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Title	Reproducing kernels for polyharmonic polynomials
Authors(s)	Render, Hermann
Publication date	2008-10
Publication information	Render, Hermann. "Reproducing Kernels for Polyharmonic Polynomials" 91, no. 2 (October, 2008).
Publisher	Springer
Item record/more information	http://hdl.handle.net/10197/5499
Publisher's statement	The final publication is available at www.springerlink.com
Publisher's version (DOI)	10.1007/s00013-008-2447-9

Downloaded 2023-04-07T10:52:29Z

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Reproducing kernels for polyharmonic polynomials

H. Render

Abstract. The reproducing kernel of the space of all homogeneous polynomials of degree k and polyharmonic order m is computed explicitly, solving a question of A. Fryant and M.K. Vemuri.

Mathematics Subject Classification (2000). Primary 31B30, Secondary 33C55.

Keywords. Polyharmonic function, reproducing kernel, zonal harmonic, pythagorean identity.

1. Introduction

Let U be an open set in the euclidean space $\mathbb{R}^d$. A function $f : U \rightarrow \mathbb{C}$ is called *polyharmonic of order m* if f is $2m$ -times differentiable and $\Delta^m f(x) = 0$ for all $x \in U$, where

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \dots + \frac{\partial^2}{\partial x_d^2}$$

is the *Laplace operator* and Δ^m is its m -th iterate. For $m = 1$ this class of functions is just the class of all *harmonic* functions, while for $m = 2$ the term *biharmonic* function is used which is important in elasticity theory. Polyharmonic functions have been studied by several mathematicians, see e.g. [20], [21], [22], [23], [31], [35], and classical work is due to E. Almansi [1], M. Nicolesco [33] and N. Aronszajn [4]. Polyharmonicity is an important tool in several areas of mathematics, e.g. in approximation theory, radial basis functions and wavelet analysis, see [6], [24], [25], [26], [28], [32].

In this paper we shall be concerned with a problem posed by A. Fryant and M.K. Vemuri in [18]. Let $\mathcal{P}(\mathbb{R}^d)$ be the space of all polynomials endowed with the scalar product

$$\langle P, Q \rangle_F := \sum_{|\alpha| \leq N} \alpha! c_\alpha \overline{d_\alpha} \quad (1.1)$$

The author is partially supported by Grant MTM2006-13000-C03-03 of the D.G.I. of Spain.

for polynomials $P(x) = \sum_{|\alpha| \leq N} c_\alpha x^\alpha$ and $Q(x) = \sum_{|\alpha| \leq N} d_\alpha x^\alpha$. An alternative way to define the scalar product (1.1) is the following:

$$\langle P, Q \rangle_F = \left[P \left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_d} \right) \bar{Q} \right] (0).$$

Let $\mathcal{P}_k(\mathbb{R}^d)$ be the space of all homogeneous polynomials of degree k . Define the Hilbert space of all homogeneous polynomials of degree k which are polyharmonic of order at most m , so we define

$$\mathcal{H}_k^m(\mathbb{R}^d) := \{h \in \mathcal{P}_k(\mathbb{R}^d) : \Delta^m h = 0\}.$$

Let $Q_k^j(x)$ with $j = 1, \dots, b_d^{k,m} := \dim \mathcal{H}_k^m(\mathbb{R}^d)$ be an orthonormal basis of $\mathcal{H}_k^m(\mathbb{R}^d)$ with respect to the inner product (1.1) and define the *reproducing kernel* Z_k^m of $\mathcal{H}_k^m(\mathbb{R}^d)$ by

$$Z_k^m(x, y) := \sum_{j=1}^{b_d^{k,m}} Q_k^j(x) \overline{Q_k^j(y)}. \quad (1.2)$$

