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THE GOURSAT PROBLEM FOR A GENERALIZED HELMHOLTZ OPERATOR IN $\mathbb{R}^2$

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1. INTRODUCTION

Let us consider in $\mathbb{R}^2$ the mixed Cauchy problem

$$(1) \quad (\text{cauchy0}) \quad \begin{cases} \Delta^p u + \sum_{|\alpha| \leq k_0} a_\alpha \frac{\partial^{|\alpha|} u}{\partial x^\alpha} = f \\ P|(u - g), \end{cases}$$

where p is a positive integer, k_0 is an integer with $0 \leq k_0 \leq 2p-1$, Δ denotes the standard Laplace operator in $\mathbb{R}^2$

$$\Delta := \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2},$$

the coefficients $a_\alpha = a_\alpha(x, y)$ as well as the data functions $f = f(x, y)$ and $g = g(x, y)$ are real-analytic functions near 0, and $P = P(x, y)$ is a homogeneous polynomial of degree $2p$. Here, the notation $P|(u - g)$ means that P divides $u - g$ in the ring of germs of real-analytic functions at 0. For instance, if $P(x, y) = L(x, y)^{2p}$ for some linear function $L(x, y)$ (which is equivalent to saying that the zero set of $P(x, y)$ consists of a single line with multiplicity $2p$), then (1) with $k_0 = 2p - 1$ is a standard Cauchy problem and the classical Cauchy-Kowalevsky Theorem guarantees that (1) has a unique real-analytic solution u near 0 for every choice of data functions f and g . In the recent paper [1], the authors show that if P is elliptic (i.e. the zero set of $P(x, y)$ consists of only the origin), then (1) with $k_0 = p$ has a unique solution u for every choice of data functions f and g . In this paper, we shall consider the case where the zero set of $P(x, y)$ is a union of $2p$ distinct lines (in which case (1) may be called a Goursat problem). This case is much more subtle and leads to a small divisor problem. We shall give a sufficient condition (which is also necessary in the case $p = 1$; see Section 7) on the divisor P (see Theorem 1 below) for the homogeneous Goursat problem

$$(2) \quad (\text{goursatp1}) \quad \begin{cases} \Delta^p u = f \\ P|(u - g) \end{cases}$$

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to have a unique real-analytic solution u for every real-analytic data f and g . We shall also give a sufficient condition on P (Theorem 3 below) for the perturbed Goursat problem

$$(3) \quad (\textbf{goursatp2}) \quad \begin{cases} \Delta^p u + cu = f \\ P|(u - g), \end{cases}$$

where $c = c(x, y)$ is a real-analytic function near 0, to have a unique real-analytic solution u for every data function f and g .

The conditions on P in Theorems 1 and 3 involve Diophantine properties of a determinant constructed from the geometry of the lines constituting the zero set of P . For instance, if $p = 1$, so that P has degree two and its zero set consists of two distinct lines, then the condition can be phrased in terms of the (acute) angle $\theta = 2\pi\alpha$ between the two lines. The necessary and sufficient condition for the homogeneous Goursat problem

$$(4) \quad (\textbf{goursat11}) \quad \begin{cases} \Delta u = f \\ P|(u - g) \end{cases}$$

to be solvable (Corollary 4) is that

$$(5) \quad (\textbf{dioleray}) \quad \liminf_{\mathbb{Z} \ni m \rightarrow \infty} \left(\inf_{n \in \mathbb{Z}} \left| \alpha - \frac{n}{m} \right| \right) > 0,$$

a condition that is satisfied by e.g. all non-Liouville numbers. Our condition for the perturbed Goursat problem

$$(6) \quad (\textbf{goursat12}) \quad \begin{cases} \Delta u + cu = f \\ P|(u - g), \end{cases}$$

to be solvable (Corollary 5) is more restrictive, namely there exists a constant $C > 0$ such that

$$(7) \quad (\textbf{diophantine} - 1) \quad \left| \alpha - \frac{n}{m} \right| \geq \frac{C}{m^2}, \quad \forall n, m \in \mathbb{Z}, m \neq 0.$$

We note that every irrational number α that satisfies an integral quadratic equation (like $\sqrt{k/l}$ for any integers k and l) satisfies (7) (by Liouville's Theorem on Diophantine approximation). We also point out that every irrational, algebraic number satisfies

$$(8) \quad (\textbf{diophantine} - 2) \quad \left| \alpha - \frac{n}{m} \right| \geq \frac{C_\mu}{m^\mu}, \quad \forall n, m \in \mathbb{Z}, m \neq 0,$$

for some constant C_μ (that depends on μ) and *every* $\mu > 2$ by the Thue-Siegel-Roth Theorem [7]). However, there are algebraic numbers that do not satisfy (7).

We also mention that it follows from our proof that (6) has a unique formal power series solution for all f and g if and only if α is irrational. Thus, as a consequence of our results, we conclude that the family of Goursat problems (6), parametrized by the angle $2\pi\alpha$ between the two lines in the zero set of P , displays "chaotic" behavior in that the

set of parameters for which (6) is solvable is dense as is the set of parameters for which there is not even a formal solution.

The Goursat problem (4) (i.e. (2) with $p = 1$) can be transformed, by a simple linear change of coordinates, into a Goursat problem considered by J. Leray in [5]. His main result is equivalent our Corollary 4. The relationship between the two Goursat problems and Leray's work is briefly explained in Section 3 below. Leray's work was extended to complex parameters and to higher dimensions by Yoshino in [10] and [11]. Other related work on mixed Cauchy and Goursat problems include that of Gårding [3] (see also Theorem 9.4.2 in Hörmander [4]), Shapiro [8], the first author and Shapiro [2], and the authors [1]. Our approach to studying the Goursat problem is inspired by ideas from [8] (see also [2]). The proofs are based on a new estimate for an associated Fischer operator in the real Fischer norm (Theorem 6). The real Fischer norm was introduced in [6] and was also used in [1].

