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The Khavinson-Shapiro conjecture and polynomial decompositions

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Abstract

The main result of the paper states the following: Let $\psi$ be a polynomial in $n$ variables of degree $t$. Suppose that there exists a constant $C > 0$ such that any polynomial $f$ has a polynomial decomposition $f = \psi q_f + h_f$ with $\Delta^k h_f = 0$ and $\deg q_f \leq \deg f + C$. Then $\deg \psi \leq 2k$. Here $\Delta^k$ is the $k$th iterate of the Laplace operator $\Delta$. As an application, new classes of domains in $\mathbb{R}^n$ are identified for which the Khavinson-Shapiro conjecture holds.

Keywords: harmonic and polyharmonic polynomials, Fischer decompositions, harmonic divisors, algebraic Dirichlet problems


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1. Introduction

A real-valued function $h$ defined on an open set $U$ in $\mathbb{R}^n$ is called \emph{k-harmonic} or \emph{polyharmonic of order $k$} if $h$ is differentiable up to the order $2k$ and satisfies the equation $\Delta^k h(x) = 0$ for all $x \in U$. Here $\Delta$ denotes the Laplacian $\frac{\partial^2}{\partial x_1^2} + \ldots + \frac{\partial^2}{\partial x_n^2}$ and $\Delta^k$ is the $k$th iterate of the Laplace operator $\Delta$. Polyharmonic functions have been studied extensively in [6], and they are useful in many branches in mathematics, see [22]. For example, in elasticity theory and dynamics of slow, viscous fluids polyharmonic functions of order 2, or more briefly, \emph{biharmonic functions}, are very important.

Before discussing our main results we still need some notation. By $\mathbb{R}[x_1, \ldots, x_n]$ we denote the space of all polynomials with real coefficients in the variables $x_1, \ldots, x_n$. Frequently we use the fact that any polynomial $\psi$ of degree $m$ can be expanded into a sum of homogeneous polynomials $\psi_j$ of degree $j$ for $j = 0, \ldots, m$, and we write shortly $\psi = \psi_0 + \ldots + \psi_m$; here $\psi_m \neq 0$ is called the \emph{principal part} or \emph{leading part} of the polynomial $\psi$. The degree of a polynomial $\psi$ is denoted by $\deg \psi$.

In this article we will be concerned with a conjecture (see below) which arises naturally from of the following statement proven in [25, Theorem 3] (for $k = 1$ see also [8]):

\textbf{Theorem 1.1.} Let $\psi \in \mathbb{R}[x_1, \ldots, x_n]$ be a polynomial of degree $2k$ such that the leading part $\psi_{2k}$ is non-negative. Then for any polynomial $f \in \mathbb{R}[x_1, \ldots, x_n]$ there exist unique polynomials $q_f$ and $h_f$ in $\mathbb{R}[x_1, \ldots, x_n]$ such that

$$f = \psi q_f + h_f \quad \text{and} \quad \Delta^k (h_f) = 0. \quad (1.1)$$

Moreover, the decomposition is degree preserving, meaning that $\deg h_f \leq \deg f$ and, consequently, $\deg q_f \leq \deg f - 2k$.

Theorem 1.1 is related to the polynomial solvability of Dirichlet-type problems. For example, let us consider the polynomial

$$\psi_0(x) = \sum_{j=1}^n \frac{x_j^2}{a_j^2} - 1, \quad (1.2)$$

so $E_0 := \{x \in \mathbb{R}^d : \psi_0(x) < 0\}$ is an ellipsoid. Then the decomposition (1.1) (where $k = 1$) shows the well known and old fact that for any polynomial $f$, ...
restricted to the boundary $\partial E_0$, there exists a harmonic polynomial $h$ which coincides with the data function $f$ on $\partial E_0$. In other words: the solutions for polynomial data functions of the Dirichlet problem for the ellipsoid are again polynomials, see [7], [9], [12], or [20].

In [20] D. Khavinson and H.S. Shapiro formulated the following two conjectures (i) and (ii) for bounded domains $\Omega$ for which the Dirichlet problem is solvable:

(KS): $\Omega$ is an ellipsoid if for every polynomial $f$ the solution of the Dirichlet problem $u_f$ is (i) a polynomial and, respectively, (ii) entire.