In [18] it was proved that there exists a constant $\gamma_d^k(m)$, depending only on the dimension d , the integer m and the degree k , such that

$$\sum_{j=1}^{b_d^{k,m}} \left| Q_k^j(x) \right|^2 = \gamma_d^k(m) \quad \text{for all } x \in \mathbb{S}^{d-1}, \quad (1.3)$$

where $\mathbb{S}^{d-1} = \{x \in \mathbb{R}^d : |x| = 1\}$ is the unit sphere and $|x|^2 = x_1^2 + \dots + x_d^2$ for $x = (x_1, \dots, x_d) \in \mathbb{R}^d$. However, the nature of this constant was not further explored. We shall show that

$$\gamma_d^k(m) = \sum_{s=0}^{\min\{k/2, m-1\}} \frac{a_{k-2s}}{2^s s! d(d+2) \dots (d+2(k-s-1))} \quad (1.4)$$

where a_k is the dimension of $\mathcal{H}_k^1(\mathbb{R}^d)$, the set of all homogeneous harmonic polynomials of degree k , given by

$$a_k := \dim \mathcal{H}_k^1(\mathbb{R}^d) = \frac{(2k+d-2)(k+d-3)!}{k!(d-2)!}, \quad (1.5)$$

see e.g. [2, p. 450]. Furthermore we shall show that the reproducing kernel $Z_k^m(x, y)$ can be described explicitly:

$$Z_k^m(x, y) = \omega_{d-1} \sum_{s=0}^{\min\{[k/2], m-1\}} \frac{|x|^{2s} |y|^{2s} Z_{k-2s}(x, y)}{2^s s! d(d+2) \dots (d+2(k-s-1))}$$

where $Z_k(x, y)$ is the *zonal harmonic of degree k with pole y* (for definition see Section 2). Formula (1.4) allows us to improve a criterion for the convergence of

the orthogonal series

$$\sum_{k=0}^{\infty} \sum_{j=1}^{b_d^{k,m}} a_{k,j} Q_k^j(x)$$

which will be presented in Section 3.

2. The reproducing kernel

The inner product defined in (1.1) is an important tool in the theory of spherical harmonics, see [3], [5], [12], [16], [24], [39]. We note that in [34] and [38] this inner product is called the *Fischer inner product*, in honour of the work of E. Fischer [14]; in [8], [9], [42] it is called the *Bombieri inner product*, and in [18] the *Calderón inner product*. However, it seems that is a classical tool in invariant theory (see [13], [36], [40]) and we shall refer to it as the *apolar inner product*.

The apolar inner product has the following property: for all polynomials f, g

$$\langle Q^*(D) f, g \rangle_F = \langle f, Q \cdot g \rangle_F \quad (2.1)$$

where $Q^*(x)$ is the polynomial obtained by conjugation the coefficients of the polynomial Q , and $Q^*(D)$ is the differential operator associated to $Q^*(x)$. Equation (2.1) says that the adjoint of the multiplication operator $g \mapsto Qg$ is just the differential operator $Q^*(D)$. In passing, we note that the apolar inner product has an integral representation:

$$\langle f, g \rangle_F = \frac{1}{\pi^d} \int_{\mathbb{C}^d} f(z) \overline{g(z)} e^{-|z|^2} dz$$

where dz is Lebesgue measure on $\mathbb{R}^{2d}$, see [7]. The space of all entire functions $f : \mathbb{C}^n \rightarrow \mathbb{C}$ which satisfy

$$\|f\|_F^2 := \frac{1}{\pi^d} \int_{\mathbb{C}^d} |f(z)|^2 e^{-|z|^2} dz < \infty \quad (2.2)$$

is called the *Bargmann space* $\mathcal{F}_n$ (also called *Fock* or *Fischer space*, see [34]).

For homogeneous polynomials f, g we define the following well-known inner product

$$\langle f, g \rangle_{\mathbb{S}^{d-1}} := \int_{\mathbb{S}^{d-1}} f(\theta) \overline{g(\theta)} d\theta \quad (2.3)$$

where $\mathbb{S}^{d-1}$ is the unit sphere and $d\theta$ is the rotation-invariant measure on $\mathbb{S}^{d-1}$. The following result follows from Theorem 5.14 in [5]; the result seems to be due to Ü. Kuran [30]:

Theorem 2.1. *For homogeneous harmonic polynomials f, g of degree k one has*

$$\langle f, g \rangle_F = d(d+2) \dots (d+2k-2) \frac{1}{\omega_{d-1}} \langle f, g \rangle_{\mathbb{S}^{d-1}}.$$

