On real-analytic recurrence relations for cardinal exponential B-splines

J.M. Aldaz

Departamento de Matemáticas y Computación, Universidad de La Rioja, Edificio Vives, Luis de Ulloa s/n., 26004 Logroño, Spain

O. Kounchev

Institute of Mathematics and Informatics, Bulgarian Academy of Sciences, 8 Acad. G. Bonchev Str., 1113 Sofia, Bulgaria

H. Render

Departamento de Matemáticas y Computación, Universidad de La Rioja, Edificio Vives, Luis de Ulloa s/n., 26004 Logroño, Spain

Abstract

Let $L_{N+1}$ be a linear differential operator of order $N+1$ with constant coefficients and real eigenvalues $\lambda_1, \ldots, \lambda_{N+1}$, let $E(\Lambda_{N+1})$ be the space of all $C^\infty$-solutions of $L_{N+1}$ on the real line. We show that for $N \geq 2$ and $n = 2, \ldots, N$, there is a recurrence relation from suitable subspaces $E_n$ to $E_{n+1}$ involving real-analytic functions, and with $E_{N+1} = E(\Lambda_{N+1})$ if and only if contiguous eigenvalues are equally spaced.

Key words: L-splines, cardinal splines, basic spline, recurrence relation.

Email addresses: aldaz@dmc.unirioja.es (J.M. Aldaz), kounchev@math.bas.bg (O. Kounchev), render@gmx.de (H. Render).

1 Partially supported by Grant BFM2003-06335-C03-03 of the D.G.I. of Spain.
2 Acknowledgement of support within the project “Institutes Partnership” with the Alexander von Humboldt Foundation, Bonn.
3 Partially supported by Grant BFM2003-06335-C03-03 of the D.G.I. of Spain. Acknowledgment of support within the project “Institutes Partnership” with the Alexander von Humboldt Foundation, Bonn.

Preprint submitted to JAT 19 March 2007
1 Introduction

Recurrence relations for generalized splines have been discussed by several authors since the appearance of the pioneering work of C. de Boor and M.G. Cox in [4] and [5] respectively, cf. also [6], [7], [11], [12], [13], [18]. In order to motivate our results, let us consider briefly the case of cardinal polynomial splines. It is well known that the cardinal B-splines $M_{N+1}$ and $M_N$ (of order $N+1$ and $N$, and support in $[0, N+1]$ and $[0, N]$ respectively) are related by the identity

$$M_{N+1}(x) = \frac{x}{N}M_N(x) + \frac{N+1-x}{N}M_N(x-1) \quad (1)$$

for all $x \in \mathbb{R}$ (see e.g. [3, p. 86]). Analogous recurrence relations were proved for trigonometric and hyperbolic B-splines in [12] and [18] respectively, cf. [11] for a unified proof. On the other hand, L. Schumaker identified the classes of generalized splines which have B-splines bases computable by recursion relations analogous to those for polynomial, trigonometric, and hyperbolic splines. He proved in [17] that, in addition to the preceding spaces, essentially the only other space of splines admitting such a basis is a certain space of Tchebycheffian splines.

Our objective is to investigate whether there exists a recurrence relation generalizing (1) to the larger class of cardinal L-splines. This question was asked independently in the Conclusion of [19], p. 1436. Cardinal L-splines also arise in a natural way in the study of the so-called cardinal polysplines, see [8], [9], [10] and [1].

Polynomial and hyperbolic cardinal splines are special cases of cardinal L-splines, also known as cardinal exponential splines; here it is assumed that $L$ is a linear differential operator of the form

$$L = \prod_{j=1}^{N+1} \left( \frac{d}{dx} - \lambda_j \right). \quad (2)$$

Throughout the paper we shall assume that the eigen-values $\lambda_1, ..., \lambda_{N+1}$ are real numbers and we shall often use the notation

$$\Lambda_{N+1} := (\lambda_1, ..., \lambda_{N+1}) \quad (3)$$

The functions in

$$E(\Lambda_{N+1}) := E(\lambda_1, ..., \lambda_{N+1}) := \{ f \in C^\infty(\mathbb{R}) : Lf = 0 \} \quad (4)$$

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are called exponential polynomials. A vector space $\mathcal{E}$ is called an exponential space of dimension $N + 1$ if there exists $(\lambda_1, \ldots, \lambda_{N+1}) \in \mathbb{R}^{N+1}$ such that

$$\mathcal{E} = E(\lambda_1, \ldots, \lambda_{N+1}).$$  (5)

A function $u : \mathbb{R} \to \mathbb{R}$ is a cardinal $L-$ spline of order $N + 1$ if $u$ is $(N - 1)$-times continuously differentiable and for every $l \in \mathbb{Z}$ there exists an $f_l \in E(\Lambda_{N+1})$ such that $u(t) = f_l(t)$ whenever $t \in (l, l + 1)$. There exists (up to a scalar factor) a unique cardinal $L$-spline $Q_{N+1}$ of order $N + 1$ and support (equal to) $[0, N + 1]$, called the B-spline of order $N + 1$, see [14]. We shall also write $Q_{\Lambda_{N+1}}$ or $Q_{(\lambda_1, \ldots, \lambda_{N+1})}$ instead of $Q_{N+1}$.