This paper is organized as follows. We present our main results in Section 2. In Section 3, we discuss the relation between our results in the case $p = 1$ and $c \equiv 0$ and those of Leray in [5]. An associated Fischer operator, which is used in the proofs of the main results, is introduced in Section 4 and a crucial estimate is proved for that operator (Theorem 6). The proof of Theorem 1 is also given in that section. The proof of Theorem 3 is given in the subsequent section. In Section 6, we consider the case $p = 2$ and present an explicit family of examples to which Theorem 3 can be applied (see Theorem 8). Finally, in Section 7, we show that our condition in Corollary 4 is also necessary in this case ($p = 1$).

2. MAIN RESULTS

(mainresults)

We shall now formulate our results more precisely. We must first introduce some notation. Let $B_R := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < R^2\}$ be the open disk of radius R in $\mathbb{R}^2$ (where $0 < R \leq \infty$). We denote by $A(B_R)$ the algebra of all infinitely differentiable functions $f : B_R \rightarrow \mathbb{C}$ such that for any compact subset $K \subset B_R$ the homogeneous Taylor series $\sum_{m=0}^{\infty} f_m(x, y)$ converges absolutely and uniformly to f on K ; here, f_m is the homogeneous polynomial of degree m defined by the Taylor series of f

$$f_m(x, y) = \sum_{k+l=m} \frac{1}{k!l!} \frac{\partial^m f}{\partial x^k \partial y^l}(0) x^k y^l.$$

Note that the functions in $A(B_R)$ are real-analytic. For a real number a , we shall define the unimodular complex number

$$(9) \quad (\mathbf{A}) \quad A = A(a) := \frac{a + i}{a - i}.$$

As a goes from $-\infty$ to ∞ , A ranges over the unit circle (from 1 to 1 in the negative direction) and, hence, there is a unique $\beta \in (0, 1)$ such that $A = e^{2\pi i\beta}$. Note that β is rational precisely when A is a root of unity. For future reference, we observe, using the fact that $2 \arctan a = i \log(1 - ia)/(1 + ia)$, that for $a \in [0, \infty)$ the acute angle between the lines $y = 0$ and $x - ay = 0$ is $\pi\beta$. Now, let us fix a positive integer p , distinct real numbers $a_1, a_2, \dots, a_{2p-1}$, and write a for the vector $a = (a_1, \dots, a_{2p-1})$. We shall denote by $P_a(x, y)$ the divisor

$$(10) \quad (\mathbf{Pa}) \quad P_a(x, y) := y \prod_{j=1}^{2p-1} (x - a_j y).$$

If the divisor P in (1) is a homogeneous polynomial of degree $2p$ with $2p$ distinct lines as its zero set, then there is no loss of generality in assuming that P is of the form (10), since the Laplace operator is rotationally invariant. We associate to the vector a a sequence of $2p \times 2p$ matrices $\{M_{m,p,a}\}_{m=0}^\infty$, where

$$(11) \quad (\mathbf{Ma}) \quad M_{m,p,a} := \begin{pmatrix} 1 & 1 & 1 & \dots & 1 & 1 & \dots & 1 \\ 1 & A_1 & A_1^2 & \dots & A_1^{p-1} & A_1^{m+p+1} & \dots & A_1^{m+2p} \\ 1 & A_2 & A_2^2 & \dots & A_2^{p-1} & A_2^{m+p+1} & \dots & A_2^{m+2p} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 1 & A_{2p-1} & A_{2p-1}^2 & \dots & A_{2p-1}^{p-1} & A_{2p-1}^{m+p+1} & \dots & A_{2p-1}^{m+2p} \end{pmatrix}.$$

Here, $A_j := A(a_j)$ where $A(a_j)$ is given by (9). We shall consider the Goursat problem

$$(12) \quad (\mathbf{goursatp}) \quad \begin{cases} \Delta^p u + cu = f \\ P_a|(u - g), \end{cases}$$

where the coefficient $c = c(x, y)$ as well as the data functions $f = f(x, y)$, $g = g(x, y)$ belong to $A(B_R)$. Our first result concerns the homogenous problem, i.e. $c \equiv 0$.

Theorem 1. (homodelp) *Let p be a positive integer and $a_1, \dots, a_{2p-1}$ real, distinct, non-zero numbers. Let $A_j := A(a_j)$, for $j = 1, \dots, 2p-1$, be the unimodular complex numbers given by (9), $P_a(x, y)$ the homogeneous polynomial given by (10), and $\{M_{m,p,a}\}_{m=0}^\infty$ given by (11). If $\det M_{m,p,a} \neq 0$ for all integers $m \geq 0$, and*

$$(13) \quad (\mathbf{leraycond1}) \quad \tau := \liminf_{m \rightarrow \infty} (\det M_{m,p,a})^{1/m} > 0,$$

then the homogeneous Goursat problem

$$(14) \quad (\mathbf{goursatp0}) \quad \begin{cases} \Delta^p u = f \\ P_a|(u - g) \end{cases}$$

has a unique solution $u \in A(B_{\tau R})$ for every $f, g \in A(B_R)$.

Remark 2. (rmkmatrix) For future reference, we note the following identity
(15)

$$(\mathbf{Ma3}) \det M_{m,p,a} = \det \begin{pmatrix} A_1 - 1 & A_1^2 - 1 & \dots & A_1^{p-1} - 1 & A_1^{m+p+1} - 1 & \dots & A_1^{m+2p} - 1 \\ A_2 - 1 & A_2^2 - 1 & \dots & A_2^{p-1} - 1 & A_2^{m+p+1} - 1 & \dots & A_2^{m+2p} - 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ A_{2p-1} - 1 & A_{2p-1}^2 - 1 & \dots & A_{2p-1}^{p-1} - 1 & A_{2p-1}^{m+p+1} - 1 & \dots & A_{2p-1}^{m+2p} - 1 \end{pmatrix},$$

for $k \geq p - 1$. In particular, for $p = 1$, we have $\det M_{m,p,a} = A_1^{m+2} - 1$.

