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Convergence of polyharmonic splines on semi-regular grids $\mathbb{Z} \times a\mathbb{Z}^n$ for $a \rightarrow 0$.

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Abstract

Let $p, n \in \mathbb{N}$ with $2p \geq n + 2$, and let I_a be a polyharmonic spline of order p on the grid $\mathbb{Z} \times a\mathbb{Z}^n$ which satisfies the interpolating conditions $I_a(j, am) = d_j(am)$ for $j \in \mathbb{Z}, m \in \mathbb{Z}^n$ where the functions $d_j : \mathbb{R}^n \rightarrow \mathbb{R}$ and the parameter $a > 0$ are given. Let $B_s(\mathbb{R}^n)$ be the set of all integrable functions $f : \mathbb{R}^n \rightarrow \mathbb{C}$ such that the integral

$$\|f\|_s := \int_{\mathbb{R}^n} |\hat{f}(\xi)| (1 + |\xi|^s) d\xi$$

is finite. The main result states that for given $\sigma \geq 0$ there exists a constant $c > 0$ such that whenever $d_j \in B_{2p}(\mathbb{R}^n) \cap C(\mathbb{R}^n)$, $j \in \mathbb{Z}$, satisfy $\|d_j\|_{2p} \leq D \cdot (1 + |j|^\sigma)$ for all $j \in \mathbb{Z}$ there exists a polyspline $S : \mathbb{R}^{n+1} \rightarrow \mathbb{C}$ of order p on strips such that

$$|S(t, y) - I_a(t, y)| \leq a^{2p-1} c \cdot D \cdot (1 + |t|^\sigma)$$

for all $y \in \mathbb{R}^n$, $t \in \mathbb{R}$ and all $0 < a \leq 1$.

Keywords. Radial Basis Functions; Interpolation; Polysplines; Polyharmonic Splines.

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1 Introduction

Let Γ be a discrete subset of $\mathbb{R}^{n+1}$ and let p be a natural number satisfying $2p \geq n+2$. We define $SH^p(\mathbb{R}^{n+1}, \Gamma)$ to be the set of all tempered distributions u on $\mathbb{R}^{n+1}$ which are $2p - n - 2$ continuously differentiable and such that

$$\Delta^p u(x) = 0 \text{ for all } x \in \mathbb{R}^{n+1} \setminus \Gamma.$$

Here Δ^p is the p -th iterate of the Laplace operator $\Delta = \sum_{j=1}^{n+1} \frac{\partial^2}{\partial x_j^2}$ for a natural number p . For the discrete subset $\Gamma = \mathbb{Z}^{n+1}$ one calls the distribution $u \in SH^p(\mathbb{R}^{n+1}, \Gamma)$ *polyharmonic cardinal spline* of order p . Fundamental results about polyharmonic cardinal splines and interpolation properties have been proved in [15]. In particular, it is proven in Proposition 2 in [15] that $u \in SH^p(\mathbb{R}^{n+1}, \Gamma)$ satisfies the equation

$$\Delta^p u(x) = \sum_{\gamma \in \Gamma} a_\gamma \delta(x - \gamma) \text{ for all } x \in \mathbb{R}^{n+1},$$

where the a_γ 's are constants and δ is the Dirac distribution for the point 0.

In the present paper we shall be concerned with interpolation problems on semi-regular grids Γ_a , i.e. grids of the form

$$\Gamma_a := \mathbb{Z} \times a\mathbb{Z}^n \tag{1}$$

where $a\mathbb{Z}^n$ is the set $\{am : m \in \mathbb{Z}^n\}$ and $a > 0$ is a positive real number. This kind of interpolation problems typically occur in practice, when the measurements of data are taken in one direction with a small step size $a > 0$ while in the second direction they are distributed with a relatively large step size. Let us illustrate this kind of problems by an instructive example from Geophysics, elaborated in [8, chapter 6]: data of the magnetic field over the Cobb Offset (California) are collected by air planes on 13 parallel lines in $\mathbb{R}^2$ with approximately 200 data points which are relatively uniformly distributed on every line and we may assume that the step size is approximately a ; the distance between each two neighbouring lines is *very large* compared with a and thus we are close to the setting of our problem where the grid is given by Γ_a . Due to the reversals of the magnetic field the neighboring layers of the magma have opposite signs and thus the data oscillate wildly. This kind of data provide a good test for every smoothing method. Interpolation for this kind of data can be subsumed under the general theory of interpolation with Radial Basis Functions for scattered data (see [5], [15], [17], [4], [20]). Indeed, the general theory allows the data even to be arbitrarily scattered, and from a theoretical point of view there seems to be no big advantage to know that the data have been taken in a very regular way. However, we can use this information in a completely different way: using univariate interpolation methods applied to the discrete data given on each line $j \times \mathbb{R}$ one can find a spline function $d_j : \mathbb{R} \rightarrow \mathbb{C}$ interpolating the data on the line. Thus we may assume that the data are not anymore given in a discrete manner but by means of functions d_j defined on the *whole* line

