<table>
<thead>
<tr>
<th><strong>Title</strong></th>
<th>Reproducing kernels for polyharmonic polynomials</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Authors(s)</strong></td>
<td>Render, Hermann</td>
</tr>
<tr>
<td><strong>Publication date</strong></td>
<td>2008-10</td>
</tr>
<tr>
<td><strong>Publication information</strong></td>
<td>Archiv der Mathematik, 91 (2): 136-144</td>
</tr>
<tr>
<td><strong>Publisher</strong></td>
<td>Springer</td>
</tr>
<tr>
<td><strong>Item record/more information</strong></td>
<td><a href="http://hdl.handle.net/10197/5499">http://hdl.handle.net/10197/5499</a></td>
</tr>
<tr>
<td><strong>Publisher's statement</strong></td>
<td>The final publication is available at <a href="http://www.springerlink.com">www.springerlink.com</a></td>
</tr>
<tr>
<td><strong>Publisher's version (DOI)</strong></td>
<td>10.1007/s00013-008-2447-9</td>
</tr>
</tbody>
</table>
Reproducing kernels for polyharmonic polynomials

H. Render

Abstract. The reproducing kernel of the space of all homogeneous polynomials of degree $k$ and polyharmonic order $m$ is computed explicitly, solving a question of A. Fryant and M.K. Vemuri.

Mathematics Subject Classification (2000). Primary 31B30, Secondary 33C55.

Keywords. Polyharmonic function, reproducing kernel, zonal harmonic, pythagorean identity.

1. Introduction

Let $U$ be an open set in the euclidean space $\mathbb{R}^d$. A function $f : U \rightarrow \mathbb{C}$ is called polyharmonic of order $m$ if $f$ is $2m$-times differentiable and $\Delta^m f (x) = 0$ for all $x \in U$, where

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \ldots + \frac{\partial^2}{\partial x_d^2}$$

is the Laplace operator and $\Delta^m$ is its $m$-th iterate. For $m = 1$ this class of functions is just the class of all harmonic functions, while for $m = 2$ the term biharmonic function is used which is important in elasticity theory. Polyharmonic functions have been studied by several mathematicians, see e.g. [20], [21], [22], [23], [31], [35], and classical work is due to E. Almansi [1], M. Nicolesco [33] and N. Aronszajn [4]. Polyharmonicity is an important tool in several areas of mathematics, e.g. in approximation theory, radial basis functions and wavelet analysis, see [6], [24], [25], [26], [28], [32].

In this paper we shall be concerned with a problem posed by A. Fryant and M.K. Vemuri in [18]. Let $\mathcal{P} (\mathbb{R}^d)$ be the space of all polynomials endowed with the scalar product

$$(P, Q) := \sum_{|\alpha| \leq N} a^\alpha \overline{b^\alpha}$$

(1.1)

The author is partially supported by Grant MTM2006-13000-C03-03 of the D.G.I. of Spain.
for polynomials \( P(x) = \sum_{|\alpha| \leq N} c_\alpha x^\alpha \) and \( Q(x) = \sum_{|\alpha| \leq N} d_\alpha x^\alpha \). An alternative way to define the scalar product (1.1) is the following:

\[
\langle P, Q \rangle_F = \left[ P \left( \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_d} \right) Q \right](0).
\]

Let \( \mathcal{P}_k(\mathbb{R}^d) \) be the space of all homogeneous polynomials of degree \( k \). Define the Hilbert space of all homogeneous polynomials of degree \( k \) which are polyharmonic of order at most \( m \), so we define

\[
\mathcal{H}^m_k(\mathbb{R}^d) := \{ h \in \mathcal{P}_k(\mathbb{R}^d) : \Delta^m h = 0 \}.
\]

Let \( Q_j^k(x) \) with \( j = 1, \ldots, b^{k,m}_d \) be an orthonormal basis of \( \mathcal{H}^m_k(\mathbb{R}^d) \) with respect to the inner product (1.1) and define the reproducing kernel \( Z^m_k(x, y) \) of \( \mathcal{H}^m_k(\mathbb{R}^d) \) by

\[
Z^m_k(x, y) := \sum_{j=1}^{b^{k,m}_d} Q_j^k(x) Q_j^k(y).
\]

In [18] it was proved that there exists a constant \( \gamma^k_d(m) \), depending only on the dimension \( d \), the integer \( m \) and the degree \( k \), such that

\[
\sum_{j=1}^{b^{k,m}_d} \left| Q_j^k(x) \right|^2 = \gamma^k_d(m) \quad \text{for all } x \in \mathbb{S}^{d-1},
\]