We mention that e.g. all numbers $a_1, \dots, a_{2p-1}$ such that $A_1, \dots, A_{2p-1}$ are algebraic and $\det M_{m,p,a} \neq 0$ for all m satisfy (13) (see [9], Lemma 2.1).

It will follow from our proof of Theorem 3 below that the Goursat problem (12), and hence in particular (14), has a unique formal solution u if and only if $\det M_{m,p,a} \neq 0$ for all integers $m \geq 0$. The Diophantine condition (13) is sufficient (and necessary for $p = 1$; see Section 7 below) for the formal solution to (14) to converge. For the formal solution to the general Goursat problem (3) to converge, we need a stronger condition. We have the following result.

Theorem 3. (helmdelp) *Let p be a positive integer and $a_1, \dots, a_{2p-1}$ real, distinct, non-zero numbers. Let $A_j := A(a_j)$, for $j = 1, \dots, 2p-1$, be the unimodular complex numbers given by (9), $P_a(x, y)$ the homogeneous polynomial given by (10), and $\{M_{m,p,a}\}_{m=0}^\infty$ given by (11). If there exists a constant $C > 0$ such that*

$$(16) \quad (\mathbf{leraycond2}) \det M_{m,p,a} \geq \frac{C}{m^p},$$

for all natural numbers $m \geq 1$ then there exists $0 < r \leq R$ such that the Goursat problem (12) has a unique solution $u \in A(B_r)$ for every $f, g \in A(B_R)$.

In Section 6 below, we give some explicit examples of a_1, a_2, a_3 such that (16) holds for the corresponding unimodular numbers A_1, A_2, A_3 .

In the case $p = 1$, the zero set of P_a is the union of the two distinct lines given by $y = 0$ and $x = ay$. By the rotational symmetry of Δ , we may also assume that $a \geq 0$. If we denote the acute angle between the two lines by $2\pi\alpha$ and by $\beta \in (0, 1/2]$ the number such that $A := A(a) = e^{2\pi i\beta}$, then as mentioned in the beginning of this section we have $\beta = 2\alpha$. As noted in Remark 2 above, we have $\det M_{m,p,a} = A^{m+2} - 1$. The condition $\det M_{m,p,a} = A^{m+2} - 1 \neq 0$ is clearly equivalent to α being irrational. Since

$$|A^{m+2} - 1| \approx \inf_{n \in \mathbb{Z}} |2\pi(m+2)\beta - 2\pi n| = 2\pi(m+2) \inf_{n \in \mathbb{Z}} \left| \beta - \frac{n}{m+2} \right|,$$

where by $E_k \approx F_k$ we mean $CF_k \leq E_k \leq DF_k$ for nonzero constants C, D , it is not difficult to see that Theorems 1 and 3, specialized to the case $p = 1$, can be formulated as follows.

Corollary 4. (homodel1) *Let Γ_1, Γ_2 be two distinct lines through the origin in $\mathbb{R}^2$, and denote by $\theta = 2\pi\alpha$ the acute angle between them. Suppose that α is irrational and satisfies the condition*

$$(17) \quad (\text{leraycond3}) \quad \tau := \liminf_{m \rightarrow \infty} \left(\inf_{n \in \mathbb{Z}} \left| \alpha - \frac{n}{m} \right| \right)^{1/m} > 0.$$

Then, the homogeneous Goursat problem

$$(18) \quad (\text{goursat10}) \quad \begin{cases} \Delta u = f \\ u = g \quad \text{on } \Gamma_1 \cup \Gamma_2 \end{cases}$$

has a unique solution $u \in A(B_{\tau R})$ for every $f, g \in A(B_R)$.

The condition (17) is also necessary for the conclusion of Corollary 4 to hold. This fact is proved in Section 7 below. As mentioned in the introduction, Corollary 4 is equivalent to the result of Leray in [5]. A more detailed explanation of this equivalence is given in Section 3 below.

We conclude this section by reformulating Theorem 3 in the case $p = 1$.

Corollary 5. (helmdel1) *Let Γ_1, Γ_2 be two distinct lines through the origin in $\mathbb{R}^2$, and denote by $\theta = 2\pi\alpha$ the acute angle between them. Suppose that α satisfies the Diophantine condition*

$$(19) \quad (\text{diophantine}) \quad \left| \alpha - \frac{n}{m} \right| \geq \frac{C}{m^2}, \quad \forall n, m \in \mathbb{Z}, m \neq 0$$

for some constant $C > 0$. Then, for any $c \in A(B_R)$, there exists $0 < r \leq R$ such that the Goursat problem

$$(20) \quad (\text{goursatp10}) \quad \begin{cases} \Delta u + cu = f \\ u = g \quad \text{on } \Gamma_1 \cup \Gamma_2, \end{cases}$$

has a unique solution $u \in A(B_r)$ for every $f, g \in A(B_R)$.

3. LERAY'S GOURSAT PROBLEM

(lerayequiv)

Consider the homogeneous Goursat problem

$$(21) \quad (\text{goursat0}) \quad \begin{cases} \lambda \frac{\partial^2 u}{\partial x \partial y} + \Delta u = f \\ xy|(u - g), \end{cases}$$

where λ is a real constant. It follows from the general theory of Goursat (or mixed Cauchy) problems that (21) has a unique real-analytic solution near 0, for all f and g , if $|\lambda| > 2$ (see Gårding [3]; see also Theorem 9.4.2 in Hörmander [4]). The case where $\lambda \in [-2, 2]$ is much more subtle, and was analyzed by Leray in [5] (see also the work of Yoshino [10], [11] for extensions to complex parameters and higher dimensions). For $\lambda \in [-2, 2]$, let $\beta \in [-1/4, 1/4]$ denote the angle such that $\lambda = 2 \sin(2\pi\beta)$. Leray showed that the unique solvability of (21) depends on Diophantine properties of β . For instance, there is a unique formal power series solution u for every f and g if and only if β is irrational. Leray also gave a necessary and sufficient Diophantine condition on irrational β guaranteeing that this formal solution u converges for all convergent f and g ,

$$(22) \quad (\text{diolera2}) \quad \liminf_{\mathbb{Z} \ni m \rightarrow \infty} \left(\inf_{n \in \mathbb{Z}} \left| \beta - \frac{n}{m} \right|^{1/m} \right) > 0.$$