Conjectures (i) and (ii) are still open, but important contributions have been made by several authors. Most of the results are proven for the two-dimensional case, see e.g. [12], [13], [23] and [17]. M. Putinar and N. Stylianopoulos have shown recently in [24] that the conjecture (i) for a simply connected bounded domain $\Omega$ in the complex plane is true if and only if the Bergman orthogonal polynomials satisfy a finite recurrence relation. D. Khavinson and N. Stylianopoulos proved among other things that the Bergman orthogonal polynomials satisfy a recurrence relation of order $N + 1$ if and only if conjecture (i) holds and a degree condition for the solution $u_f$ is satisfied, for details and further discussion see [21]. In [25] the second author has given a solution for (i) and (ii) for arbitrary dimension and for a large but not exhaustive class of domains.

The authors believe that the validation of the following conjecture for the case $k = 1$ would be an important step for proving the Khavinson-Shapiro conjecture (e.g. confer the proof of Theorem 27 in [25]):

**Conjecture 1.2.** Suppose $\psi \in \mathbb{R}[x_1, \ldots, x_n]$ is a polynomial, such that every polynomial $f \in \mathbb{R}[x_1, \ldots, x_n]$ has a decomposition $f = \psi q_f + h_f$, where $h_f$ is polyharmonic of order $k$. Then $\deg \psi \leq 2k$.

We are able to prove the conjecture if we add a degree condition on the involved polynomials which is in the spirit of the above-mentioned work [21]. More precisely, the main result of the present paper is the following:

**Theorem 1.3.** Let $\psi \in \mathbb{R}[x_1, \ldots, x_n]$ be a polynomial. Suppose that there exists a constant $C > 0$ such that for any polynomial $f \in \mathbb{R}[x_1, \ldots, x_n]$ there exists a decomposition $f = \psi q_f + h_f$ with $\Delta^k h_f = 0$ and

$$\deg q_f \leq \deg f + C.$$  \hspace{1cm} (1.3)

Then $\deg \psi \leq 2k$.  

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Theorem 1.3 will be a consequence of a somewhat stronger result proved in Section 3 after a short discussion of harmonic divisors in Section 2. In passing we note that the conjecture 1.2 does not hold for polynomials \( \psi \) with complex coefficients, see [18].

It is a natural question under which conditions at the given polynomial \( \psi(x) \) the degree condition in Theorem 1.3 is automatically satisfied. In other words, can we conclude from the equation

\[
f = \psi q_f + h_f \quad \text{with} \quad \Delta^k h_f = 0
\]

a restriction on the degree of \( q_f \) or \( h_f \) in terms of the degree of \( f \)? For the case \( k = 1 \) we shall prove in Section 4 that the degree condition (1.3) is satisfied if (i) the leading part \( \psi_t \) of \( \psi \) contains a non-negative non-constant factor or (ii) \( \psi \) has a homogeneous expansion of the form \( \psi = \psi_t + \psi_s + ... + \psi_0 \) where \( \psi_s \neq 0 \) contains a non-negative non-constant factor. An extension of these results for arbitrary \( k \) is also given. These results allow to identify new types of domains in \( \mathbb{R}^n \) for which the Khavinson-Shapiro conjecture is true.

2. Fischer operators and harmonic divisors

For \( Q \in \mathbb{R}[x_1, ..., x_n] \) let us define \( Q(D) \) as the differential operator replacing a monomial \( x^\alpha \) appearing in \( Q \) by the differential operator \( \partial^\alpha / \partial x^\alpha \), where \( \alpha \) is a multi-index. For two polynomials \( Q \) and \( \psi \) we call the operator \( F_{Q}^{\psi} : \mathbb{R}[x_1, ..., x_n] \rightarrow \mathbb{R}[x_1, ..., x_n] \) defined by

\[
F_{\psi}^{Q}(q) := Q(D)(\psi q) \quad q \in \mathbb{R}[x_1, ..., x_n]
\]

the Fischer operator; for the significance of this notion we refer to the excellent exposition [26], or [8], [25]. We shall need the following result due to E. Fischer [16] which is in a slightly modified form valid for polynomials with complex coefficients, see [26]:

**Theorem 2.1.** Let \( Q \in \mathbb{R}[x_1, ..., x_n] \) be a homogeneous polynomial. Then the operator \( q \mapsto Q(D)(Qq) \) is bijective.