(1.3)

where \( \mathbb{S}^{d-1} = \{ x \in \mathbb{R}^d : |x| = 1 \} \) is the unit sphere and \( |x|^2 = x_1^2 + \ldots + x_d^2 \) for \( x = (x_1, \ldots, x_d) \in \mathbb{R}^d \). However, the nature of this constant was not further explored. We shall show that

\[
\gamma^k_d(m) = \min \left( k/2, m-1 \right) \frac{a_{k-2s}}{2^s s! d (d+2) \ldots (d+2 (k-s-1))}
\]

(1.4)

where \( a_k \) is the dimension of \( \mathcal{H}^1_k(\mathbb{R}^d) \), the set of all homogeneous harmonic polynomials of degree \( k \), given by

\[
a_k := \dim \mathcal{H}^1_k(\mathbb{R}^d) = \frac{(2k + d - 2) (k + d - 3)!}{k! (d-2)!},
\]

(1.5)

see e.g. [2, p. 450]. Furthermore we shall show that the reproducing kernel \( Z^m_k(x, y) \) can be described explicitly:

\[
Z^m_k(x, y) = \omega_{d-1} \sum_{s=0}^{\min(\lfloor k/2 \rfloor, m-1)} \frac{|x|^{2s} |y|^{2s} Z_{k-2s}(x, y)}{2^s s! d (d+2) \ldots (d+2 (k-s-1))}
\]

where \( Z_k(x, y) \) is the zonal harmonic of degree \( k \) with pole \( y \) (for definition see Section 2). Formula (1.4) allows us to improve a criterion for the convergence of
the orthogonal series
\[ \sum_{k=0}^{\infty} \sum_{j=1}^{b_{d,m}} a_{k,j} Q_k^j(x) \]
which will be presented in Section 3.

2. The reproducing kernel

The inner product defined in (1.1) is an important tool in the theory of spherical harmonics, see [3], [5], [12], [16], [24], [39]. We note that in [34] and [38] this inner product is called the Fischer inner product, in honour of the work of E. Fischer [14]; in [8], [9], [42] it is called the Bombieri inner product, and in [18] the Calderón inner product. However, it seems that is a classical tool in invariant theory (see [13], [36], [40]) and we shall refer to it as the apolar inner product.

The apolar inner product has the following property: for all polynomials \( f, g \)
\[ \langle Q^* (D) f, g \rangle_F = \langle f, Q \cdot g \rangle_F \] (2.1)
where \( Q^* (x) \) is the polynomial obtained by conjugation the coefficients of the polynomial \( Q \), and \( Q^* (D) \) is the differential operator associated to \( Q^* (x) \). Equation (2.1) says that the adjoint of the multiplication operator \( g \mapsto Qg \) is just the differential operator \( Q^* (D) \). In passing, we note that the apolar inner product has an integral representation:
\[ \langle f, g \rangle_F = \frac{1}{\pi^d} \int_{\mathbb{C}^d} f(z) \overline{g(z)} e^{-|z|^2} dz \]
where \( dz \) is Lebesgue measure on \( \mathbb{R}^{2d} \), see [7]. The space of all entire functions \( f : \mathbb{C}^n \to \mathbb{C} \) which satisfy
\[ \|f\|^2_F := \frac{1}{\pi^d} \int_{\mathbb{C}^d} |f(z)|^2 e^{-|z|^2} dz < \infty \] (2.2)
is called the Bargmann space \( F_n \) (also called Fock or Fischer space, see [34]).

For homogeneous polynomials \( f, g \) we define the following well-known inner product
\[ \langle f, g \rangle_{S^{d-1}} := \int_{S^{d-1}} f(\theta) \overline{g(\theta)} d\theta \] (2.3)
where \( S^{d-1} \) is the unit sphere and \( d\theta \) is the rotation-invariant measure on \( S^{d-1} \). The following result follows from Theorem 5.14 in [5]; the result seems to be due to Ü. Kuran [30]:

**Theorem 2.1.** For homogeneous harmonic polynomials \( f, g \) of degree \( k \) one has
\[ \langle f, g \rangle_F = d(d+2) \cdots (d+2k-2) \frac{1}{\omega_{d-1}} \langle f, g \rangle_{S^{d-1}} . \]
It is well known and it follows by a quick computation that the following formula holds for any harmonic homogeneous polynomial shows that for 

\[ Z_k(x,y) := \sum_{l=1}^{\alpha_k} Y_{k,l}(x) \overline{Y_{k,l}(y)} \]  