Let us show that this result, for $\lambda \in (-2, 2)$, is equivalent to our Corollary 4 above. Consider the linear change of variables

$$(23) \quad (\text{trans}) \quad x \rightarrow -\sqrt{1 - \frac{\lambda^2}{4}}x + \frac{\lambda}{2}y.$$

As the reader can easily verify, this change of variables leads to the following transformation for the principal symbol of the operator

$$(24) \quad \lambda \frac{\partial^2}{\partial x \partial y} + \Delta \rightarrow \Delta.$$

Hence, the Goursat problem (21) is transformed into the following

$$(25) \quad (\text{goursat01}) \quad \begin{cases} \Delta u = f \\ y(x - ay)|(u - g), \end{cases}$$

where

$$(26) \quad (\text{b}) \quad a := \frac{\lambda/2}{\sqrt{1 - (\lambda/2)^2}}.$$

If we let $\theta = 2\pi\alpha$ denote the acute angle between the two lines $L_1 := \{y = 0\}$ and $L_2 := \{x = by\}$ and β the angle such that $\lambda := 2 \sin(2\pi\beta)$, then we have

$$\alpha = \frac{1 - 2\beta}{4}.$$

Clearly, we have

$$\liminf_{\mathbb{Z} \ni m \rightarrow \infty} \left(\inf_{n \in \mathbb{Z}} \left| \beta - \frac{n}{m} \right| \right)^{1/m} = \liminf_{\mathbb{Z} \ni m \rightarrow \infty} \left(\inf_{n \in \mathbb{Z}} \left| \alpha - \frac{n}{m} \right| \right)^{1/m}.$$

This shows, as mentioned in the introduction, that Leray's result, with $\lambda \in (-2, 2)$, is equivalent to our Corollary 4, with $0 < a < \infty$.

4. AN ESTIMATE FOR AN ASSOCIATED FISCHER OPERATOR AND THE PROOF OF THEOREM 1

(s:est)

Let $\mathbb{C}[x, y]$ denote the space of polynomials in x, y with complex coefficients. For each integer $m \geq 0$, we shall let $\mathcal{P}_m$ denote the subspace of homogeneous polynomials of degree m . We endow $\mathbb{C}[x, y]$ with the real Fischer inner product

$$(27) \quad \langle f, g \rangle := \int_{\mathbb{R}^2} f(x, y) \overline{g(x, y)} e^{-(x^2+y^2)} dx dy,$$

and denote by $\|\cdot\|$ the corresponding norm (see [6]). We shall fix a positive integer p and distinct real numbers $a_1, \dots, a_{2p-1}$ and consider the Fischer operator $F_a(q) := \Delta^p(P_a q)$, where P_a is given by (10). Observe that F_a is a linear operator sending $\mathcal{P}_m$ into $\mathcal{P}_m$. Our main result in this section is the following, in which the notation introduced above is used.

Theorem 6. (estimate) *Let p be a positive integer and $a_1, \dots, a_{2p-1}$ real, distinct, non-zero numbers. Let $A_j := A(a_j)$, for $j = 1, \dots, 2p-1$, be the unimodular complex numbers given by (9) and $P_a(x, y)$ the homogeneous polynomial given by (10). Then the Fischer operator $F_a: \mathcal{P}_m \rightarrow \mathcal{P}_m$, for $m \geq 0$, is a bijection if and only if $\det M_{m,p,a} \neq 0$, where $M_{m,p,a}$ is given by (11). Moreover, if $\det M_{m,p,a} \neq 0$, then we have the estimate*

$$(28) \quad \|P_a q\| \leq \frac{C}{|\det M_{m,p,a}|} \|\Delta^p(P_a q)\|, \quad \forall q \in \mathcal{P}_m,$$

for some $C \geq 0$ (independent of m).

For the proof of Theorem 6, we shall need the following lemma. To state the lemma, we observe the well known fact that any homogeneous polynomial $f(x, y)$ of degree m can be expressed in the following way

$$(29) \quad (\text{expand}) \quad f(x, y) = \sum_{k+l=m} f_{kl} z^k \bar{z}^l,$$

where $z = x + iy$ and $\bar{z} = x - iy$.

Lemma 7. (Fischer) *Let $f(x, y)$ be a homogeneous polynomial of degree m given by (29). Then, we have*

$$(30) \quad (\mathbf{normid}) \quad \|f\|^2 = \pi m! \sum_{k+l=m} |f_{kl}|^2.$$

Proof. As in [6] (see also [1]), we observe that for any homogeneous polynomial $f(x, y)$ of degree m , we have

$$(31) \quad (\mathbf{norm1}) \quad \|f\|^2 = I_{2m+1} \int_{\mathbb{T}} |f(\eta)|^2 ds_\eta$$

where $\mathbb{T}$ denotes the unit circle in $\mathbb{R}^2$, ds arclength, and I_k the integral

$$I_k := \int_0^\infty e^{-r^2} r^k dr.$$

A simple substitution argument gives

$$(32) \quad (\mathbf{int1}) \quad I_{2m+1} = \int_0^\infty e^{-r^2} r^{2m+1} dr = \frac{1}{2} \int_0^\infty e^{-x} x^m dx = \frac{1}{2} m!.$$