At first we observe that the conjecture 1.2 is equivalent to the surjectivity of the Fischer operator with \( Q = (\sum_{i=1}^{n} x_i^2)^k \); this fact is well known but for convenience of the reader we include the short proof.
Proposition 2.2. Suppose $k \in \mathbb{N}$ and $\psi$ is a polynomial. The operator
\[ F^k_\psi(q) := \Delta^k (\psi q) \]
is surjective if and only if every polynomial $f$ can be decomposed as $f = \psi q_f + h_f$, where $h$ is polyharmonic of order $k$.

Proof. Taking $\Delta^k$ of both sides of $f = \psi q + h$ gives $\Delta^k f = F^k_\psi(q)$. Given $g$ we can find $f$ such that $g = \Delta^k f$, showing $F^k_\psi$ is surjective. Conversely, if $F^k_\psi$ is surjective, then given $f$ there is a $q$ such that $\Delta^k f = F^k_\psi(q)$, showing that $h = f - \psi q$ is polyharmonic of order $k$. \qed

A polynomial $f_m$ is called homogeneous of degree $m$ if $f_m(rx) = r^m f_m(x)$ for all $r > 0$ and for all $x \in \mathbb{R}^n$. We will use $\mathbb{P}^N$ to denote the space of polynomials of degree at most $N$, and $\mathbb{P}^N_{\text{hom}}$ the space of homogeneous polynomials of degree $N$. For a homogeneous polynomial $\psi$ we define the space of all homogeneous $k$-harmonic divisors of degree $m$ of $\psi$ by
\[ D^m_k(\psi) = \{ q \in \mathbb{P}^m_{\text{hom}} : \Delta^k (\psi q) = 0 \} . \]

For $k = 1$ we obtain the definition of a harmonic divisor (of degree $m$) which arises in the investigation of stationary sets for the wave and heat equation, see [2], [3], and the injectivity of the spherical Radon transform, see [4], [1].

It is an interesting but difficult problem to compute the dimension of the space $D^m_k(\psi)$ in dependence of the polynomial $\psi$. In the proof of our main result Theorem 1.3 we shall use the rough upper estimate provided in the next proposition and the remarks following:

Proposition 2.3. Let $\psi \in \mathbb{R}[x_1, \ldots, x_n]$ be a homogeneous polynomial. Then
\[ \dim D^m_k(\psi) \leq \dim \{ f \in \mathbb{P}^m_{\text{hom}} : \Delta^k f = 0 \} . \]

Proof. Let $q \in D^m_k(\psi)$. Then $q \in \mathbb{P}^m_{\text{hom}}$ and $q \psi = h$ for some $h \in \mathbb{P}^{m+t}_{\text{hom}}$ with $\Delta^k h = 0$, where $t$ is the degree of $\psi$. Clearly we have $\psi(D) (\psi q) = \psi(D) h$ and
\[ 0 = \psi(D) \left( \Delta^k h \right) = \Delta^k \left( \psi(D) h \right) . \tag{2.2} \]

By Theorem 2.1 the operator $F$ defined by $F(q) = \psi(D)(\psi q)$ is bijective, and from $\psi q = h$ we infer that $q = F^{-1}(w)$ with $w := \psi(D) h$. Equation (2.2) shows that $w \in \{ f \in \mathbb{P}^m_{\text{hom}} : \Delta^k f = 0 \}$. Thus
\[ D^m_k(\psi) \subset F^{-1} \left( \{ f \in \mathbb{P}^m_{\text{hom}} : \Delta^k f = 0 \} \right) . \]