(2.4)
is the reproducing kernel of \( \mathcal{H}_k^*(\mathbb{R}^d) \) with respect to (2.3); the function \( x \mapsto Z_k(x,y) \) is also called the zonal harmonic of degree \( k \) with pole \( y \). The addition theorem says that

\[ Z_k(x,y) = \frac{\alpha_k}{\omega_{d-1}} |x|^k |y|^k P_k\left(\frac{\langle x,y \rangle}{|x| |y|}\right) \]  

(2.5)

where \( P_k \) is a polynomial of degree \( k \) with \( P_k(1) = 1 \) (see [2, p. 455]) and \( \omega_{d-1} \) is the surface area of \( S^{d-1} \). The polynomial \( P_k \) is up to a factor equal to the ultraspherical polynomial \( C_k^{(d-2)/2}(t) \), (see [2, p. 456]). Since \( P_k(1) = 1 \) one has

\[ P_k(t) = \frac{C_k^{(d-2)/2}(t)}{C_k^{(d-2)/2}(1)} \]

Observe that the property \( P_k(1) = 1 \) also implies that

\[ Z_k(x,x) = \frac{\alpha_k}{\omega_{d-1}} |x|^{2k}. \]  

(2.6)

We now prove

**Theorem 2.2.** Let \( Y_{k,l}(x) \) , \( l = 1, ..., a_k \), be an orthonormal basis of \( \mathcal{H}_k^*(\mathbb{R}^d) \) with respect to the scalar product (2.3). Then the polynomials \( |x|^{2s} Y_{k,l}(x) \) for \( s,k \in \mathbb{N}_0 \) and \( l = 1, ..., a_k \) are orthogonal with respect to the apolar inner product and

\[ \omega_{d-1} \left\| |x|^{2s} Y_{k,l}(x) \right\|_F^2 = 2^s s! d (d + 2) \cdots (d + 2 (k + s - 1)). \]  

(2.7)

**Proof.** Let \( |x|^{2s} Y_{k,l} \) and \( |x|^{2s_1} Y_{k_1,l_1} \) be two basis functions. Without loss of generality we may assume that \( s \leq s_1 \). By property (2.1) we obtain

\[ \left\langle |x|^{2s} Y_{k,l}, |x|^{2s_1} Y_{k_1,l_1} \right\rangle_F = \left\langle \Delta^s \left[ |x|^{2s} Y_{k,l} \right], |x|^{2s_1-2s} Y_{k_1,l_1} \right\rangle_F. \]

It is well known and it follows by a quick computation that the following formula

\[ \Delta^m \left[ |x|^{2s} h \right] = 2^s [2s + d - 2 + 2 \text{ deg } h] \cdot |x|^{2s-2m} h \]  

(2.8)

holds for any harmonic homogeneous polynomial \( h \). A simple induction argument shows that for \( m \leq 2s \)

\[ \Delta^m \left[ |x|^{2s} h \right] = |x|^{2s-2m} \cdot h \cdot (2s) \cdots (2s - 2 (m - 1)) \cdot [2s + d - 2 + 2 \text{ deg } h] \cdots [2s - 2 (m - 1) + d - 2 + 2 \text{ deg } h]. \]  

(2.9)
In particular, for \( m = s \) we obtain that \( \Delta^s \left( |x|^{2s} f \right) = d_s (\deg h) f \) where \( d_s (\deg h) \) is the number
\[
2^s! \cdot (2s + d - 2 + 2 \deg h) \cdot (2s - 2 + d - 2 + 2 \deg h) \cdots (d + 2 \deg h).
\]
Thus we have
\[
\left\langle |x|^{2s} Y_{k,l} , |x|^{2s_1} Y_{k_1,l_1} \right\rangle_f = d_s (k) \cdot \left\langle Y_{k,l} , |x|^{2s_1-2s} Y_{k_1,l_1} \right\rangle_f .
\]
(2.11)
If \( s_1 > s \) we can use again (2.1) and we see that
\[
\left\langle |x|^{2s} Y_{k,l} , |x|^{2s_1} Y_{k_1,l_1} \right\rangle_f = d_s (k) \cdot \left\langle \Delta Y_{k,l} , |x|^{2s_1-2s-2} Y_{k_1,l_1} \right\rangle_f = 0.
\]
If \( s_1 = s \), and \( k \neq k_1 \) or \( l \neq l_1 \), we see from (2.11) that \( \left\langle |x|^{2s} Y_{k,l} , |x|^{2s_1} Y_{k_1,l_1} \right\rangle_f = 0 \) since \( Y_{k,l} \) and \( Y_{k_1,l_1} \) are orthogonal according to Theorem 2.1.

