψ_s is non-zero and contains a non-negative non-constant factor. Let f be a polynomial and assume that there exists a decomposition*

$$f = \psi q + h$$

where h is harmonic and q is a polynomial. Then $\deg q \leq 2 - s + \deg f$ and $\deg h \leq t + 2 - s + \deg f$.

Before proving Theorem 4.2 we notice the following conclusion:

Corollary 4.3. *Suppose that ψ is a polynomial with a non-zero second-highest degree term that contains a non-negative factor. If every polynomial f has a Fischer decomposition $f = \psi q_f + h_f$ with h_f harmonic, then $\deg(\psi) \leq 2$.*

Proof. Suppose $\deg(\psi) > 2$. By Theorem 4.2, $\deg q_f - \deg f$ is bounded. Now we can apply Theorem 1.3, to obtain $\deg \psi \leq 2$. $\square$

The following lemma is needed for the proof of Theorem 4.2:

Lemma 4.4. *Suppose that ψ is a polynomial of degree $t > 2$ and $\psi = \psi_t + \psi_s + \psi_{s-1} + ... + \psi_0$ is the decomposition into a sum of homogeneous polynomials. Assume that $g \in \mathbb{P}^m$ and q is a polynomial of degree M such that $F_\psi^k(q) := \Delta(\psi q) = g$ and $M + s > m$. Then for every $p \in \mathbb{P}^{s-1}$,*

$$\int_{\mathbb{S}^{n-1}} q_M^2 \cdot \psi_s \cdot p \; d\theta = 0,$$

where $q_M \neq 0$ is the senior term of q.

Proof (of lemma). Write $q = q_M + ... + q_0$ with homogeneous polynomials q_j of degree $j = 0, ..., M$. Expand the product ψq into a sum of homogeneous polynomials,

$$\psi q = \psi_t q_M + ... + \psi_t q_{M-t+s+1} + (\psi_t q_{M-t+s} + \psi_s q_M) + R(x) \tag{4.1}$$

where $R(x)$ is a polynomial of degree $< M+s$. Since $\Delta(\psi q) = g$ and $M+s > m$, we conclude that $\Delta(\psi_t q_M) = 0$ and $\Delta(\psi_t q_{M-t+s} + \psi_s q_M) = 0$. Thus, we can write

$$\psi_t q_M = h_{M+t} \tag{4.2}$$

$$\psi_t q_{M-t+s} + \psi_s q_M = h_{M+s}, \tag{4.3}$$

where h_{M+t} and h_{M+s} are homogeneous harmonic polynomials.

Take $p \in \mathbb{P}^{s-1}$, and multiply equation (4.3) by $q_M p$ and integrate over the unit sphere, $\mathbb{S}^{n-1}$. Then

$$\int_{\mathbb{S}^{n-1}} \psi_t q_{M-t+s} \cdot q_M p \; d\theta + \int_{\mathbb{S}^{n-1}} \psi_s q_M^2 \cdot p \; d\theta = \int_{\mathbb{S}^{n-1}} h_{M+s} \cdot q_M p \; d\theta.$$

Since $\deg(q_M p) < M+s$ and h_{M+s} is harmonic, the integral on the right-hand side is zero. Indeed, homogeneous harmonics of different degree are orthogonal in the space $L^2(\mathbb{S}^{n-1})$ (see [7]), and, moreover, $q_M p$ can be matched on $\mathbb{S}^{n-1}$ by a harmonic polynomial of not higher degree. Substituting equation 4.2 into the first integral on the left-hand side gives $\int_{\mathbb{S}^{n-1}} h_{M+t} \cdot p \cdot q_{M-t+s} d\theta$, which is also zero, since $\deg(p q_{M-t+s}) < M + t$. $\square$

Proof of Theorem 4.2. By assumption we may write $\psi_s = \phi P$ where ϕ is non-negative and P has degree $< s$. Suppose that $M + s > \deg f + 2$. We have $\Delta(\psi q) = \Delta f$ and $M + s > deg(\Delta f)$. Then, q, ψ satisfy Lemma 4.4 with $g = \Delta f$, and thus $\int_{\mathbb{S}^{n-1}} q_M^2 \cdot \psi_s \cdot p \, d\theta = 0$, for all p of degree $< s$. In particular, this is true for $p = P$. Hence,

$$0 = \int_{\mathbb{S}^{n-1}} q_M^2 \cdot \psi_s \cdot P d\theta = \int_{\mathbb{S}^{n-1}} q_M^2 \cdot \phi \cdot P^2 d\theta.$$

Since $P \neq 0$, $\phi \neq 0$, and $\phi(\theta) \geq 0$ for all $\theta \in \mathbb{S}^{n-1}$, we have the contradiction $q_M = 0$. □