Substituting (29) in (31), using the parametrization $z = e^{i\theta}$ for $\mathbb{T}$ and the identity (32), yields

$$(33) \quad (\mathbf{norm2}) \quad \|f\|^2 = \frac{1}{2} m! \sum_{k+l=m} \sum_{i+j=m} f_{kl} \overline{f_{ij}} \int_0^{2\pi} e^{i(k+j-l-i)\theta} d\theta,$$

from which (30) readily follows. □

Proof of Theorem 6. We fix $f \in \mathcal{P}_m$ and consider the equation

$$(34) \quad (\mathbf{Faisf}) \quad F_a(q) := \Delta^p(P_a q) = f,$$

for $q \in \mathcal{P}_m$. Note that $q \in \mathcal{P}_m$ solves (34) if and only if $u = P_a q$ solves the Goursat problem

$$(35) \quad (\mathbf{goursat2}) \quad \begin{cases} \Delta^p u = f \\ u(x, 0) = u(a_1 y, y) \dots u(a_{2p-1} y, y) = 0. \end{cases}$$

We shall look for u of the form $u = v + w$, where $w(x, y) = (x^2 + y^2)^p s(x, y)$ for some $s \in \mathcal{H}_m$ such that

$$(36) \quad (\mathbf{classic}) \quad \Delta^p w(x, y) = \Delta^p((x^2 + y^2)^p s(x, y)) = f(x, y)$$

and $v \in \mathcal{H}_{m+2p}$ satisfies

$$(37) \quad (\mathbf{goursat3}) \quad \begin{cases} \Delta^p v = 0 \\ v(x, 0) = -w(x, 0) \\ v(a_j y, y) = -w(a_j y, y), \quad j = 1, \dots, 2p-1. \end{cases}$$

It is well known that (36) has a unique solution $w(x, y) = (x^2 + y^2)^p s(x, y)$ (see e.g. [8] and references therein). Moreover, in view of the results in [1], we have

$$(38) \quad \|w\| \leq C_1 \|f\|$$

for some constant $C_1 > 0$. Thus, to complete the proof of the theorem it suffices to show that (37) has a solution $v \in \mathcal{P}_{m+2p}$ for every $f \in \mathcal{P}_m$ if and only if $\det M_{m,p,a} \neq 0$, and that, in this case,

$$(39) \quad (\mathbf{goal1}) \quad \|v\| \leq \frac{C}{|\det M_{m,p,a}|} \|f\|$$

for some constant $C > 0$. To this end, we shall actually need the exact form of the solution to (36). Using $z = x + iy$ and $\bar{z} = x - iy$, we may write

$$(40) \quad (\mathbf{id1}) \quad w(x, y) = W(z, \bar{z}) = z^p \bar{z}^p \sum_{k+l=m} s_{kl} z^k \bar{z}^l = \sum_{k+l=m} s_{kl} z^{k+p} \bar{z}^{l+p}.$$

We observe that $\Delta = 4\partial^2/\partial z \partial \bar{z}$. Thus, if we write $f(x, y) = \sum_{k+l=m} f_{kl} z^k \bar{z}^l$, then (36) is equivalent to

$$(41) \quad (\mathbf{id2}) \quad s_{kl} = \frac{f_{kl}}{4^p (k+1) \dots (k+p)(l+1) \dots (l+p)}, \quad \forall k+l=m.$$

Now, we note that every function $v(x, y)$ that satisfies $\Delta^p v = 0$ is of the form

$$(42) \quad (\mathbf{vform}) \quad v(x, y) = \sum_{t=0}^{p-1} (\bar{z}^t \phi_t(z) + z^t \psi_t(\bar{z})),$$

where $\phi_t(z)$ and $\psi_t(\bar{z})$ are holomorphic functions of z and $\bar{z}$, respectively. The function v is a homogeneous polynomial of degree $m+2p$ if and only if $\phi_t(z) = b_{p-1-t} z^{m+2p-t}$ and $\psi_t(\bar{z}) = c_t \bar{z}^{m+2p-t}$, for constants b_{p-1-t} and c_t and $t = 0, \dots, p-1$. Using this notation, equation (37) is equivalent to finding monomials

$$(43) \quad (\mathbf{id3}) \quad \phi_t(z) = b_{p-1-t} z^{m+2p-t}, \quad \psi_t(\bar{z}) = c_t \bar{z}^{m+2p-t},$$

for $t = 0, 1, \dots, p-1$, such that

$$(44) \quad (\mathbf{get1}) \quad \sum_{t=0}^{p-1} (x^t \phi_t(x) + x^t \psi_t(x)) = -W(x, x)$$

and

$$(45) \quad (\mathbf{get2}) \quad \sum_{t=0}^{p-1} (((a_j - i)y)^t \phi_t((a_j + i)y) + ((a_j + i)y)^t \psi_t((a_j - i)y)) = \\ -W((a_j + i)y, (a_j - i)y), \quad j = 1, \dots, 2p-1.$$

In (45), we use the fact that ϕ_t is homogeneous of degree $m + 2p - t$ and we divide the equation by $(a_j - i)^{m+2p}$. With $A_j := A(a_j)$ and $A(a)$ given by (9), the equation becomes

$$(46) \quad (\text{get3}) \quad \sum_{t=0}^{p-1} (A_j^{m+2p-t} \phi_t(y) + A_j^t \psi_t(y)) = -W(A_j y, y), \quad j = 1, \dots, 2p-1.$$

Substituting (41) and (43) in (44) and (46), we obtain the following system of linear equations for the coefficients $b_0, \dots, b_{p-1}, c_0, \dots, c_{p-1}$

$$(47) \quad \begin{aligned} & \sum_{t=0}^{p-1} (b_{p-1-t} + c_t) = - \sum_{k+l=m} \frac{f_{kl}}{4^p(k+1) \dots (k+p)(l+1) \dots (l+p)} \\ (\text{system1}) \quad & \sum_{t=0}^{p-1} (A_1^{m+2p-t} b_{p-1-t} + A_1^t c_j) = - \sum_{k+l=m} \frac{f_{kl} A_1^{m+2p}}{4^p(k+1) \dots (k+p)(l+1) \dots (l+p)} \\ & \vdots \\ & \sum_{t=0}^{p-1} (A_{2p-1}^{m+2p-t} b_{p-1-t} + A_{2p-1}^t c_j) = - \sum_{k+l=m} \frac{f_{kl} A_{2p-1}^{m+2p}}{4^p(k+1) \dots (k+p)(l+1) \dots (l+p)} \end{aligned}$$