We will study whether for a given fixed natural number $N$ there exist “good” functions $a_N, b_N : \mathbb{R} \to \mathbb{R}$ such that the recurrence relation

$$Q_{N+1}(x) = a_N(x)Q_N(x) + b_N(x)Q_N(x-1)$$  (6)

holds for all $x \in \mathbb{R}$. Note that $a_N$ necessarily coincides with $Q_{N+1}/Q_N$ on $(0, 1)$ and $b_N$ with $Q_{N+1}/Q_N(\cdot - 1)$ on $(N, N + 1)$. Moreover, if $a_N$ is known for $x \in [1, N]$, then the function $b_N$ must be of the form

$$b_N(x) = \frac{Q_{N+1}(x) - a_N(x)Q_N(x)}{Q_N(x-1)}$$  (7)

for $x \in (1, N)$. These arguments show that there exist many possibilities for $a_N$ and $b_N$. However, if we require $a_N$ to be real-analytic on $\mathbb{R}$ then it is uniquely determined by its values on $(0, 1)$, and then $b_N$ is also uniquely determined on $(0, N + 1)$. If in addition $b_N$ is real-analytic on $(-\infty, 2)$ and $(N, \infty)$ then $b_N$ is completely determined on $\mathbb{R}$. An analogous statement can be made by interchanging the roles of $a_N$ and $b_N$. On the other hand, it is not enough to require that $a_N$ and $b_N$ be $C^\infty$ to obtain uniqueness, as Example 6 shows.

The main purpose of the paper is to find out under which conditions both $a_N$ and $b_N$ can be chosen to be real-analytic on $\mathbb{R}$. Let us introduce the following terminology: we say that there exists a real-analytic recurrence relation from $E(\Lambda_N)$ to $E(\Lambda_{N+1})$ if there exist real-analytic functions $a_N, b_N$ defined on $\mathbb{R}$ such that (6) holds for all $x \in \mathbb{R}$. The following is our main result:

**Theorem 1** Let $(\lambda_1, \ldots, \lambda_{N+1}) \in \mathbb{R}^{N+1}$. Then there exists a sequence of exponential spaces $\mathcal{E}_j$ of dimension $j$, $j = 1, \ldots, N + 1$,

$$\mathcal{E}_1 \subset \mathcal{E}_2 \subset \ldots \subset \mathcal{E}_N \subset \mathcal{E}_{N+1} = E(\lambda_1, \ldots, \lambda_{N+1})$$  (8)
with real-analytic recurrence relations from \( \mathcal{E}_n \) for \( \mathcal{E}_{n+1} \) for \( n = 2, \ldots, N \), if and only if there exist \( \alpha, \beta \in \mathbb{R} \) and a permutation \( \sigma \) of \( \{1, 2, \ldots, N + 1\} \) such that 
\[
\lambda_{\sigma(k)} = \alpha + (k - 1) \beta, \quad 1 \leq k \leq N + 1
\]

Let us note that the sufficiency part follows from [18] or [11] in the setting of L-splines with arbitrary knots; but it is also an easy byproduct of our methods of proof. This makes the paper self-contained.

Finally we mention that recurrence relations of a different nature were obtained by N. Dyn and A. Ron in [6] and [7]. When specialized to cardinal L-splines, and under the assumption \( \lambda_1 \neq \lambda_{N+1} \), their results yield the following four-term recurrence relation (see e.g. [8]):
\[
Q_{(\lambda_1, \lambda_2, \ldots, \lambda_{N+1})}(x) = e^{-\lambda_{N+1}}Q_{(\lambda_2, \ldots, \lambda_N)}(x) - \frac{e^{-\lambda_1}Q_{(\lambda_1, \ldots, \lambda_N)}(x)}{\lambda_1 - \lambda_{N+1}} - \frac{Q_{(\lambda_2, \ldots, \lambda_{N+1})}(x - 1)}{\lambda_1 - \lambda_{N+1}} + \frac{Q_{(\lambda_1, \ldots, \lambda_N)}(x - 1)}{\lambda_1 - \lambda_{N+1}}.
\]

Acknowledgment. We are indebted very much to Marie-Laurence Mazure for several helpful remarks and for pointing out an error in an earlier version. We thank both referees for their very thorough job, which led to a complete rewriting of the paper and to a greatly improved presentation.

2 Preliminaries

The general theory of cardinal L-splines was developed by Micchelli [14], cf. also [8, Ch. 13]. Let \( (\lambda_1, \ldots, \lambda_{N+1}) \in \mathbb{C}^{N+1} \). We define the function \( \varphi_{N+1} \) for the operator \( L \) given in (2) as the unique function in the space \( E(\Lambda_{N+1}) \) such that
\[
\frac{d^m}{dx^m} \varphi_{N+1}(0) = 0 \text{ for } m = 0, \ldots, N - 1 \text{ and } \frac{d^N}{dx^N} \varphi_{N+1}(0) = 1. \tag{9}
\]

We shall also write \( \varphi_{(\lambda_1, \ldots, \lambda_{N+1})} \) instead of \( \varphi_{N+1} \). Another useful way to explain properties of \( \varphi_{N+1} \) is the identity
\[
\varphi_{(\lambda_1, \ldots, \lambda_{N+1})}(x) := [\lambda_1, \ldots, \lambda_{N+1}] h_x
\]
where \( h_x \) is the function defined by \( h_x(t) = e^{xt} \) and \([\lambda_1, \ldots, \lambda_{N+1}]\) is the divided difference operator with respect to the variable \( t \), see [16]. Recall that for
pairwise distinct $\lambda_1, \ldots, \lambda_{N+1}$ and for any suitable function $f$

$$[\lambda_1, \ldots, \lambda_{N+1}] f = \sum_{j=1}^{N+1} d_j f(\lambda_j), \quad d_j := \prod_{k=1, k \neq j}^{N+1} (\lambda_j - \lambda_k)^{-1}. \quad (11)$$

Note that $\varphi_{(\lambda_1)}(x) = e^{\lambda_1 x}$, furthermore $\varphi_{(\lambda_1, \lambda_1)}(x) = xe^{\lambda_1 x}$ for $\lambda_1 = \lambda_2$ and

$$\varphi_{(\lambda, \lambda_2)}(x) = \frac{e^{\lambda_1 x} - e^{\lambda_2 x}}{\lambda_1 - \lambda_2} \text{ for } \lambda_1 \neq \lambda_2. \quad (12)$$

From identity (11) one obtains the following simple consequence:

**Lemma 2** Let $\lambda_1, \lambda_2, \ldots, \lambda_{N+1}$ be pairwise distinct complex numbers and let $N \geq 1$. Then there exist nonzero constants $c_j$, $j = 2, \ldots, N+1$, such that

$$\varphi_{N+1}(x) = \sum_{j=2}^{N+1} c_j \varphi_{(\lambda_1, \lambda_j)}(x). \quad (13)$$

The last lemma can be generalized to the case of arbitrary $\lambda_1, \ldots, \lambda_{N+1}$ : For $0 \leq k \leq N$ one has the identity

$$[\lambda_1, \ldots, \lambda_{N+1}] f = [\lambda_{k+1}, \ldots, \lambda_{N+1}] F_k \text{ with } F_k(y) := [\lambda_1, \ldots, \lambda_k, y] f. \quad (14)$$

This is easy to check for pairwise distinct $\lambda_{k+1}, \ldots, \lambda_{N+1}$, using the classical recurrence relation for divided differences. The continuity of divided differences gives then the general case. Using this the following is easily established:

**Lemma 3** Let $\lambda_1, \ldots, \lambda_{N+1}$ be complex numbers, define $F_x(\lambda) = \varphi_{(\lambda_1, \lambda)}(x)$ and denote by $F_x^{(l)}$ its $l$-th derivative with respect to the variable $\lambda$. Suppose that, up to a permutation, $(\lambda_2, \ldots, \lambda_{N+1})$ is equal to $(\mu_1, \ldots, \mu_1, \ldots, \mu_r, \ldots, \mu_r)$ where $\mu_1, \ldots, \mu_r$ are pairwise distinct and $\mu_j$ has multiplicity $\alpha_j > 0$ for $j = 1, \ldots, r$. Then there exist nonzero constants $c_{j,l}$, $j = 1, \ldots, r$; $l = 1, \ldots, \alpha_j - 1$, such that

$$\varphi_{N+1}(x) = \sum_{j=1}^{r} \sum_{l=0}^{\alpha_j-1} c_{j,l} F_x^{(l)}(\mu_j). \quad (15)$$

Set $\varphi_{N+1}^+(x) := \varphi_{N+1}(x)$ for $x \geq 0$ and $\varphi_{N+1}^+(x) := 0$ for $x < 0$. The basic cardinal $L$-spline $Q_{N+1}$ is defined (up to a factor) as the unique cardinal $L$-spline of order $N + 1$ with support in $[0, N + 1]$. The basic spline $Q_{N+1}$ can
be introduced via divided differences, see [11] and [14]. We use the formula

\[ Q_{N+1}(x) = \sum_{j=0}^{N+1} s_{N+1,j} \varphi_{N+1}^j(x - j) \]  

(16)

where the coefficients \( s_{N+1,j} \) are defined by the equation

\[ \prod_{j=1}^{N+1} \left( e^{-\lambda_j} - z \right) = \sum_{j=0}^{N+1} s_{N+1,j} z^j. \]  

(17)

Later we shall use the identity

\[ \sum_{j=0}^{N+1} s_{N+1,j} \varphi_{N+1}(x - j) = 0 \]  

(18)

which implies that \( Q_{N+1}(x) = 0 \) for all \( x \geq N + 1 \). Further we need the formulas \( s_{N+1,N+1} = (-1)^{N+1} \), and

\[ s_{N+1,0} = e^{-(\lambda_1 + \cdots + \lambda_{N+1})}, \quad s_{N+1,1} = -e^{-(\lambda_1 + \cdots + \lambda_{N+1})} \sum_{i=1}^{N+1} e^{\lambda_i}. \]  

(19)

3 Real-analytic recurrence relations: necessary conditions

First, note that for \( N = 1 \) there exists always a real-analytic recurrence relation from \( E(\lambda_1) \) to \( E(\lambda_1, \lambda_2) \). Indeed, \( Q_{(\lambda_1)} \) is given by \( Q_{(\lambda_1)}(x) = e^{\lambda_1 x} 1_{[0,1]} \), where \( 1_{[0,1]} \) denotes the characteristic function of the interval \([0,1]\). Then

\[ Q_{(\lambda_1, \lambda_2)}(x) = a_1(x) Q_{(\lambda_1)}(x) + b_1(x) Q_{(\lambda_1)}(x - 1) \]  

(20)

where \( a_1 \) and \( b_1 \) are defined by real-analytic continuation of the functions \( Q_{(\lambda_1, \lambda_2)}/Q_{(\lambda_1)} \) on \((0, 1)\) and \( Q_{(\lambda_1, \lambda_2)}/Q_{(\lambda_1)}(\cdot - 1) \) on \((1, 2)\) respectively.

3.1 Uniqueness

We shall assume that \( L \) is of the form (2), where all \( \lambda_j \) are real if not otherwise stated. Then \( \varphi_{N+1}(x) \neq 0 \) for all \( x \in \mathbb{R} \setminus \{0\} \) since \( \varphi_{N+1} \) has at most \( N \) real zeros on \( \mathbb{R} \). Further we know that \( Q_{N+1}(x) > 0 \) for all \( x \in (0, N + 1) \).
Proposition 4 For any \( N \geq 2 \), uniqueness of the functions \( a_N \) and \( b_N \) satisfying (6) is guaranteed by requiring either \( a_N \) to be real-analytic on \( \mathbb{R} \) and \( b_N \) to be real-analytic on \( (-\infty, 2) \) and \( (N, \infty) \), or \( b_N \) to be real-analytic on \( \mathbb{R} \) and \( a_N \) to be real-analytic on \( (-\infty, 1) \) and \( (N - 1, \infty) \).