For \( (s, k, l) = (s_1, k_1, l_1) \) Theorem 2.1 and (2.11) show that \( \omega_{d-1} \sum_{s=0}^{\min \{k/2, m-1\}} \| |x|^{2s} Y_{k,l} \|_f^2 \) is equal to the product of \( d (d + 2) \cdots (d + 2k - 2) \) and \( d_s (k) \) (so (2.10) for \( \deg h = k \)). This product is equal to
\[
2^s s! d (d + 2) \cdots (d + 2 (k + s - 1)).
\]

\( \square \)

**Proposition 2.3.** The system \( |x|^{2s} Y_{k-2s,l} (x) \) for \( s = 0, 1, \ldots, \min \{k/2, m-1\} \) and \( l = 1, \ldots, a_{k-2s} \) is an orthogonal basis for \( \mathcal{H}_k^m (\mathbb{R}^d) \).

**Proof.** The polynomial \( f (x) = |x|^{2s} Y_{k,l} (x) \) satisfies \( \Delta^m f = 0 \) if and only if \( s \leq m - 1 \). Hence \( |x|^{2s} Y_{k-2s,l} (x) \) is a homogeneous polynomial of degree \( k \) which satisfies \( \Delta^m f = 0 \), and by Theorem 2.2 these functions are orthogonal. In order to see that it is basis, let \( f \in \mathcal{H}_k^m (\mathbb{R}^d) \).

Then \( f \) can be written uniquely in the form
\[
f = \sum_{k=0}^{\min \{k/2, m-1\}} |x|^{2s} h_{k-2s} \text{ with harmonic homogeneous polynomials } h_{k-2s} \text{ of degree } k - 2s,
\]
see [5]. Formula (2.9) shows that \( \Delta^m |x|^{2s} h_{k-2s} = C_{m,k,s} |x|^{2s-2m} h_{k-2s} \) for \( m \leq s \) and for some nonzero constant \( C_{m,k,s} \). The condition \( \Delta^m f = 0 \) implies that the summation in the last sum ranges only over indices \( s \) with \( s \leq m - 1 \). So \( f \) is a linear combination of the above basis functions.

\( \square \)

**Theorem 2.4.** The reproducing kernel \( Z_k^m (x, y) \) for the Hilbert space \( \mathcal{H}_k^m (\mathbb{R}^d) \) endowed with the apolar inner product is given by
\[
Z_k^m (x, y) = \sum_{s=0}^{\min \{k/2, m-1\}} \frac{|x|^{2s} |y|^{2s} Z_{k-2s} (x, y)}{2^s s! d (d + 2) \cdots (d + 2 (k + s - 1))}.
\]

(2.12)

**Proof.** We use formula (1.2) for the system \( |x|^{2s} Y_{k-2s,l} (x) \), \( l = 1, \ldots, a_{k-2s}, s = 0, 1, \ldots, \min \{k/2, m-1\} \), by taking into account the normalization constants.
given in (2.7). This gives

\[
Z_m^k (x, y) = \omega_{d-1} \sum_{s=0}^{\min\{[k/2], m-1\}} \frac{a_{k-2s}}{2^{s}s!d^2 (d + 2) \ldots (d + 2(k - s - 1))}. \]

Now (2.4) completes the proof. \(\square\)

Corollary 2.5. The values \(Z_m^k (x, x)\) for \(|x| = 1\) of the reproducing kernel \(Z_m^k\) of \(H_m^k (\mathbb{R}^d)\) are constant equal to

\[
\gamma_k^d (m) := \min\{[k/2], m-1\} \sum_{s=0}^{a_{k-2s}} \frac{1}{2^{s}s!d^2 (d + 2) \ldots (d + 2(k - s - 1))}. \quad (2.13)
\]

Proof. Insert \(y = x\) in formula (2.12) and use (2.6). \(\square\)