If we write d for the column vector of coefficients $d = (c_0, \dots, c_{p-1}, b_0, \dots, b_{p-1})^t$ and e for the column vector whose $(j+1)$ th component, $j = 0, \dots, 2p-1$, is given by

$$- \sum_{k+l=m} \frac{f_{kl} A_j^{m+2p}}{4^p(k+1) \dots (k+p)(l+1) \dots (l+p)},$$

where we let $A_0 := 1$, then (47) can be written

$$(48) \quad (\text{matrixeq}) \quad M_{m,p,a} d = e,$$

where $M_{m,p,a}$ is given by (11). We conclude, as claimed above, that (37) has a unique solution $v \in \mathcal{P}_{m+2p}$ for every $f \in \mathcal{P}_m$ if and only if $\det M_{m,p,a} \neq 0$.

Let us now suppose that $\det M_{m,p,a} \neq 0$ and write d_i for the i th component of d , $i = 1, \dots, 2p$. Using Cramer's rule and the fact that $|A_j| = 1$, we conclude from (48) that

$$(49) \quad (\text{coeffest}) \quad |d_i| \leq C_1 |\det M_{m,p,a}|^{-1} \sum_{k+l=m} \frac{|f_{kl}|}{(k+1) \dots (k+p)(l+1) \dots (l+p)}.$$

By the Cauchy-Schwarz inequality, we conclude that

$$(50) \quad (\text{coeffest2}) \quad |d_i| \leq C_1 |\det M_{m,p,a}|^{-1} \left(\sum_{k+l=m} |f_{kl}|^2 \right)^{1/2} S_m,$$

where S_m denotes the sum

$$(51) \quad (\mathbf{Sm}) \quad S_m := \left(\sum_{k+l=m} \frac{1}{(k+1)^2 \dots (k+p)^2 (l+1)^2 \dots (l+p)^2} \right)^{1/2}.$$

By setting $l = m - k$, we obtain

$$(52) \quad (\mathbf{Smest}) \quad \begin{aligned} S_m^2 &= \sum_{k=0}^m \left(\prod_{j=1}^p (k+j)^2 (m-k+j)^2 \right)^{-1} \\ &\leq 2 \sum_{k=0}^{[m/2]+1} \left(\prod_{j=1}^p (k+j)^2 (m-k+j)^2 \right)^{-1} \\ &= 2m^{-2p} \sum_{k=0}^{[m/2]+1} \left(\prod_{j=1}^p (k+j)^2 \left((1+(j-k)/m)^2 \right) \right)^{-1} \end{aligned}$$

Now, note that, for $j = 1, \dots, p$ and $k = 0, \dots, [m/2] + 1$, we have $(j-k)/m \geq -3/4$ when $m \geq 2$ and, hence, $(1+(j-k)/m)^{-2} \leq 16$. Consequently, we have

$$(53) \quad (\mathbf{Smest2}) \quad S_m^2 \leq \frac{32}{m^{2p}} \sum_{k=0}^{[m/2]+1} \left(\prod_{j=1}^p (k+j)^2 \right)^{-1} \leq \frac{32}{m^{2p}} \sum_{k=0}^{\infty} \frac{1}{(k+1)^{2p}} \leq \frac{C_2}{m^{2p}},$$

for some $C_2 > 0$ independent of m . Thus, by Lemma 7, we obtain from (50) and (53) the following estimates for the functions $\tilde{\phi}_t(z, \bar{z}) := \bar{z}^t \phi_t(z)$, where ϕ_t is given by (43),

$$(54) \quad (\mathbf{estphipsi}) \quad \begin{aligned} \|\tilde{\phi}_t\| &= \sqrt{(m+2p)!} |b_{p-1-t}| \\ &\leq C_1 C_2 |\det M_{m,p,a}|^{-1} \sqrt{(m+1) \dots (m+2p)} \|f\| m^{-p} \\ &\leq C_3 |\det M_{m,p,a}|^{-1} \|f\|. \end{aligned}$$

We obtain a similar estimate for $\tilde{\psi}_t(z, \bar{z}) := z^t \psi_t(\bar{z})$. These estimates yields (39) since v is given by (42). This completes the proof of Theorem 6. $\square$

The arguments in the proof above also yield a proof of Theorem 1. We conclude this section by giving this proof.

Proof of Theorem 1. It is well known that to prove Theorem 1 it suffices to show that the equation

$$(55) \quad (\mathbf{PDE}) \quad \Delta^p(Pq) = f$$

has a unique solution $q \in A(B_{\tau R})$ for every $f \in A(B_R)$ (see e.g. [1]). As in the proof of Theorem 6, we shall look for the solution $u := P_a q$ in the form $u = v + w$, where $w(x, y) = (x^2 + y^2)^p s(x, y)$ satisfies (36) and v solves (37). It is well known that $w \in A(B_R)$ (see [8]; see also [1]). Thus, to complete the proof, it suffices to show that $v \in A(B_{\tau R})$. We

expand v as a series $v = \sum_m v_m$, where the v_m are the homogeneous Taylor polynomials of degree m of v . Similarly, we expand $w = \sum_m w_m$ and $f = \sum_m f_m$. By homogeneity, we observe that the homogeneous polynomials v_m, w_m, f_m satisfy (37) (with $v = v_m, w = w_m$, and $f = f_m$). The fact that $v \in A(B_{\tau R})$ now follows easily from the definition (13) of τ , the form (42) of v , and the estimate (54). The details are left to the reader. $\square$