PROOF. By (6), (16), (19), we have for all \( x \in (0, 1) \)
\[
a_N (x) = \frac{Q_{N+1} (x)}{Q_N (x)} = \frac{s_{N+1,0} \varphi_{N+1} (x)}{s_{N,0} \varphi_N (x)} = e^{-\lambda_{N+1}} \frac{\varphi_{N+1} (x)}{\varphi_N (x)},
\] (21)
and for all \( x \in (N, N + 1) \) using (18) and \( s_{N+1,N+1} = (-1)^{N+1} \)
\[
b_N (x) = \frac{Q_{N+1} (x)}{Q_N (x - 1)} = -\frac{\varphi_{N+1} (x - N - 1)}{\varphi_N (x - N - 1)}.
\] (22)
Since \( \varphi_n \) vanishes only at 0, with multiplicity \( n \), the function \( \varphi_{N+1}/\varphi_N \) has a real-analytic extension to all \( \mathbb{R} \). Thus, if we require \( a_N \) to be real-analytic on \( \mathbb{R} \), then \( a_N \) is uniquely defined by (21) on \( \mathbb{R} \). Since (6) implies (7) for all \( x \in (1, N + 1) \), the function \( b_N \) is uniquely defined on \( (1, N + 1) \). If we want \( b_N \) to be real-analytic on \( (N, \infty) \) we have to define \( b_N (x) \) on \( (N, \infty) \) by (22). If we want it to be real-analytic on \( (-\infty, 2) \), we have to define \( b_N \) as the real-analytic extension of \( b_N \) restricted to \( (1, 2) \). Using (7), (16) for \( x \in (1, 2) \), and (21) it is simple to see that for \( x \in (1, 2) \)
\[
b_N (x) = \frac{s_{N+1,1} \varphi_{N+1} (x - 1)}{s_{N,0} \varphi_N (x - 1)} - \frac{s_{N,1} s_{N+1,0} \varphi_{N+1} (x)}{s_{N,0} \varphi_N (x)}.
\] (23)
An entirely analogous argument works in the second case of the proposition.

3.2 Non-analytic recurrence relations

The preceding proof also yields the following result.

Theorem 5 Let \( N \geq 2 \), be a natural number. Then there exist a real-analytic function \( a_N : \mathbb{R} \to \mathbb{R} \) and a function \( b_N \in C^{N-2} (\mathbb{R}) \), real-analytic on \( \mathbb{R} \setminus \{2, ..., N\} \), such that for all \( x \in \mathbb{R} \)
\[
Q_{N+1} (x) = a_N (x) Q_N (x) + b_N (x) Q_N (x - 1).
\] (24)
Positivity over the interval \( (0, N + 1) \) of the functions \( a_N \) and \( b_N \) appearing in the recurrence relation is always desirable from the viewpoint of stability,
cf. also the polynomial case in (1). From (21) it is clear that \( a_N \) in Theorem 5 is always positive on the half line \((0, \infty)\). Moreover (23) implies that \( b_N(1) = -\frac{s_{N,0}}{s_{N,1}} a_N(1) > 0 \) since \( a_N(1) > 0 \), \( s_{N,0} > 0 \), and \( s_{N,1} < 0 \), cf. (19). However, in general \( b_N \) is not positive on \((1, N+1)\), cf. Example 12.

The following example shows that the functions \( a_N \) and \( b_N \) are not unique if they are only required to be \( C^\infty \), even in the polynomial case.

**Example 6** Let \( \Lambda = (0, 0, 0) \) and take \( N = 2 \) in (1), i.e. \( M_3(x) = \frac{5}{2} M_2(x) + \frac{3-x}{2} M_2(x-1) \). Then there exist \( c, d \in C^\infty(\mathbb{R}) \), \( c \neq 0, d \neq 0 \) such that

\[
0 = c(x) M_2(x) + d(x) M_2(x-1).
\]

Thus, \( M_3(x) = \left( \frac{5}{2} + c(x) \right) M_2(x) + \left( \frac{3-x}{2} + d(x) \right) M_2(x-1) \) is a different decomposition with \( C^\infty \)-coefficients.

To see why such \( c \) and \( d \) exist, one may simply take \( c \neq 0 \) to be a \( C^\infty \)-function with support contained in the open interval \((1, 2)\). Define \( d \) just by the equation \( d(x) = -c(x) M_2(x) / M_2(x-1) \) for \( x \in (1, 2) \) and 0 otherwise. Then \( d \) is a \( C^\infty \)-function.

### 3.3 Necessary conditions

**Lemma 7** Let \( L_{\Lambda_{N+1}} = \prod_{j=1}^{N+1} \left( \frac{d}{dx} - \lambda_j \right) \) and \( (\lambda_1, \ldots, \lambda_{N+1}) \in \mathbb{R}^{N+1} \). If \( \varphi \neq 0 \) is a solution of \( L_{\Lambda_{N+1}} \varphi = 0 \), then there exists an \( M > 0 \) such that \( \varphi \) has zeros only in a strip \( |\text{Re} \, z| \leq M \).

**PROOF.** This follows from the asymptotics of \( \varphi \), since it is a sum of exponentials and all \( \lambda_j \) are real.

**Theorem 8** Let \( N \geq 2 \) and \( F_N := \varphi_{N+1}/\varphi_N \). Then each property below implies the next one:

(i) There exists a real-analytic recurrence relation from \( E(\Lambda_N) \) to \( E(\Lambda_{N+1}) \);

(ii) there exist nonzero constants \( A_N, B_N \) such that for all \( x \in \mathbb{R} \)

\[
A_N F_N(x) + B_N F_N(x - 1) + F_N(x - N - 1) = 0; \quad (25)
\]

(iii) the function \( F_N \) has an entire extension.

**PROOF.** For (i) \( \Rightarrow \) (ii) suppose that there exist real-analytic functions \( a_N \) and \( b_N \) on the real line satisfying the recurrence relation (6). Comparing (22)
with (23) one obtains (25) where

$$A_N = \frac{s_{N+1,0}s_{N,1}}{s_{N,0}^2}, \quad B_N := \frac{s_{N+1,1}}{s_{N,0}}.$$  \tag{26}$$

It is clear from (19) that $A_N$ and $B_N$ are nonzero.