$j \times \mathbb{R}$; thus the data d_j may have smoothness up to a certain prescribed order. From the general theory of polysplines, developed by the first author in [8], one can find a polyspline S interpolating the data function d_j on each line; let us recall that in this case the polyspline S is a function polyharmonic of order p in the open strips $(j, j+1) \times \mathbb{R}$, which is continuously differentiable up to order $2p-2$; note that the smoothness is independent of the dimension. We refer to [8, chapter 6] where polysplines are defined as piecewise solutions of elliptic equations of order $2p$ which are smooth of order $2p-2$. In [8, chapter 6] the performance of the polysplines is compared with other smoothing methods in the above-mentioned example of magnetic data; in [13] one can find applications of polysplines to Medical Imaging.

It is an essential feature of the polyspline theory that the data are given on hypersurfaces and not in a discrete way. This fact implies a remarkable advantage of the polysplines: The polysplines may be defined for any order p (independent of the dimension n) while the polyharmonic splines require the condition $2p \geq n+1$ for $\mathbb{R}^{n+1}$. In particular, the biharmonic case $p=2$ already gives a C^2 interpolation result.

The smoothness order $2p-n-2$ for the polyharmonic splines on $\mathbb{R}^{n+1}$ is related to the smoothness order of the fundamental solution R_p of the polyharmonic operator Δ^p which satisfies $\Delta^p R_p(x) = \delta(x)$; for the explicit form of R_p see [1, p. 2] or [8, Theorem 10.37]. Polysplines have smoothness of order $2p-2$. Roughly speaking, polysplines may be considered by means of the Green formula as simple layer polyharmonic potentials, i.e. as convolutions of the fundamental solution R_p with some continuous measures which are supported on the data hypersurfaces; such convolutions increase the smoothness of R_p , cf. [7, p. 346], [8].

In this paper we want to compare the interpolation methods for polyharmonic splines and polysplines. We shall idealize and simplify the above-described framework of practical examples by assuming that the measurements have provided data values $(d_\gamma)_{\gamma \in \mathbb{Z} \times a\mathbb{Z}^n}$ for the grid Γ_a defined in (1). This assumption has the advantage that the interpolation polyharmonic splines I_a (satisfying $I(\gamma) = d_\gamma$ for all $\gamma \in \Gamma_a = \mathbb{Z} \times a\mathbb{Z}^n$) can be defined in an analytical way using Fourier methods. Similarly, interpolation with polysplines can be performed in an explicit way since the geometry of the data sets is very regular, see [2], [8], [9], [10], [12]. In the present paper we shall need the definition of a polyspline only in a very special setting: a function $u : \mathbb{R}^{n+1} \rightarrow \mathbb{C}$ is a *cardinal polyspline of order p on strips* if u is $2p-2$ times continuously differentiable and $\Delta^p u(x) = 0$ for all x in the open strips $(j, j+1) \times \mathbb{R}^n$ with $j \in \mathbb{Z}$.

The interpolation polyharmonic spline I_a depends on the step size $a > 0$ in the direction of the second variable. *The main problem* which we address is the convergence of the polyharmonic interpolants I_a on the grid Γ_a for given data and for $a \rightarrow 0$. In order that the problem is well-posed we need an assumption that the data depend on the parameter $a > 0$ in a reasonable way. This is achieved by assuming that functions $d_j : \mathbb{R}^n \rightarrow \mathbb{R}$, $j \in \mathbb{Z}$, are given such that the data for step size $a > 0$ are provided by the values $d_j(am)$ for

$(j, m) \in \mathbb{Z}^{n+1}$, and that these data are of polynomial growth. Then there exists a unique polyharmonic spline $I_a \in SH^p(\mathbb{R}^{n+1}, \mathbb{Z} \times a\mathbb{Z}^n)$ such that

$$I_a(j, am) = d_j(am) \text{ for all } m \in \mathbb{Z}^n, j \in \mathbb{Z}, \quad (2)$$

see [15] or [11]. Later we shall assume that the data functions $d_j : \mathbb{R}^n \rightarrow \mathbb{R}$, $j \in \mathbb{Z}$ for the interpolation problem are sufficiently smooth.