Since $F^{-1}$ is a bijective operator the claim is now obvious. \qed
Let us define \( H^m_k := \{ f \in \mathbb{P}^m_\text{hom} : \Delta^k f = 0 \} \). By Theorem 2.1 for \( Q(x) = |x|^{2k} \) it follows that any polynomial \( f \) has a Fischer decomposition \( f = |x|^{2k} q + h \) where \( h \) is \( k \)-harmonic. Moreover, \( h \) and \( q \) are homogeneous iff \( f \) is. So we have

\[
\mathbb{P}^m_\text{hom} = |x|^{2k} \mathbb{P}^m_\text{hom} - 2k \oplus H^m_k.
\]

Thus we obtain

\[
\dim D^m_k(\psi) \leq \dim H^m_k = \dim \mathbb{P}^m_\text{hom} - \dim \mathbb{P}^{m-2k}_\text{hom}.
\] (2.3)

The following question was posed by M. Agranovsky for the case \( k = 1 \) in [1], where it was also answered in the case that \( \psi \) factors completely into linear factors.

**Question 2.4** (Agranovksy). *What is the asymptotic behavior of \( \dim D^m_k(\psi) \), as \( m \to \infty \)?*

We expect that a full answer to this question would allow to relax the assumption on degree appearing in Theorem 1.3.

3. Proof of the main result

Assume that \( 2k \leq t \) and let \( \psi \) be a polynomial of degree \( \leq t \) and let \( F^k_\psi \) be the Fischer operator defined in Proposition 2.2. The following technical notion is a crucial tool for proving our main result Theorem 1.3: For a natural number \( M \) define \( S_i \subset \mathbb{P}^i \) as the subspace whose image under \( F^k_\psi \) is contained in \( \mathbb{P}^{M+t-2k} \), i.e.,

\[
S_i := \{ q \in \mathbb{P}^i : F^k_\psi(q) \in \mathbb{P}^{M+t-2k} \}
\]

for \( i \in \mathbb{N}_0 \). Since \( \psi \) has degree \( \leq t \) it follows that

\[
\mathbb{P}^M = S_M \subset S_{M+1} \subset \ldots \subset S_{M+j}
\]

for all \( j \geq 1 \).

**Proposition 3.1.** Let \( \psi = \psi_t + \ldots + \psi_0 \) be a polynomial of degree \( \leq t \) and let \( M \) be a natural number. Then for all \( j \in \mathbb{N} \)

\[
\dim S_{M+j} \leq \dim S_{M+j-1} + \dim D^{M+j}_k(\psi_t).
\]
Proof. For given \( j \in \mathbb{N} \) we will construct a space \( Q_j \) such that \( S_{M+j} = S_{M+j-1} \oplus Q_j \), and \( \dim Q_j \leq \dim D_k^{M+j} (\psi_t) \). First define \( Q_{H,j} := \{ q_{M+j} : q_{M+j} \) is the degree-(\( M + j \)) homogeneous term of some \( q \in S_{M+j} \} \). Choose (finitely many) polynomials in \( S_{M+j} \) whose leading terms form a basis for \( Q_{H,j} \), and define \( Q_j \) to be the subspace of \( S_{M+j} \) spanned by these polynomials. Suppose \( \hat{q} \in S_{M+j} \). The degree-(\( M + j \)) homogeneous term \( \hat{q}_{M+j} \) (possibly zero) can be matched by the leading homogeneous term of some \( q \in Q_j \) so that \( \hat{q} - q \in S_{M+j-1} \). This shows that \( S_{M+j} = S_{M+j-1} \oplus Q_j \).

Now, we will establish \( \dim Q_j \leq \dim D_k^{M+j} (\psi_t) \). It suffices to show that \( Q_{H,j} \subset D_k^{M+j} (\psi_t) \), since \( \dim Q_j = \dim Q_{H,j} \) by construction. Suppose \( q_{M+j} \in Q_{H,j} \) is nonzero, i.e., there is a \( q \in S_{M+j} \) and \( \deg q = M + j \) such that \( q_{M+j} \) is the leading homogeneous term of \( q \). Since \( F^k_\psi(q) \in \mathbb{P}^{M+t-2k} \), we have \( \deg(\Delta^k(q)) \leq M + t - 2k \). This implies that the leading term, \( \Delta^k(q_{M+j}) \), of \( \Delta^k(q) \) is zero (since it has degree \( M + j + t - 2k \)). i.e., \( q_{M+j} \) is \( k \)-harmonic. Therefore, \( Q_{H,j} \subset D_k^{M+j} \).