Let us prove now (ii) $\Rightarrow$ (iii). Clearly $\varphi_{N+1}/\varphi_N$ is a meromorphic function. Hence we can write $\varphi_{N+1}/\varphi_N = \psi_{N+1}/\psi_N$, where $\psi_{N+1}$ and $\psi_N$ are entire functions without any common zero, and for $j = N, N + 1$, each zero of $\psi_j$ is a zero of $\varphi_j$. Now (25) implies that for each $z \in \mathbb{C}$

$$0 = A_N\psi_{N+1}(z)\psi_N(z - 1)\psi_N(z - N - 1) + B_N\psi_{N+1}(z - 1)\psi_N(z)\psi_N(z - N - 1) + \psi_{N+1}(z - N - 1)\psi_N(z)\psi_N(z - 1).$$

We show that $\psi_N$ has no zero in the complex plane, so $\psi_{N+1}/\psi_N$ is entire. Suppose there exists a zero of $\psi_N$. By Lemma 7 there exists an $K \in \mathbb{R}$ such that all zeros of $\varphi_N$ (and hence of $\psi_N$) satisfy $\text{Re} z \geq K$. Let $K_0$ be the infimum of $\{\text{Re} z : \psi_N(z) = 0\}$. Then there exists a zero $z_0$ of $\psi_N$ with $\text{Re} z_0 < K_0 + \frac{1}{2}$. It follows that $\psi_N(z_0 - 1) \neq 0$ and $\psi_N(z_0 - N - 1) \neq 0$. Then the equation above shows that $0 = A_N\psi_{N+1}(z_0)\psi_N(z_0 - 1)\psi_N(z_0 - N - 1)$. By (ii), $A_N \neq 0$, so we conclude that $\psi_{N+1}(z_0) = 0$. This contradicts the fact that $\psi_{N+1}$ and $\psi_N$ have no common zeros.

**Theorem 9** Assume that $\lambda_1, ..., \lambda_{N+1}$ are given with $\lambda_1 \neq \lambda_2$. Suppose that for each $n = 2, ..., N$ there exists a real-analytic recurrence relation from $E(\lambda_1, ..., \lambda_n)$ to $E(\lambda_1, ..., \lambda_{n+1})$. Then there exist pairwise distinct nonzero integers $m_3, ..., m_{N+1}$ such that

$$\lambda_j - \lambda_1 = m_j (\lambda_2 - \lambda_1) \text{ for } j = 3, ..., N + 1. \tag{27}$$

We first prove the following two lemmas:

**Lemma 10** With the notations of Lemma 3, given $(\lambda_1, ..., \lambda_{N+1})$, the following holds: All functions $\varphi_n/\varphi_2$, $2 \leq n \leq N + 1$, have entire extensions if and only if so do all functions $F_x^{(l)}(\mu_j)/\varphi_2$ for $0 \leq l \leq \alpha_j - 1$ and $j = 1, ..., r$.

**Proof.** Sufficiency is clear since by Lemma 3, $\varphi_n$ is a linear combination of the functions $F_x^{(l)}(\mu_j)$. For the necessity, use induction over $N$. For $N = 1$ the statement is trivial. Suppose now that $\varphi_n/\varphi_2$, $2 \leq n \leq N + 1$, have entire extensions, so they have entire extensions for $2 \leq n \leq N$. By the induction hypothesis each summand (necessarily non-zero) of $\varphi_n/\varphi_2$ in the
corresponding sum arising from (15) has an entire extension. By Lemma 3, \( \varphi_{N+1}/\varphi_2 \) is a linear combination of multiples of the same summands and one more term with a nonzero coefficient, either the value \( F_x(\mu_j)/\varphi_2 \) for a new \( \mu_j \), or of the type \( F_x^{(l)}(\mu_j)/\varphi_2 \) at an old one. Since the other summands and \( \varphi_{N+1}/\varphi_2 \) have entire extensions it follows that the new term also has an entire extension.

**Lemma 11** Suppose that \( \lambda_1 \neq \lambda_2 \). Given \( \lambda \in \mathbb{C} \), the function \( x \mapsto \varphi_{(\lambda_1,\lambda)} (x) /\varphi_{(\lambda_1,\lambda_2)} (x) \) has an entire extension if and only if there exists a nonzero \( m \in \mathbb{Z} \) such that

\[
\lambda - \lambda_1 = m (\lambda_2 - \lambda_1). \tag{28}
\]

Moreover, if \( x \mapsto \varphi_{(\lambda_1,\lambda)} (x) /\varphi_{(\lambda_1,\lambda_2)} (x) \) has an entire extension, it cannot be so for \( x \mapsto \frac{d}{dx} \varphi_{(\lambda_1,\lambda)} (x) /\varphi_{(\lambda_1,\lambda_2)} (x) \).

**PROOF.** Suppose that \( x \mapsto \varphi_{(\lambda_1,\lambda)} (x) /\varphi_{(\lambda_1,\lambda_2)} (x) \) has an entire extension. Then by (12), any nonzero complex zero \( z_0 \) of \( e^{\lambda_1 z} - e^{\lambda_2 z} \) must be a zero of \( z \mapsto \varphi_{(\lambda_1,\lambda)} (z) \). Since \( z_0 := 2\pi i / (\lambda_2 - \lambda_1) \) is a zero of \( e^{\lambda_1 z} - e^{\lambda_2 z} \) we conclude that \( 0 = \varphi_{(\lambda_1,\lambda)} (z_0) \). This implies that \( \lambda \neq \lambda_1 \), and \( e^{\lambda_2 z_0} - e^{\lambda_1 z_0} = 0 \). The existence of some nonzero integer \( m \) satisfying (28) follows immediately.