The following instructive example is due to L. Hansen and H.S. Shapiro [17]; it was also suggested in [19] as a simple example for which the Khavinson-Shapiro conjecture is unresolved (whenever φ is a cubic): Let $\varphi \in \mathbb{R}[x_1, ..., x_n]$ be a homogeneous harmonic polynomial of degree > 2, in particular φ does not contain a nonnegative non-constant factor, see [11]. We perturb the equation for the unit ball $|x|^2 - 1$ by $\varepsilon\varphi$, i.e. we consider

$$\psi_\varepsilon(x) := |x|^2 - 1 + \varepsilon\varphi(x) \text{ for } \varepsilon > 0. \tag{4.4}$$

If $\varepsilon > 0$ is small enough, then the component of $E_\varepsilon := \{\psi_\varepsilon < 0\}$ containing 0 is a bounded domain in $\mathbb{R}^d$. Then the Dirichlet problem for the data function $|x|^2 = x_1^2 + ... + x_n^2$ restricted to ∂E_ε has a harmonic polynomial solution $u_f(x) = 1 - \varepsilon\varphi(x)$ since

$$|x|^2 = \psi_\varepsilon(x) \cdot 1 + 1 - \varepsilon\varphi(x).$$

Note that in this example the degree of the solution u_f for the Dirichlet problem is higher than the degree of the data function f.

The question arises whether any polynomial data function may have a polynomial solution. If this is the case, and ψ_ε is irreducible and changes the sign in a neighborhood of some point in ∂E_ε then the proof of Theorem 27 in [25] implies that for any polynomial f there exists a decomposition $f = \psi_\varepsilon q_f + h_f$ where h_f is harmonic. By Corollary 4.3 $\deg \psi_\varepsilon \leq 2$. Thus we have proved that for this class of examples the Khavinson-Shapiro conjecture is true.

In the rest of this section we extend Theorem 4.2 to the case $k \geq 1$. We consider the following inner product

$$\langle f, g \rangle := \int_{\mathbb{R}^n} f(x)g(x)e^{-|x|^2} dx \tag{4.5}$$

and the following orthogonality condition established in [25].

Theorem 4.5. *Suppose that f is a homogeneous polynomial, and let $k \in \mathbb{N}$ with $2(k-1) \leq$ deg f. Then $\Delta^k f = 0$ if and only if $\langle f, g \rangle = 0$ for all polynomials g with $2(k-1) +$ deg $g <$ deg f.*

Theorem 4.6. *Suppose that ψ is a polynomial of degree $t > s$ and $\psi = \psi_t + \psi_s + \psi_{s-1} + ... + \psi_0$ is the decomposition into a sum of homogeneous polynomials. Assume the polynomial $\psi_s \neq 0$ is non-negative. If the polynomial f has the decomposition*

$$f = \psi q + h$$

where h is k-harmonic, then $\deg(q) \leq 2k - s + \deg f$.

Proof. Suppose that $M + s > 2k + \deg f$, where $f = \psi q + h$ and $M = \deg q$. We will derive a contradiction. We proceed as in the proof of Lemma 4.4 writing $q = q_M + ... + q_0$ with homogeneous polynomials q_j of degree $j = 0, ..., M$. Expand the product ψq as in (4.1). Then we conclude that $\Delta^k(\psi_t q_M) = 0$ and $\Delta^k(\psi_t q_{M-t+s} + \psi_s q_M) = 0$. Thus, we can write

$$\psi_t q_M = H_{M+t} \tag{4.6}$$

$$\psi_t q_{M-t+s} + \psi_s q_M = H_{M+s}, \tag{4.7}$$

where H_{M+t} and H_{M+s} are homogeneous k-harmonic polynomials. Next take the inner product (4.5) of both sides of equation (4.7) with q_M. Then

$$\langle q_{M-t+s}, q_M \psi_t \rangle + \langle \psi_s, q_M^2 \rangle = \langle H_{M+s}, q_M \rangle$$

Using equation 4.6, we arrive at $\langle q_{M-t+s}, H_{M+t} \rangle + \langle \psi_s, q_M^2 \rangle = \langle H_{M+s}, q_M \rangle$. Now we use Theorem 4.5. Since $\deg H_{M+t} > \deg q_{M-t+s} + 2(k-1)$ and $\deg H_{M+s} > \deg q_M + 2(k-1)$, the first term on the left and the term on the right are both zero. Thus, $\langle \psi_s, q_M^2 \rangle = 0$ implies $q_M = 0$ (since $\psi \neq 0$ is non-negative), a contradiction. $\square$

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