The main result of this paper, Theorem 1.3, follows now from the following more general result by taking \( \alpha = 1 \):

**Theorem 3.2.** Let \( \psi \) be a polynomial of degree \( t \). Suppose that there exist constants \( \alpha \geq 1, C > 0 \) such that for any polynomial \( f \) there exists a decomposition \( f = \psi q_f + h_f \) with \( \Delta^k h_f = 0 \) and
\[
\deg q_f \leq \alpha \deg f + C.
\]

Then \( t \leq 2k \cdot \alpha^{n-1} \).

**Proof.** Suppose \( t \geq 2k \). (If \( t < 2k \), there is nothing to prove.) Let \( f \in \mathbb{P}^{M+t-2k} \) and suppose that \( M > 2k \). Choose a polynomial \( g \in \mathbb{P}^{M+t} \) with \( \Delta^k g = f \). By assumption there exists \( q_f \) and \( h_f \) with \( g = \psi q_f + h_f \) and \( \Delta^k h_f = 0 \) and \( \deg q_f \leq \alpha (M + t) + C \). Then \( f = \Delta^k g = F^k_\psi(q_f) \) and we infer the inclusion
\[
\mathbb{P}^{M+t-2k} \subset F^k_\psi(\mathbb{P}^{BM})
\] (3.1)
with \( B_M := \alpha M + \alpha t + C \geq M \). Using the above notation \( S_{BM} = \{ q \in \mathbb{P}^{BM} : F^k_\psi(q) \in \mathbb{P}^{M+t-2k} \} \) we see that (3.1) implies that \( \mathbb{P}^{M+t-2k} \subset F^k_\psi(S_{BM}) \). Since \( F^k_\psi \) is a linear operator, we have
\[
\dim \mathbb{P}^{M+t-2k} \leq \dim F^k_\psi(S_{BM}) \leq \dim S_{BM}.
\] (3.2)
Applying Proposition 3.1 inductively we obtain
\[
\dim S_{BM} \leq \dim(P^M) + \sum_{j=M+1}^{BM} \dim D^j_k (\psi_t) \quad (3.3)
\]

Since \( P^{M+t-2k} = P^M \oplus P^{M+1} \oplus \ldots \oplus P^{M+t-2k} \) and \( \dim P^{M+1} \leq \dim P^M \) for \( j \geq 1 \) we infer from (3.2) and (3.3) the interesting formula
\[
(t - 2k) \dim P^M_{\text{hom}} \leq \sum_{j=M+1}^{BM} \dim D^j_k (\psi_t). \quad (3.4)
\]

Further we know from (2.3) that \( \dim D^j_k (\psi_t) \leq \dim P^j_{\text{hom}} - \dim P^{j-2k}_{\text{hom}} \). Thus the right hand side in (3.4) is a telescoping sum. Using that \( \dim P^j_{\text{hom}} \leq \dim P^{BM}_{\text{hom}} \) for \( j = BM - 2k + 1, \ldots, BM \) and \( \dim P^{M+1-2k}_{\text{hom}} \leq \dim P^j_{\text{hom}} \) for the lower indices we can estimate
\[
\sum_{j=M+1}^{BM} \dim D^j_k (\psi_t) \leq 2k \dim P^{BM}_{\text{hom}} - 2k \dim P^{M+1-2k}_{\text{hom}}.
\]

Thus we infer from (3.4) and the well known fact
\[
\dim P^{M+1}_{\text{hom}} = \binom{n + M}{n - 1} = \binom{n + M}{M + 1},
\]
proven in [7] that
\[
(t - 2k) \frac{(M + 2) \ldots (M + n)}{(n - 1)!} \leq 2k \frac{(BM + 1) \ldots (BM + n - 1) - (M + 2 - 2k) \ldots (M + n - 2k)}{(n - 1)!}
\]