In the following we need some facts from the theory of spherical harmonics. Let $Y_{k,l}(x), l = 1, \dots, a_k$ be a basis of $\mathcal{H}_k^1(\mathbb{R}^d)$ which is orthonormal with respect to the scalar product (2.3). Then

$$Z_k(x, y) := \sum_{l=1}^{a_k} Y_{k,l}(x) \overline{Y_{k,l}(y)} \quad (2.4)$$

is the *reproducing kernel* of $\mathcal{H}_k^1(\mathbb{R}^d)$ with respect to (2.3); the function $x \mapsto Z_k(x, y)$ is also called the *zonal harmonic* of degree k with pole y . The addition theorem says that

$$Z_k(x, y) = \frac{a_k}{\omega_{d-1}} |x|^k \cdot |y|^k P_k \left(\frac{\langle x, y \rangle}{|x| \cdot |y|} \right) \quad (2.5)$$

where P_k is a polynomial of degree k with $P_k(1) = 1$ (see [2, p. 455]) and ω_{d-1} is the surface area of $\mathbb{S}^{d-1}$. The polynomial P_k is up to a factor equal to the ultraspherical polynomial $C_k^{(d-2)/2}(t)$, (see [2, p. 456]). Since $P_k(1) = 1$ one has

$$P_k(t) = \frac{C_k^{(d-2)/2}(t)}{C_k^{(d-2)/2}(1)}.$$

Observe that the property $P_k(1) = 1$ also implies that

$$Z_k(x, x) = \frac{a_k}{\omega_{d-1}} |x|^{2k}. \quad (2.6)$$

We now prove

Theorem 2.2. *Let $Y_{k,l}(x)$, $l = 1, \dots, a_k$, be an orthonormal basis of $\mathcal{H}_k^1(\mathbb{R}^d)$ with respect to the scalar product (2.3). Then the polynomials $|x|^{2s} Y_{k,l}(x)$ for $s, k \in \mathbb{N}_0$ and $l = 1, \dots, a_k$ are orthogonal with respect to the apolar inner product and*

$$\omega_{d-1} \left\| |x|^{2s} Y_{k,l}(x) \right\|_F^2 = 2^s s! d(d+2) \dots (d+2(k+s-1)). \quad (2.7)$$

Proof. Let $|x|^{2s} Y_{k,l}$ and $|x|^{2s_1} Y_{k_1,l_1}$ be two basis functions. Without loss of generality we may assume that $s \leq s_1$. By property (2.1) we obtain

$$\left\langle |x|^{2s} Y_{k,l}, |x|^{2s_1} Y_{k_1,l_1} \right\rangle_F = \left\langle \Delta^s \left[|x|^{2s} Y_{k,l} \right], |x|^{2s_1-2s} Y_{k_1,l_1} \right\rangle_F.$$

It is well known and it follows by a quick computation that the following formula

$$\Delta \left(|x|^{2s} h \right) = 2s[2s + d - 2 + 2 \deg h] \cdot |x|^{2s-2} h \quad (2.8)$$

holds for any harmonic homogeneous polynomial h . A simple induction argument shows that for $m \leq 2s$

$$\begin{aligned} \Delta^m \left(|x|^{2s} h \right) &= |x|^{2s-2m} \cdot h \cdot (2s) \dots (2s - 2(m-1)) \cdot \\ &\quad [2s + d - 2 + 2 \deg h] \dots [2s - 2(m-1) + d - 2 + 2 \deg h]. \end{aligned} \quad (2.9)$$

In particular, for $m = s$ we obtain that $\Delta^s (|x|^{2s} h) = d_s(\deg h) \cdot h$ where $d_s(\deg h)$ is the number

$$2^s s! \cdot (2s + d - 2 + 2 \deg h) (2s - 2 + d - 2 + 2 \deg h) \dots (d + 2 \deg h). \quad (2.10)$$

Thus we have

$$\left\langle |x|^{2s} Y_{k,l}, |x|^{2s_1} Y_{k_1,l_1} \right\rangle_F = d_s(k) \cdot \left\langle Y_{k,l}, |x|^{2s_1-2s} Y_{k_1,l_1} \right\rangle_F. \quad (2.11)$$

If $s_1 > s$ we can use again (2.1) and we see that

$$\left\langle |x|^{2s} Y_{k,l}, |x|^{2s_1} Y_{k_1,l_1} \right\rangle_F = d_s(k) \cdot \left\langle \Delta Y_{k,l}, |x|^{2s_1-2s-2} Y_{k_1,l_1} \right\rangle_F = 0.$$

If $s_1 = s$, and $k \neq k_1$ or $l \neq l_1$, we see from (2.11) that $\left\langle |x|^{2s} Y_{k,l}, |x|^{2s_1} Y_{k_1,l_1} \right\rangle_F = 0$ since $Y_{k,l}$ and Y_{k_1,l_1} are orthogonal according to Theorem 2.1.