In [11] we have obtained a *qualitative* solution of the convergence problem: the polyharmonic splines I_a (with respect to the grid Γ_a) of order p interpolating given data functions $d_j : \mathbb{R}^n \rightarrow \mathbb{R}$, $j \in \mathbb{Z}$, on the grid Γ_a , converge pointwise to a limit function S , and this limit function S is a polyspline of order p on strips. In this sense we see that *polysplines* are a *continuous version of polyharmonic splines*. However, the method used in [11] permitted only data satisfying the restrictive assumption

$$d_j = 0 \quad \text{for } |j| > N, \quad (3)$$

where N is some positive integer. In the present paper we shall give a *quantitative* answer to the convergence problem by replacing (3) by the weaker assumption of a sufficient decay. Roughly speaking, we shall prove that the polyharmonic splines I_a of order p (on the grid Γ_a) interpolating given data functions $d_j : \mathbb{R}^n \rightarrow \mathbb{R}$, $j \in \mathbb{Z}$, on the grid Γ_a , converge to a limit function S with a rate of convergence a^{2p-1} for $a \rightarrow 0$, see Theorem 1 below. Thus in the present paper we establish the link between the Schoenberg type interpolation for polyharmonic splines proved in [15] and the Schoenberg type interpolation for polysplines proved in [2] and [3].

In order to formulate the main result precisely, let us introduce now some notations. By $B_s(\mathbb{R}^n)$ we denote the set of all *integrable* functions $f : \mathbb{R}^n \rightarrow \mathbb{C}$ such that the integral

$$\|f\|_s := \int_{\mathbb{R}^n} |\widehat{f}(\xi)| (1 + |\xi|^s) d\xi \quad (4)$$

is finite. Here $\widehat{f}$ denotes as usual the Fourier transform of a function $f : \mathbb{R}^n \rightarrow \mathbb{C}$, defined by $\widehat{f}(\omega) := \int_{\mathbb{R}^n} e^{-i\langle x, \omega \rangle} f(x) dx$. For an extended study of similar function spaces see Definition 10.1.6 in Hörmander [6]. Further $C^k(\mathbb{R}^n)$ denotes the set of all k -times continuously differentiable functions $f : \mathbb{R}^n \rightarrow \mathbb{C}$ for $k \in \mathbb{N}_0 \cup \{\infty\}$. We shall write $C(\mathbb{R}^n)$ for $C^0(\mathbb{R}^n)$.

The main result of the present paper is the following:

Theorem 1 *Let $\sigma \geq 0$ and $p \in \mathbb{N}$ with $2p \geq n + 2$. Then there exists a constant $c > 0$ such that whenever $d_j \in B_{2p}(\mathbb{R}^n) \cap C(\mathbb{R}^n)$, $j \in \mathbb{Z}$, satisfy*

$$\|d_j\|_{2p} \leq D \cdot (1 + |j|^\sigma) \quad \text{for all } j \in \mathbb{Z}, \quad (5)$$

there exists a polyspline $S : \mathbb{R}^{n+1} \rightarrow \mathbb{C}$ of order p on strips such that

$$|S(t, y) - I_a(t, y)| \leq a^{2p-1} c \cdot D \cdot (1 + |t|^\sigma)$$

for all $y \in \mathbb{R}^n$, $t \in \mathbb{R}$ and all $0 < a \leq 1$. Here $I_a \in SH^p(\mathbb{R}^{n+1}, \mathbb{Z} \times a\mathbb{Z}^{n-1})$ is the polyharmonic spline satisfying (2).

The result of Theorem 1 may be compared with the error estimates available in [14, Theorem 2] for cardinal polyharmonic splines: if f is in the Sobolev space $L_s^{2p}(\mathbb{R}^{n+1})$ (namely, the space of order $2p$, see [14] for the precise definition) and $S_\sigma f$ its polyharmonic interpolant on the grid $\sigma^{-1} \cdot \mathbb{Z}^{n+1}$, then there exists a constant C such that for all $\sigma > 0$

$$\left(\int |f(x) - (S_\sigma f)(x)|^s dx \right)^{\frac{1}{s}} \leq \sigma^{-2p} C \left(\int |\Delta^{2p} f(x)|^s dx \right)^{\frac{1}{s}}.$$

Our result is of different nature since we consider a limit process for the grid $\mathbb{Z} \times a\mathbb{Z}^n$, $a \rightarrow 0$. General results concerning the error estimates of f and its interpolant Sf can also be found in [16].

The paper is organized in the following way: in the first Section we recall and improve some facts from [11] concerning the representation of polyharmonic splines $L_{p,a,f}$ which interpolate data at the points $(0, am)$, $m \in \mathbb{Z}^n$, given by the values $f(am)$, $m \in \mathbb{Z}^n$, of a function f , and zero data on (j, am) , $m \in \mathbb{Z}^n$, $j \in \mathbb{Z} \setminus \{0\}$, see (8) and (9). The main result in Section 2 is the following: if $f \in B_{2p}(\mathbb{R}^n) \cap C(\mathbb{R}^n)$ then there exist constants $\eta > 0$ and $C > 0$ (independent of f) and there exists a cardinal polyspline $L_{p,f}$ on strips such that the following estimate holds

$$|L_{p,a,f}(t, y) - L_{p,f}(t, y)| \leq a^{2p-1} C e^{-\eta t} \cdot \int |\hat{f}(\xi)| (1 + |\xi|^2)^p d\xi \quad (6)$$

for all $t \in \mathbb{R}$ and $y \in \mathbb{R}^n$ and for all $0 < a \leq 1$. Due to the exponential decay in estimate (6) we can give in Section 3 an estimate for the case of infinitely many data functions d_j as described in Theorem 1. In the Appendix we provide the somewhat technical proofs of Theorems 5, 6 and 7 which contain the basic estimates.