Clearly the term \((n - 1)!\) can be canceled in the inequality. Divide the inequality by \( M^{n-1} \) on both sides and recall that \( BM = \alpha M + \alpha t + C \). Now take the limit \( M \to \infty \) and we obtain
\[
t - 2k \leq 2k (\alpha^{n-1} - 1).
\]

This implies \( t \leq 2k \alpha^{n-1} \) and the proof is complete. \( \square \)
4. Criteria for degree-related decompositions

We are now turning to the question under which conditions the degree condition is automatically satisfied. The first criterion is simple to prove:

**Proposition 4.1.** Suppose that $\psi$ is a polynomial of degree $t > 2$ and $\psi = \psi_t + \ldots + \psi_0$ is the decomposition into a sum of homogeneous polynomials. Assume the polynomial $\psi_t$ contains a non-negative non-constant factor. Let $f$ be a polynomial and assume that there exists a decomposition

$$f = \psi q + h$$

where $h$ is harmonic and $q$ is a polynomial. Then $\deg q \leq \deg f - t$ and $\deg h \leq \deg f$.

**Proof.** Write $q = q_M + \ldots + q_0$ with homogeneous polynomials $q_j$ of degree $j = 0, \ldots, M$. Expand the product $\psi q$ into a sum of homogeneous polynomials, so $\psi q = \psi_t q_M + R(x)$ where $R(x)$ is a polynomial of degree $< M + t$. Suppose that $M + t > \deg f$. Since $\Delta f = \Delta (\psi q)$ we conclude that $\Delta (\psi_t q_M) = 0$, so $\psi_t q_M$ is harmonic. By the Brelot-Choquet theorem, a harmonic polynomial cannot have non-negative factors, see [11]. Thus $\psi_t q_M = 0$, and we obtain a contradiction. $\square$

The next criterion is more difficult to prove and uses again ideas from the proof of the Brelot-Choquet theorem:

**Theorem 4.2.** Suppose that $\psi$ is a polynomial of degree $t > 2$ and $\psi = \psi_t + \psi_s + \psi_{s-1} + \ldots + \psi_0$ is the decomposition into a sum of homogeneous polynomials. Assume the polynomial $\psi_s$ is non-zero and contains a non-negative non-constant factor. Let $f$ be a polynomial and assume that there exists a decomposition

$$f = \psi q + h$$

where $h$ is harmonic and $q$ is a polynomial. Then $\deg q \leq 2 - s + \deg f$ and $\deg h \leq t + 2 - s + \deg f$.

Before proving Theorem 4.2 we notice the following conclusion:

**Corollary 4.3.** Suppose that $\psi$ is a polynomial with a non-zero second-highest degree term that contains a non-negative factor. If every polynomial $f$ has a Fischer decomposition $f = \psi q_f + h_f$ with $h_f$ harmonic, then $\deg(\psi) \leq 2$. 

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Proof. Suppose \( \deg(\psi) > 2 \). By Theorem 4.2, \( \deg q_f - \deg f \) is bounded. Now we can apply Theorem 1.3, to obtain \( \deg \psi \leq 2 \). \( \square \)

The following lemma is needed for the proof of Theorem 4.2:

**Lemma 4.4.** Suppose that \( \psi \) is a polynomial of degree \( t > 2 \) and \( \psi = \psi_t + \psi_s + \psi_{s-1} + \ldots + \psi_0 \) is the decomposition into a sum of homogeneous polynomials. Assume that \( g \in \mathbb{P}^m \) and \( q \) is a polynomial of degree \( M \) such that \( F^k_{\psi}(q) := \Delta(\psi q) = g \) and \( M + s > m \). Then for every \( p \in \mathbb{P}^{s-1} \),

\[
\int_{S^{n-1}} q_M^2 \cdot \psi_s \cdot p \, d\theta = 0,
\]

where \( q_M \neq 0 \) is the senior term of \( q \).