For $(s, k, l) = (s_1, k_1, l_1)$ Theorem 2.1 and (2.11) show that $\omega_{d-1} \left\| |x|^{2s} Y_{k,l} \right\|_F^2$ is equal to the product of $d(d+2) \dots (d+2k-2)$ and $d_s(k)$ (so (2.10) for $\deg h = k$). This product is equal to

$$2^s s! d(d+2) \dots (d+2(k+s-1)).$$

□

Proposition 2.3. *The system $|x|^{2s} Y_{k-2s,l}(x)$ for $s = 0, 1, \dots, \min\{[k/2], m-1\}$ and $l = 1, \dots, a_{k-2s}$ is an orthogonal basis for $\mathcal{H}_k^m(\mathbb{R}^d)$.*

Proof. The polynomial $f(x) = |x|^{2s} Y_{k,l}(x)$ satisfies $\Delta^m f = 0$ if and only if $s \leq m-1$. Hence $|x|^{2s} Y_{k-2s,l}(x)$, $s = 0, 1, \dots, \min\{[k/2], m-1\}$ and $l = 1, \dots, a_{k-2s}$, are homogeneous polynomials of degree k which satisfy $\Delta^m f = 0$, and by Theorem 2.2 these functions are orthogonal. In order to see that it is basis, let $f \in \mathcal{H}_k^m(\mathbb{R}^d)$. Then f can be written uniquely in the form $f = \sum_{s=0}^{[k/2]} |x|^{2s} h_{k-2s}$ with harmonic homogeneous polynomials h_{k-2s} of degree $k-2s$, see [5]. Formula (2.9) shows that $\Delta^m (|x|^{2s} h_{k-2s}) = C_{m,k,s} |x|^{2s-2m} h_{k-2s}$ for $m \leq s$ and for some nonzero constant $C_{m,k,s}$. The condition $\Delta^m f = 0$ implies that the summation in the last sum ranges only over indices s with $s \leq m-1$. So f is a linear combination of the above basis functions. □

Theorem 2.4. *The reproducing kernel $Z_k^m(x, y)$ for the Hilbert space $\mathcal{H}_k^m(\mathbb{R}^d)$ endowed with the apolar inner product is given by*

$$Z_k^m(x, y) = \omega_{d-1} \sum_{s=0}^{\min\{[k/2], m-1\}} \frac{|x|^{2s} |y|^{2s} Z_{k-2s}(x, y)}{2^s s! d(d+2) \dots (d+2(k-s-1))}. \quad (2.12)$$

Proof. We use formula (1.2) for the system $|x|^{2s} Y_{k-2s,l}(x)$, $l = 1, \dots, a_{k-2s}$, $s = 0, 1, \dots, \min\{[k/2], m-1\}$, by taking into account the normalization constants

given in (2.7). This gives

$$Z_k^m(x, y) = \omega_{d-1} \sum_{s=0}^{\min\{[k/2], m-1\}} \sum_{l=1}^{a_{k-2s}} \frac{|x|^{2s} Y_{k-2s,l}(x) |y|^{2s} Y_{k-2s,l}(y)}{2^s s! d(d+2) \dots (d+2(k-s-1))}.$$

Now (2.4) completes the proof. $\square$

Corollary 2.5. *The values $Z_k^m(x, x)$ for $|x| = 1$ of the reproducing kernel Z_k^m of $\mathcal{H}_k^m(\mathbb{R}^d)$ are constant equal to*

$$\gamma_d^k(m) := \sum_{s=0}^{\min\{[k/2], m-1\}} \frac{a_{k-2s}}{2^s s! d(d+2) \dots (d+2(k-s-1))}. \quad (2.13)$$