5. PROOF OF THEOREM 3

Proof of Theorem 3. We fix $a = (a_1, \dots, a_{2p-1})$ as in the theorem. For brevity, we denote P_a simply by P . To prove Theorem 3, it suffices to show that there is $0 < r \leq R$ such that the equation

$$(56) \quad (\mathbf{PDE1}) \quad (\Delta^p + c)(Pq) = f$$

has a unique solution $q \in A(B_r)$ for every $f \in A(B_R)$. We shall look for the solution $u = Pq$ as a series $u = \sum_m u_m = \sum_m Pq_{m-2p}$, where the u_m are the homogeneous Taylor polynomials of degree m of u . To this end, we expand, similarly, both f and c as Taylor series $f = \sum_m f_m$ and $c = \sum_m c_m$. The equation (55) then implies

$$(57) \quad (\mathbf{basic0}) \quad \Delta^p(Pq_j) = f_j, \quad j = 0, 1, \dots, 2p-1,$$

and, for each $m \geq 2p$,

$$(58) \quad (\mathbf{basic1}) \quad \Delta^p(Pq_m) = f_m - \sum_{k=0}^{m-2p} c_{m-k-2p} Pq_k.$$

Since the Fischer operator $F = F_a$, given by $F(q) = \Delta^p(Pq)$, is bijective : $\mathcal{P}_m \rightarrow \mathcal{P}_m$ for every m (by Theorem 6), we can solve, uniquely, (57) and (58) inductively for q_m . This gives us a unique formal power series solution $u = \sum_m u_m$ with $u_m = Pq_{m-2p}$. It remains to prove that there is $r > 0$ such that this series converges to a function in $A(B_r)$. For this, we observe that Theorem 6 and the assumption (16) implies the following estimate

$$(59) \quad (\mathbf{basic2}) \quad \|u_{m+2p}\| \leq Cm^p \|\Delta(Pq_m)\| \leq Cm^p \left(\|f_m\| + \sum_{k=0}^{m-2p} \|c_{m-k-2p} u_{k+2p}\| \right)$$

To prove that $u \in A(B_r)$, we must show (see Proposition 16 in [1]) that for every $0 < \rho < r$ there is a constant $B > 0$ such that

$$(60) \quad (\mathbf{ind}) \quad \|u_k\| \leq B\rho^{-k} \sqrt{k!}$$

for every $k \geq 0$. Let us pick $\rho < \sigma < R$. In view of Proposition 16 in [1], we may assume that there are constants D and E such that

$$(61) \quad (\mathbf{assump}) \quad \max_{\theta \in \mathbb{T}} |c_k(\theta)| \leq D\sigma^{-k}, \quad \|f_k\| \leq E\rho^{-k} \sqrt{k!},$$

for all $k \geq 0$. We shall prove (60) by induction. Thus, assume that (60) holds for all $k \leq m + 2p - 1$. We shall prove that (60) holds also for $k = m + 2p$, provided that m

is large enough. By using (61), the induction hypothesis, and Proposition 8 in [1] (see also Proposition 7 in that paper), we conclude from (59) the following estimate, for some constant $F > 0$,

$$\begin{aligned}
 (62) \quad \|u_{m+2p}\| &\leq C m^p \left(E \rho^{-m} \sqrt{m!} + \sum_{k=0}^{m-2p} F \sigma^{-(m-k-2p)} [(k+2p+1) \dots (m-1)m]^{1/2} \|u_{k+2p}\| \right) \\
 &\leq C m^{\mu-1} \left(E \rho^{-m} \sqrt{m!} + \sum_{k=0}^{m-2p} B F \sigma^{-(m-k-2p)} \rho^{-(k+2p)} \sqrt{m!} \right) \\
 &= B \rho^{-(m+2p)} \sqrt{(m+2p)!} T_m,
 \end{aligned}$$

where

$$\begin{aligned}
 (63) \quad T_m &:= C m^p \frac{\rho^{2p}}{\sqrt{(m+1)(m+2)}} \left(E/B + F \sum_{k=0}^{m-2p} \left(\frac{\rho}{\sigma} \right)^{m-k-2p} \right) \\
 &\leq C m^p \frac{\rho^{2p}}{\sqrt{(m+1) \dots (m+2p)}} \left(E/B + F \frac{1}{1 - \rho/\sigma} \right).
 \end{aligned}$$

Since $\rho < r$, we can make $T_m \leq 1$ for all m by requiring $0 < r \leq R$ small enough (and keeping $\sigma < R$ fixed). This proves Theorem 3. $\square$

6. EXAMPLES OF SOLVABLE GOURSAT PROBLEMS FOR $\Delta^2 + c$

(ex)

In this section, we shall consider the following one-parameter family of Goursat problems

$$(64) \quad \textbf{(goursat22)} \quad \begin{cases} \Delta^2 u + cu = f \\ P_t | (u - g), \end{cases}$$

where $P_t(x, y)$, for $t > 0$, denotes the divisor

$$(65) \quad \textbf{(Pt)} \quad P_t(x, y) := xy(x - ty)(x - y/t).$$

Recall that $A = A(t)$ denotes the unimodular number given by (9) (with $a = t$). Let us denote by $\beta = \beta(t)$ the number $\beta \in (0, 2\pi)$ such that $A = e^{2\pi i \beta}$. We shall prove the following result.

Theorem 8. (helmdel2) *Let $t > 0$ and $\beta := \beta(t)$ as defined above. Suppose that β satisfies the Diophantine condition*

$$(66) \quad \textbf{(diophantine2)} \quad \left| \beta - \frac{n}{m} \right| \geq \frac{C}{m^2}, \quad \forall n, m \in \mathbb{Z}, m \neq 0,$$

for some constant $C > 0$. Then, for any $c \in A(B_R)$, there exists $0 < r \leq R$ such that the Goursat problem (64) has a unique solution $u \in A(B_r)$ for every $f, g \in A(B_R)$.

Theorem 8 is a direct consequence of Theorem 3, with $p = 2$, and the following proposition.