Conversely, from (28) and (12), one may derive that

\[
\varphi_{(\lambda_1,\lambda)} (x) /\varphi_{(\lambda_1,\lambda_2)} (x) = \frac{1}{m} \frac{e^{m(\lambda_2 - \lambda_1)x} - 1}{e^{(\lambda_2 - \lambda_1)x} - 1}. \tag{29}
\]

Since \( \frac{X^{m-1}}{X-1} = 1 + X + \ldots + X^{m-1} \) we conclude that \( x \mapsto \varphi_{(\lambda_1,\lambda)} (x) /\varphi_{(\lambda_1,\lambda_2)} (x) \) has an entire extension.

Finally, suppose that (28) holds for some nonzero \( m \). Then, with \( z_0 \) as above, we get

\[
\frac{d}{d\lambda} \varphi_{(\lambda_1,\lambda)} (z_0) \neq 0. \tag{30}
\]

Since \( \varphi_{(\lambda,\lambda_2)} (z_0) = 0 \) it follows that \( \frac{d}{d\lambda} \varphi_{(\lambda_1,\lambda)} (x) /\varphi_{(\lambda_1,\lambda_2)} (x) \) is not entire.

**PROOF OF THEOREM 9.** Suppose that for each \( n = 2, \ldots, N \) there exists a real-analytic recurrence relation from \( E(\lambda_1, \ldots, \lambda_n) \) to \( E(\lambda_1, \ldots, \lambda_{n+1}) \). By Theorem 8 all functions \( \varphi_{n+1}/\varphi_n \), \( 2 \leq n \leq N \) have entire extensions. Thus, so do all functions \( \varphi_{n}/\varphi_2 \), \( 2 \leq n \leq N \). The previous two lemmas prove that, if
\( \lambda_1 \neq \lambda_2 \), (27) holds for nonzero integers. Furthermore \( \lambda_1, \ldots, \lambda_{N+1} \) are pairwise distinct by the second statement of Lemma 11.

We have already seen that the coefficient function \( a_N \) in (24) is positive on \((0, \infty)\). It is a natural question whether the coefficient function \( b_N \) is also positive on \([1, N+1]\). By example we show that \( b_N(x) \) can be negative on the interval \((1, 2)\).

**Example 12** Let \( \Lambda = (0, 1, \lambda_3) \) with \( \lambda_3 > 1 \), and set

\[
C_{\lambda_3}(x) := \lambda_3(\lambda_3 - 1) b_2(x) \varphi_2(x) \varphi_2(x - 1).
\]  

Then \( C_{\lambda_3} \) and \( b_2 \) have the same sign on \((1, 2)\), and a computation shows that

\[
C_{\lambda_3}(x) = (1 + e)(e^x - e^{x-1}) + (1 - e^x)(1 + e)e^{(x-1)\lambda_3} + (e^{x-1} - e)e^{\lambda_3} - (1 - e^x)e^{\lambda_3} + \lambda_3(1 - e^{x-1})(1 - e^x)e^{\lambda_3}.
\]

Take \( x = 3/2 \). Then, since \( e^{1.5\lambda_3} \) is the dominating term and the coefficient \( (e^0.5 - e) \) is negative, \( C_{\lambda_3}(3/2) < 0 \) whenever \( \lambda_3 \) is large enough. So \( b_2(3/2) \) is also negative.

4 Existence of real-analytic recurrence relations: a characterization

At first we notice the following simple observation:

**Proposition 13** If there is a real-analytic recurrence relation from the exponential space \( E(\lambda_1, \ldots, \lambda_N) \) to \( E(\lambda_1, \ldots, \lambda_{N+1}) \), then there is also one from \( E(c + \lambda_1, \ldots, c + \lambda_N) \) to \( E(c + \lambda_1, \ldots, c + \lambda_{N+1}) \) for any \( c \in \mathbb{R} \).

**Proof.** For simplicity sake put \( c + \Lambda_N = (c + \lambda_1, \ldots, c + \lambda_N) \). Using the fact that \( \varphi_{c+\Lambda_N}(x) = e^{cx}\varphi_{\Lambda_N}(x) \) it is not difficult to see that \( Q_{c+\Lambda_N}(x) = c_{N}e^{cx}Q_{\Lambda_N}(x) \) for some nonzero constant \( c_N \). Assuming the recurrence relation \( Q_{\Lambda_{N+1}}(x) = a_N (x) Q_{\Lambda_N}(x) + b_N (x) Q_{\Lambda_N}(x - 1), \) it is obvious that

\[
c_{N+1}a_N Q_{c+\Lambda_{N+1}}(x) = a_N (x) Q_{c+\Lambda_N}(x) + e^{c}b_N (x) Q_{c+\Lambda_N}(x - 1).
\]  

In the following we shall make use of a general remark: Let \( U_{N+1} \) be the linear space of functions over an open interval \( I \), spanned by the functions \( 1, X, \ldots, X^{N-1} \) and a real-analytic function \( u(X) \) over \( I \). Then, given \( a \in I \), one
can define an element $\Phi_u$ in $U_{N+1}$ which satisfies $\Phi_u(a) = \ldots = \Phi_u^{(N-1)}(a) = 0$ by

$$\Phi_u(X) = u(X) - \sum_{k=0}^{N-1} \frac{u^{(k)}(a)}{k!} (X - a)^k.$$  \hfill (33)

By expanding $u(X)$ in a Taylor series about $a$ this implies

$$\Phi_u(X) = (X - a)^N \sum_{k=0}^{\infty} \frac{u^{(k+N)}(a)}{(k + N)!} (X - a)^k.$$  \hfill (34)

**Lemma 14** Suppose $\Lambda_N = (0, \lambda_2, \ldots, (N-1) \lambda_2)$ and $\Lambda_{N+1} = (\Lambda_N, M \lambda_2)$, with a natural number $M \geq N \geq 1$, and let $\varphi_N, \varphi_{N+1}$ be defined by (9). Then $\varphi_{N+1}/\varphi_N$ is an entire function of the form

$$\frac{\varphi_{N+1}(x)}{\varphi_N(x)} = cR(e^{\lambda_2 x})$$  \hfill (35)

for some non-zero constant $c$ and a polynomial $R$ defined by

$$R(X) = (X-1)^M \sum_{k=0}^{M-N} \binom{M}{k+N} (X-1)^k.$$  \hfill (36)