Proof. Insert $y = x$ in formula (2.12) and use (2.6). $\square$

3. Convergence of orthogonal series

Suppose that $Q_k^j(x)$, $j = 1, \dots, b_d^{k,m}$ is an orthonormal basis of $\mathcal{H}_k^m(\mathbb{R}^d)$ for each $k = 0, 1, 2, \dots$, and let $a_{k,j}$, $j = 1, \dots, b_d^{k,m}$ be complex numbers. A. Fryant and M.K. Vemuri discuss in [18] conditions for the numbers $a_{k,j}$, $j = 1, \dots, b_d^{k,m}$ such that the series

$$f(x) = \sum_{k=0}^{\infty} \sum_{j=1}^{b_d^{k,m}} a_{k,j} Q_k^j(x) \quad (3.1)$$

converges absolutely and uniformly on compact subsets of the open ball B_R with radius R and center 0. It is shown in [18] that the series (3.1) converges compactly in B_R for

$$R^{-1} = \lim_{k \rightarrow \infty} \sup \left(\sqrt{\gamma_d^k(m)} \|a_k\| \right)^{1/k} \quad \text{and} \quad \|a_k\|^2 := \sum_{j=1}^{b_d^{k,m}} |a_{k,j}|^2.$$

By the next Theorem we obtain the more precise description

$$R^{-1} = \frac{1}{\sqrt{2}} \lim_{k \rightarrow \infty} \sup \left(\frac{\|a_k\|}{\sqrt{k!}} \right)^{1/k}$$

improving the upper bound for R^{-1} in [18] by a factor $1/\sqrt{2}$.

Theorem 3.1. *Let M_k , $k \in \mathbb{N}_0$, be positive numbers and $\gamma_d^k(m)$ as in (2.13). Then*

$$\lim_{k \rightarrow \infty} \sup \left(\sqrt{\gamma_d^k(m)} M_k \right)^{1/k} = \frac{1}{\sqrt{2}} \lim_{k \rightarrow \infty} \sup \left(\frac{M_k}{\sqrt{k!}} \right)^{1/k}. \quad (3.2)$$

Proof. Let us define $D_k(d, s) := d(d+2) \dots (d+2(k-s-1))$. From the identity

$$D_k(d, s) = 2^{k-s} \frac{d}{2} \left(\frac{d}{2} + 1 \right) \dots \left(\frac{d}{2} + (k-s-1) \right) \quad (3.3)$$

we see that

$$D_k(d, s) \geq 2^{k-s-1} (k-s-1)! \geq 2^{k-s-1} k! \frac{1}{(k+1)^{s+1}}. \quad (3.4)$$

Observe that the inequality $a_{k-2s} \leq a_k \leq 2(k+1)^{d-2}$ is obtained from rewriting the formula (1.5) for a_k as

$$a_k = 2(k+1) \left(\frac{k}{2} + 1\right) \dots \left(\frac{k}{d-3} + 1\right) \left(\frac{k}{d-2} + \frac{1}{2}\right)$$

for $d > 2$; for $d = 2$ it is well known that $a_k = 2$ for all $k \in \mathbb{N}$. Thus we obtain

$$\gamma_d^k(m) \leq \sum_{s=0}^{m-1} \frac{2(k+1)^{d-1+s}}{s! 2^{k-1} k!} \leq \frac{(k+1)^{d-2+m}}{2^{k-2} k!} \sum_{s=0}^{\infty} \frac{1}{s!}.$$

Now take the square root, multiply the inequality with M_k , take the k -th root and then the limes superior. Hence the $\leq$ in (3.2) is proved.

For the other inequality we estimate $\gamma_d^k(m)$ below by taking only the summand for $s = 0$, so

$$\gamma_d^k(m) \geq \frac{a_k}{d(d+2) \dots (d+2(k-1))}.$$

Now (3.3) yields $D_k(d, 0) \leq 2^k (d+k)! \leq 2^k k! (d+k)^d$. Using that $a_k \geq 1$ we obtain

$$\gamma_d^k(m) \geq \frac{1}{2^k k! (d+k)^d}.$$

Again, take the square root, multiply the inequality with M_k , take the k -th root and then the limes superior. $\square$

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