Proposition 9. (matrixcomp) *Let $t > 0$, $a = (a_1, a_2, a_3) := (0, t, 1/t)$, and let $M_{m,p,a}$ be the matrix defined by (11) with $p = 2$. If $\beta = \beta(t)$ satisfies*

$$(67) \quad (\text{diophantine3}) \quad \left| \beta - \frac{n}{m} \right| \geq \frac{C}{m^\mu}, \quad \forall n, m \in \mathbb{Z}, m \neq 0,$$

for some constant $C > 0$, then

$$(68) \quad (\text{detAk}) \quad |\det M_{m,p,a}| \geq \frac{D}{m^{2\mu-2}},$$

for some $D > 0$.

Proof. It is easy to check that the unimodular numbers (A_1, A_2, A_3) that correspond to the vector a is $(-1, A, B)$, where $AB = -1$ and, in view of the discussion preceding Corollary 4,

$$(69) \quad (\mathbf{Am} - 1) |A^m - 1| \geq \frac{C'}{m^{\mu-1}}.$$

(Of course, A is given by (9), but only the above two facts will be needed in the proof.) To prove the proposition, it suffices, in view of Remark 2, to show that $|N_m| \geq C'/m^{2\mu-2}$, where

$$(70) \quad (\text{newmatrix}) \quad N_m := M_{m-4,2,a} = \det \begin{pmatrix} -2 & (-1)^{m-1} - 1 & (-1)^m - 1 \\ A - 1 & A^{m-1} - 1 & A^m - 1 \\ B - 1 & B^{m-1} - 1 & B^m - 1 \end{pmatrix}.$$

We obtain, since $AB = -1$,

$$A^m N_m = \det \begin{pmatrix} -2 & (-1)^{m-1} - 1 & (-1)^m - 1 \\ A - 1 & A^{m-1} - 1 & A^m - 1 \\ -A^{m-1} - A^m & A(-1)^{m-1} - A^m & (-1)^m - A^m \end{pmatrix}.$$

If m is even, then

$$A^m N_m = \det \begin{pmatrix} -2 & -2 & 0 \\ A - 1 & A^{m-1} - 1 & A^m - 1 \\ -A^{m-1} - A^m & -A - A^m & 1 - A^m \end{pmatrix}.$$

A straightforward calculation shows that

$$(71) \quad (\text{even}) \quad A^m N_m = 4A(A^m - 1)(A^{m-2} - 1).$$

If m is odd, then

$$A^m N_m = \det \begin{pmatrix} -2 & 0 & -2 \\ A-1 & A^{m-1}-1 & A^m-1 \\ -A^{m-1}-A^m & A-A^m & -1-A^m \end{pmatrix}.$$

This time we get

$$(72) \quad (\text{odd}) \quad A^m N_M = -2(A^{m-1}-1)^2(A^2+1).$$

The conclusion $|N_m| \geq C'/m^{2\mu-2}$ follows easily from (71) and (72). This completes the proof of the proposition. $\square$

7. DIVERGENCE OF FORMAL SOLUTIONS WHEN $p = 1$ AND $\tau = 0$.

(nec)

We now show that, for $p = 1$ and irrational angles α between the two lines Γ_1 and Γ_2 , the formal solution u to (18), with f convergent and $g \equiv 0$, need not converge when τ , given by (13), is zero. Using the notation and setup in the proof of Theorem 6, let us choose f such that for each m we have, for $k + l = m$,

$$(73) \quad f_{kl} = \begin{cases} R^{-m}, & k = 0 \\ 0, & k > 0. \end{cases}$$

Note that $f \in A(B_R)$. Let us consider the Goursat problem (18) with $g = 0$. By following the argument in the proof of Theorem 6 above, we conclude that the formal solution is of the form $u = v + w$, where w is the formal solution to (36) and $v(x, y)$ is the formal solution to (37). Hence, v is of the form $v(x, y) = \phi(z) + \psi(\bar{z})$. It is well known that the solution w to (36) converges to a function in $A(B_R)$ (see [8]; see also [1]). Thus, the solution u to the Goursat problem converges if and only if the two power series $\phi(z) = \sum_m b_m z^m$ and $\psi(\bar{z}) = \sum_m c_m \bar{z}^m$ converge. With $p = 1$, it is easy to solve the system of equations (47) for b_m and c_m explicitly and we obtain

$$(74) \quad b_m = \frac{1}{(1-A^m)} \frac{A-1}{2R^{m-2}(m-1)},$$

(A similar identity holds, of course, for c_m .) The radius of convergence of the series $\phi(z) = \sum_m b_m z^m$ is

$$(75) \quad R \liminf_{m \rightarrow \infty} |1 - A^m|^{1/m} = 0,$$

proving the assertion above that the solution u does not converge. We conclude this paper by giving an example of a number β in $A = e^{2\pi i \beta}$ such that $\tau = 0$.

Example 10. Let us define

$$(76) \quad \beta := \sum_{k=1}^{\infty} 10^{-p_k},$$

where p_k is defined recursively by $p_1 = 1$ and $p_{k+1} = p_k + k 10^{p_k}$. Note that, for every N , the rational number

$$r_N := \sum_{k=1}^N 10^{-p_k} = \frac{q_N}{10^{p_N}}$$

satisfies

$$|\beta - r_N| \leq \frac{2}{10^{p_{N+1}}}.$$

Consider the subsequence $m_N := 10^{p_N}$ and note that

$$|A^{m_N} - 1| \leq C \inf_{p,q \in \mathbb{Z}_+} q \left| \beta - \frac{p}{q} \right| \leq 2 \frac{10^{p_N}}{10^{p_{N+1}}} = \frac{2}{10^{p_{N+1} - p_N}}$$

Thus, we have

$$|A^{m_N} - 1|^{1/m_N} \leq \frac{C}{10^{(p_{N+1} - p_N)/10^{p_N}}} = \frac{C}{10^N} \rightarrow 0,$$

which shows that $\tau = \liminf_{k \rightarrow \infty} |A^k - 1|^{1/k} = 0$.

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