**Proof.** By the assumptions of the lemma, the space $E(\Lambda_{N+1})$ is generated by $1, e^{\lambda_2 x}, \ldots, e^{\lambda_2 (N-1)x}$ and $e^{M \lambda_2 x}$. So we are working, up to a change of variable $X := e^{\lambda_2 x}$, in the space $1, X, \ldots, X^{N-1}, X^M$ over the interval $I = (0, \infty)$. Use now the above notations $\Phi_u$ for $u(X) = X^M$ and $a = 1$ in (33). Then there exists a nonzero constant $d_{N+1}$ with

$$\varphi_{N+1}(x) = d_{N+1} \Phi_u(e^{\lambda_2 x}).$$  \hfill (37)

Similarly, for the system $1, X, \ldots, X^{N-2}, v(X)$ with $v(X) = X^{N-1}$, one has that $\Phi_v(X) = (X - 1)^{N-1}$ and $\varphi_N(x) = d_N \Phi_v(e^{\lambda_2 x})$ for some $d_N \neq 0$. An immediate consequence of (34) is that $\Phi_u(X)/\Phi_v(X)$ is equal to $R(X)$ defined in (36). This completes the proof of the lemma.

The following proposition provides the central step in the proof of our main theorem. In particular it shows that for the exponential space $E(0, \ldots, N)$ there exist two different exponential spaces $E_1$ and $E_2$ admitting a real-analytic recurrence relation from $E(0, \ldots, N)$ to $E_j$ for $j = 1, 2$, namely $E_1 = E(0, \ldots, N+1)$ and $E_2 = E(-1, 0, \ldots, N)$.
Proposition 15  Given two real numbers $\alpha, \beta$ with $\beta \neq 0$, an integer $N \geq 1$, and an integer $M$, $M \notin \{0, \ldots, N-1\}$, let us set

$$\Lambda_N := (\alpha, \alpha + \beta, \ldots, \alpha + (N-1)\beta)$$
$$\Lambda_{N+1} := (\alpha, \alpha + \beta, \ldots, \alpha + (N-1)\beta, \alpha + \beta M).$$

Then, the following assertions are equivalent:

(i) There exists a real-analytic recurrence relation from $E(\Lambda_N)$ to $E(\Lambda_{N+1})$;

(ii) $M = N$ or $M = -1$.

PROOF. Assume that (i) holds. Due to Proposition 13, we may assume that $\alpha = 0$. We will show that that $M = N$ if $M > 0$ and $M = -1$ if $M < 0$.

First, assume that $M > 0$. Then $M > N$ by our assumptions and we can use the last Lemma: If $M > N$, then the polynomial $R$ defined in (36) has degree $M - N + 1 \geq 2$. Now (35) and Theorem 8 yield

$$A_N R \left(e^{\lambda_2 x}\right) + B_N R \left(e^{\lambda_2 (x-1)}\right) + C_N R \left(e^{\lambda_2 (x-N-1)}\right) = 0. \quad (38)$$

Putting $\gamma = e^{-\lambda_2}$ and $X = e^{\lambda_2 x}$ one arrives at

$$A_N R (X) + B_N R (\gamma X) + C_N R \left(\gamma^{N+1} X\right) = 0 \quad (39)$$

for all $X > 0$, hence for all $X \in \mathbb{R}$. Then $(A_N + B_N + C_N) R(0) = 0$, and differentiation gives the following two relations:

$$(A_N + \gamma B_N + C_N \gamma^{N+1}) R'(0) = (A_N + \gamma^2 B_N + C_N \gamma^{2N+2}) R''(0) = 0. \quad (40)$$

Since $R(0)$, $R'(0)$ and $R''(0)$ are nonzero and $\lambda_2 \neq 0$, this implies $A_N = B_N = C_N = 0$, a contradiction. Hence $M = N$.

Now assume that $M < 0$. We will see that this is reduced to the previous case. We apply Proposition 13 with $c := -(N - 1)\beta$; so assumption (i) with $\alpha = 0$ implies that there exists a real-analytic recurrence relation from $E(c + \Lambda_N)$ to $E(c + \Lambda_{N+1})$. Now $c + \Lambda_N$ consists of the values

$$-(N - 1)\beta + j\beta = (-\beta)(N - 1 - j) \quad (41)$$

for $j = 0, \ldots, N-1$ and

$$c + \lambda_{N+1} = -(N - 1)\beta + M\beta = (-\beta)(N - 1 - M). \quad (42)$$
Since \( M < 0 \) we know that \( \tilde{M} := N - 1 - M > 0 \). By the first case applied to \( c + \Lambda_N \) and \( c + \Lambda_{N+1} \) we conclude that \( \tilde{M} = N \) which clearly implies that \( M = -1 \).

For (ii) \( \Rightarrow \) (i) we assume at first that \( M = N \). Then the real change of variable \( X = e^{\lambda x} \) transforms the cardinal spline spaces based on \( E(\Lambda_N) \) and \( E(\Lambda_{N+1}) \) into the polynomial splines of degree \( N \) and \( N + 1 \) on \((0, \infty)\) relative to the simple knots \( t_j := e^{\lambda j} \). Recurrence relations are known in such spaces, and their coefficients are real-analytic. This implies the statement by taking the inverse transform \( x = \lambda^{-1} \ln X \). The case \( M = -1 \) is handled in a similar way.

**Proposition 16** Let \( \alpha \) be a real number. Suppose \( \Lambda_N = (\alpha, ..., \alpha) \) and \( \Lambda_{N+1} = (\Lambda_N, \lambda) \) for \( \lambda \in \mathbb{R} \). Then the following assertions are equivalent:

(i) There exists a real-analytic recurrence relation from \( E(\Lambda_N) \) to \( E(\Lambda_{N+1}) \);

(ii) \( \lambda = \alpha \).