3. Convergence of orthogonal series

Suppose that \(Q_j^k (x), j = 1, \ldots, b_{d,k}^m\) is an orthonormal basis of \(H_m^k (\mathbb{R}^d)\) for each \(k = 0, 1, 2, \ldots\), and let \(a_{k,j}, j = 1, \ldots, b_{d,k}^m\) be complex numbers. A. Fryant and M.K. Vemuri discuss in [18] conditions for the numbers \(a_{k,j}, j = 1, \ldots, b_{d,k}^m\) such that the series

\[
f(x) = \sum_{k=0}^{\infty} \sum_{j=1}^{b_{d,k}^m} a_{k,j} Q_j^k (x) \quad (3.1)
\]

converges absolutely and uniformly on compact subsets of the open ball \(B_R\) with radius \(R\) and center 0. It is shown in [18] that the series (3.1) converges compactly in \(B_R\) for

\[
R^{-1} = \lim_{k \to \infty} \sup_{k} \left( \sqrt{\gamma_k^d (m)} \|a_k\| \right)^{1/k} \text{ and } \|a_k\|^2 := \sum_{j=1}^{b_{d,k}^m} |a_{k,j}|^2.
\]

By the next Theorem we obtain the more precise description

\[
R^{-1} = \frac{1}{\sqrt{2}} \lim_{k \to \infty} \sup_{k} \left( \frac{\|a_k\|}{\sqrt{k!}} \right)^{1/k}
\]

improving the upper bound for \(R^{-1}\) in [18] by a factor \(1/\sqrt{2}\).

Theorem 3.1. Let \(M_k, k \in \mathbb{N}_0\), be positive numbers and \(\gamma_k^d (m)\) as in (2.13). Then

\[
\lim_{k \to \infty} \sup_{k} \left( \sqrt{\gamma_k^d (m)} M_k \right)^{1/k} = \frac{1}{\sqrt{2}} \lim_{k \to \infty} \sup_{k} \left( \frac{M_k}{\sqrt{k!}} \right)^{1/k}. \quad (3.2)
\]

Proof. Let us define \(D_k (d, s) := d(d + 2) \ldots (d + 2(k - s - 1))\). From the identity

\[
D_k (d, s) = 2^{k-s} \left( \frac{d}{2} + 1 \right) \ldots \left( \frac{d}{2} + (k - s - 1) \right) \quad (3.3)
\]
we see that
\[ D_k (d, s) \geq 2^{k-s-1} (k-s-1)! \geq 2^{k-s-1}k! \frac{1}{(k+1)^{s+1}}. \] (3.4)

Observe that the inequality \( a_{k-2s} \leq a_k \leq 2 (k+1)^{d-2} \) is obtained from rewriting the formula (1.5) for \( a_k \) as
\[ a_k = 2 (k+1) \left( \frac{k}{2} + 1 \right) \ldots \left( \frac{k}{d-3} + 1 \right) \left( \frac{k}{d-2} + \frac{1}{2} \right) \]
for \( d > 2 \); for \( d = 2 \) it is well known that \( a_k = 2 \) for all \( k \in \mathbb{N} \). Thus we obtain
\[ \gamma^k_d (m) \leq \sum_{s=0}^{m-1} \frac{2 (k+1)^{d-1+s}}{s! 2^{k-1}k!} \leq \frac{(k+1)^{d-2+m}}{2^{k-2}k!} \sum_{s=0}^{\infty} \frac{1}{s!}. \]

Now take the square root, multiply the inequality with \( M_k \), take the \( k \)-th root and then the limes superior. Hence the \( \leq \) in (3.2) is proved.

For the other inequality we estimate \( \gamma^k_d (m) \) below by taking only the summand for \( s = 0 \), so
\[ \gamma^k_d (m) \geq \frac{a_k}{d (d+2) \ldots (d+2 (k-1))}. \]

Now (3.3) yields \( D_k (d, 0) \leq 2^k (d+k)! \leq 2^{k+1}k! (d+k)^d \). Using that \( a_k \geq 1 \) we obtain
\[ \gamma^k_d (m) \geq \frac{1}{2^{k+1}k! (d+k)^d}. \]

Again, take the square root, multiply the inequality with \( M_k \), take the \( k \)-th root and then the limes superior. \( \square \)

References


Reproducing kernels for polyharmonic polynomials


H. Render
Departamento de Matemáticas y Computación
Universidad de La Rioja
Edificio Vives, Luis de Ulloa s/n.
26004 Logroño
e-mail: render@gmx.de