*Proof (of lemma).* Write \( q = q_M + \ldots + q_0 \) with homogeneous polynomials \( q_j \) of degree \( j = 0, \ldots, M \). Expand the product \( \psi q \) into a sum of homogeneous polynomials,

\[
\psi q = \psi_t q_M + \ldots + \psi_t q_{M-t+s+1} + (\psi_t q_{M-t+s} + \psi_s q_M) + R(x) \tag{4.1}
\]

where \( R(x) \) is a polynomial of degree \( < M + s \). Since \( \Delta(\psi q) = g \) and \( M + s > m \), we conclude that \( \Delta(\psi_t q_M) = 0 \) and \( \Delta(\psi_t q_{M-t+s} + \psi_s q_M) = 0 \). Thus, we can write

\[
\psi_t q_M = h_{M+t} \tag{4.2}
\]

\[
\psi_t q_{M-t+s} + \psi_s q_M = h_{M+s}, \tag{4.3}
\]

where \( h_{M+t} \) and \( h_{M+s} \) are homogeneous harmonic polynomials.

Take \( p \in \mathbb{P}^{s-1} \), and multiply equation (4.3) by \( q_M p \) and integrate over the unit sphere, \( S^{n-1} \). Then

\[
\int_{S^{n-1}} \psi_t q_{M-t+s} \cdot q_M p \, d\theta + \int_{S^{n-1}} \psi_s q_M^2 \cdot p \, d\theta = \int_{S^{n-1}} h_{M+s} \cdot q_M p \, d\theta.
\]

Since \( \deg(q_M p) < M + s \) and \( h_{M+s} \) is harmonic, the integral on the right-hand side is zero. Indeed, homogeneous harmonics of different degree are orthogonal in the space \( L^2(S^{n-1}) \) (see [7]), and, moreover, \( q_M p \) can be matched on \( S^{n-1} \) by a harmonic polynomial of not higher degree. Substituting equation 4.2 into the first integral on the left-hand side gives \( \int_{S^{n-1}} h_{M+t} \cdot p \cdot q_{M-t+s} \, d\theta \), which is also zero, since \( \deg(pq_{M-t+s}) < M + t \). \( \square \)
Proof of Theorem 4.2. By assumption we may write $\psi_s = \phi P$ where $\phi$ is non-negative and $P$ has degree $< s$. Suppose that $M + s > \deg f + 2$. We have $\Delta(\psi q) = \Delta f$ and $M + s > \deg(\Delta f)$. Then, $q$, $\psi$ satisfy Lemma 4.4 with $g = \Delta f$, and thus $\int_{S^{n-1}} q_M^2 \cdot \psi_s \cdot p \, d\theta = 0$, for all $p$ of degree $< s$. In particular, this is true for $p = P$. Hence,

$$0 = \int_{S^{n-1}} q_M^2 \cdot \psi_s \cdot P \, d\theta = \int_{S^{n-1}} q_M^2 \cdot \phi \cdot P^2 \, d\theta.$$ 

Since $P \neq 0$, $\phi \neq 0$, and $\phi(\theta) \geq 0$ for all $\theta \in S^{n-1}$, we have the contradiction $q_M = 0$. 

The following instructive example is due to L. Hansen and H.S. Shapiro [17]; it was also suggested in [19] as a simple example for which the Khavinson-Shapiro conjecture is unresolved (whenever $\varphi$ is a cubic): Let $\varphi \in \mathbb{R}[x_1, ..., x_n]$ be a homogeneous harmonic polynomial of degree $> 2$, in particular $\varphi$ does not contain a nonnegative non-constant factor, see [11]. We perturb the equation for the unit ball $|x|^2 - 1$ by $\varepsilon \varphi$, i.e. we consider

$$\psi_\varepsilon(x) := |x|^2 - 1 + \varepsilon \varphi(x) \quad \text{for } \varepsilon > 0. \quad (4.4)$$

If $\varepsilon > 0$ is small enough, then the component of $E_\varepsilon := \{\psi_\varepsilon < 0\}$ containing 0 is a bounded domain in $\mathbb{R}^d$. Then the Dirichlet problem for the data function $|x|^2 = x_1^2 + ... + x_n^2$ restricted to $\partial E_\varepsilon$ has a harmonic polynomial solution $u_f(x) = 1 - \varepsilon \varphi(x)$ since

$$|x|^2 = \psi_\varepsilon(x) \cdot 1 + 1 - \varepsilon \varphi(x).$$

Note that in this example the degree of the solution $u_f$ for the Dirichlet problem is higher than the degree of the data function $f$.