**PROOF.** Due to Proposition 13, we may assume that \( \alpha = 0 \). By Theorem 8 and assumption (i) there exist nonzero constants \( A_N, B_N, C_N \) such that for all \( x \in \mathbb{R} \)

\[
A_N \frac{\varphi_{N+1}}{\varphi_N}(x) + B_N \frac{\varphi_{N+1}}{\varphi_N}(x - 1) + C_N \frac{\varphi_{N+1}}{\varphi_N}(x - N - 1) = 0. \tag{43}
\]

Suppose that \( \lambda \neq 0 \). Note that \( \varphi_N(x) = x^{N-1}/(N-1)! \) and, according to (33) and Lemma 14, there exists a nonzero constant \( d_{N+1} \) such that

\[
\varphi_{N+1}(x) = d_{N+1}(e^{\lambda x} - R(x)), \quad R(x) = \sum_{k=0}^{N-1} \frac{(\lambda x)^k}{k!}. \tag{44}
\]

Multiply (43) with \([x(x-1)(x-N-1)]^{N-1}\). It follows that there exists a polynomial \( Q \) such that

\[
e^{\lambda x} P(x) - Q(x) = 0 \quad \text{for all } x \in \mathbb{R}, \tag{45}
\]

where the polynomial \( P \) is defined by

\[
P(x) = A_N [(x-1)(x-N-1)]^{N-1} + B_N e^{-\lambda} [x(x-N-1)]^{N-1}
+ C_N e^{-\lambda(N+1)} [x(x-1)]^{N-1}.
\]

This is impossible unless \( P = Q = 0 \). But \( P = 0 \) implies \( A_N = B_N = C_N = 0 \). Thus we cannot have \( \lambda \neq 0 \).
For \((ii) \Rightarrow (i)\) note that \(E(\Lambda_{N+1})\) is the classical polynomial spline space.

Now we are going to prove our main result stated as Theorem 17.

**PROOF OF THEOREM 1.** Proof of the necessity by induction. For \(n = 2\) there is nothing to prove. Suppose that there exists a sequence of exponential spaces \(E_1 \subset E_2 \subset \ldots \subset E_N \subset E_{N+1} = E(\Lambda_{N+1})\) with real-analytic recurrence relations from \(E_n\) to \(E_{n+1}\) for \(n = 2, \ldots, N\). The recursive assumption enables us to assume, without loss of generality, that \(\lambda_j = \alpha + (j - 1)\beta\) for \(1 \leq j \leq N\).

Suppose that \(\beta \neq 0\). From Theorem 9 we can deduce that \(\lambda_{N+1} = \alpha + \beta M\) for some integer \(M\) different from \(0, \ldots, N - 1\). Proposition 15 ensures that either \(M = N\) or \(M = -1\). If \(M = N\), then the equality \(\lambda_j = \alpha + (j - 1)\beta\) is valid for \(j = N + 1\) too. If \(M = -1\), then \(\lambda_{\sigma(j)} = \hat{\alpha} + (j - 1)\beta\) for \(1 \leq j \leq N + 1\), with \(\hat{\alpha} := \alpha - \beta\), and with \(\sigma(1) = N + 1\), \(\sigma(j) := j - 1\) for \(j = 2, \ldots, N + 1\).

The case \(\beta = 0\) follows from Proposition 16.

Sufficiency follows from Proposition 15 and 16.

Consider the exponential space \(E(\lambda_1, \lambda_2)\). For simplicity assume that \(\lambda_1 = 0\), and put \(\beta = \lambda_2 - \lambda_1\). Then the proof of our main theorem shows how to construct all increasing sequences of exponential spaces admitting analytic relations, starting from \(E(\lambda_1, \lambda_2) = E(0, \beta)\) in the following (uncomplete) scheme:

\[
\begin{array}{ccc}
E(0, \beta) & \nearrow & \searrow \\
& E(-\beta, 0, \beta) & E(0, \beta, 2\beta) \\
& \searrow & \nearrow \\
E(-2\beta, -\beta, 0, \beta) & E(-\beta, 0, \beta, 2\beta) & E(0, \beta, 2\beta, 3\beta)
\end{array}
\] (46)

Let us look at the particular case that \(\Lambda_{N+1} = (\lambda_1, \ldots, \lambda_{N+1})\) is ordered, so \(\lambda_1 \leq \ldots \leq \lambda_{N+1}\). Then there exists a real-analytic recurrence relation from \(E(\lambda_1, \ldots, \lambda_n)\) to \(E(\lambda_1, \ldots, \lambda_{n+1})\) for \(n = 2, \ldots, N\), if and only if

\[
\lambda_n = \lambda_1 + (n - 1)(\lambda_2 - \lambda_1).
\] (47)

The following description is obvious from the above scheme:
Theorem 17 Let \((\lambda_1, \ldots, \lambda_{N+1}) \in \mathbb{R}^{N+1}\). Then there exist real-analytic recurrence relations from \(E(\lambda_1, \ldots, \lambda_n)\) to \(E(\lambda_1, \ldots, \lambda_{n+1})\) for \(n = 1, 2, \ldots, N\) if and only if for \(3 \leq j \leq N + 1\)
\[
\lambda_j = \lambda_1 + m_j(\lambda_2 - \lambda_1)
\]

with either \(m_{j+1} = \min\{m_1, \ldots, m_j\} - 1\) or \(m_{j+1} = \max\{m_1, \ldots, m_j\} + 1\), and with \(m_1 = 0, m_2 = 1\).

It follows from our results that the only exponential spaces admitting real-analytic recurrence relations are either the classical polynomial spaces, or transformations of polynomial spaces via an exponential map, cf. the discussion in section 6 in [17].

References