The question arises whether any polynomial data function may have a polynomial solution. If this is the case, and $\psi_\varepsilon$ is irreducible and changes the sign in a neighborhood of some point in $\partial E_\varepsilon$ then the proof of Theorem 27 in [25] implies that for any polynomial $f$ there exists a decomposition $f = \psi_\varepsilon q_f + h_f$ where $h_f$ is harmonic. By Corollary 4.3 $\deg \psi_\varepsilon \leq 2$. Thus we have proved that for this class of examples the Khavinson-Shapiro conjecture is true.

In the rest of this section we extend Theorem 4.2 to the case $k \geq 1$. We consider the following inner product

$$\langle f, g \rangle := \int_{\mathbb{R}^n} f(x) g(x) e^{-|x|^2} \, dx \quad (4.5)$$
and the following orthogonality condition established in [25].

**Theorem 4.5.** Suppose that \( f \) is a homogeneous polynomial, and let \( k \in \mathbb{N} \) with \( 2(k - 1) \leq \deg f \). Then \( \Delta^k f = 0 \) if and only if \( \langle f, g \rangle = 0 \) for all polynomials \( g \) with \( 2(k - 1) + \deg g < \deg f \).

**Theorem 4.6.** Suppose that \( \psi \) is a polynomial of degree \( t > s \) and \( \psi = \psi_t + \psi_s + \psi_{s-1} + ... + \psi_0 \) is the decomposition into a sum of homogeneous polynomials. Assume the polynomial \( \psi_s \neq 0 \) is non-negative. If the polynomial \( f \) has the decomposition
\[
f = \psi q + h
\]
where \( h \) is \( k \)-harmonic, then \( \deg(q) \leq 2k - s + \deg f \).

**Proof.** Suppose that \( M + s > 2k + \deg f \), where \( f = \psi q + h \) and \( M = \deg q \). We will derive a contradiction. We proceed as in the proof of Lemma 4.4 writing \( q = q_M + ... + q_0 \) with homogeneous polynomials \( q_j \) of degree \( j = 0, ..., M \). Expand the product \( \psi q \) as in (4.1). Then we conclude that \( \Delta^k(\psi q_M) = 0 \) and \( \Delta^k(\psi q_{M-t+s} + \psi_s q_M) = 0 \). Thus, we can write
\[
\psi q_M = H_{M+t}
\]
\[
\psi q_{M-t+s} + \psi_s q_M = H_{M+s},
\]
where \( H_{M+t} \) and \( H_{M+s} \) are homogeneous \( k \)-harmonic polynomials. Next take the inner product (4.5) of both sides of equation (4.7) with \( q_M \). Then
\[
\langle q_{M-t+s}, q_M \psi_t \rangle + \langle \psi_s, q_M^2 \rangle = \langle q_{M+t}, q_M \psi_t \rangle + \langle \psi_s, q_M^2 \rangle = \langle q_{M+s}, q_M \psi_t \rangle + \langle \psi_s, q_M^2 \rangle = \langle H_{M+s}, q_M \psi_t \rangle + \langle \psi_s, q_M^2 \rangle = \langle H_{M+s}, q_M \rangle
\]
Using equation 4.6, we arrive at \( \langle q_{M-t+s}, q_M \psi_t \rangle + \langle \psi_s, q_M^2 \rangle = \langle H_{M+s}, q_M \rangle \).

Now we use Theorem 4.5. Since \( \deg H_{M+t} > \deg q_{M-t+s} + 2(k - 1) \) and \( \deg H_{M+s} > \deg q_M + 2(k - 1) \), the first term on the left and the term on the right are both zero. Thus, \( \langle \psi_s, q_M^2 \rangle = 0 \) implies \( q_M = 0 \) (since \( \psi \neq 0 \) is non-negative), a contradiction.

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