2 Basic facts for interpolation with polyharmonic splines

Let us remind that throughout the paper we assume $2p \geq n + 2$.

A function $L_{p,a}$ is a *fundamental* polyharmonic spline of order p for the grid Γ_a whenever $L_{p,a}$ is in $SH_p(\mathbb{R}^{n+1}, \Gamma_a)$ and

$$L_{p,a}(0) = 1, \text{ and } L_{p,a}(\gamma) = 0 \quad \text{for all } \gamma \in \Gamma_a \setminus \{0\}.$$

Define the function φ by putting $\hat{\varphi}(\omega) := |\omega|^{-2p}$ for $\omega = (s, \xi)$ with $s \in \mathbb{R}$ and $\xi \in \mathbb{R}^n$, and put

$$S_{\varphi,a}(\omega) := \sum_{k \in \mathbb{Z}} \sum_{m \in \mathbb{Z}^n} \frac{1}{((s + 2\pi k)^2 + |\xi - 2\pi a^{-1}m|^2)^p}. \quad (7)$$

Then

$$L_{p,a}(x) = \frac{a^{n-1}}{(2\pi)^{n+1}} \int_{\mathbb{R}^{n+1}} e^{i\langle x, \omega \rangle} \frac{\hat{\varphi}(\omega)}{S_{\varphi,a}(\omega)} d\omega \quad \text{for } x \in \mathbb{R}^{n+1}.$$

Assume that $f : \mathbb{R}^n \rightarrow \mathbb{C}$ is a function of polynomial growth: we define $L_{p,a,f}$ as the polyharmonic spline of order p for the grid Γ_a such that

$$L_{p,a,f}((0, am)) = f(am) \text{ and } L_{p,a,f}((j, am)) = 0 \text{ for } j \in \mathbb{Z} \setminus \{0\} \quad (8)$$

and for all $m \in \mathbb{Z}^n$. Indeed, $L_{p,a,f}$ is given by

$$L_{p,a,f}(t, y) := \sum_{m \in \mathbb{Z}^n} f(am) L_{p,a}((t, y - am)). \quad (9)$$

The following result was proven in [11]:

Theorem 2 *Let $a > 0$ and suppose that for the continuous function $f : \mathbb{R}^n \rightarrow \mathbb{C}$ there exist constants $A > 0$ and $\delta > 0$ such that for all $x \in \mathbb{R}^n$ and $\omega \in \mathbb{R}^n$*

$$|f(x)| \leq A(1 + |x|)^{-n-\delta} \quad \text{and} \quad |\widehat{f}(\omega)| \leq A(1 + |\omega|)^{-n-\delta}. \quad (10)$$

Then the polyharmonic spline $L_{p,a,f}$ of order p for the grid Γ_a , defined in (9), is given by

$$L_{p,a,f}(t, y) = \frac{1}{2\pi} \int_{\mathbb{R}^n} \int_{\mathbb{R}} e^{its} e^{i\langle y, \xi \rangle} \widehat{f}(\xi) \frac{B_{a,p}(s, \xi, y)}{S_{\varphi,a}(s, \xi)} ds d\xi. \quad (11)$$

Here the function $B_{a,p}$ is defined for all $(s, \xi) \in \mathbb{R}^{n+1} \setminus (\{0\} \times \frac{2\pi}{a}\mathbb{Z}^n)$ by

$$B_{a,p}(s, \xi, y) = \sum_{m \in \mathbb{Z}^n} \frac{1}{\left(s^2 + |\xi - 2\pi a^{-1}m|^2\right)^p} e^{-\frac{2\pi}{a}i\langle y, m \rangle}. \quad (12)$$

Clearly, for each $s \neq 0$ the function $\xi \mapsto B_{a,p}(s, \xi, y)$ is well-defined since $2p \geq n+2$. Note that $B_{a,p}(s, \xi, y)$ has poles at $\{0\} \times \frac{2\pi}{a}\mathbb{Z}^n$. Since $|B_{a,p}(s, \xi, y)| \leq B_{a,p}(s, \xi, 0) \leq S_{\varphi,a}(s, \xi)$ we have for all $\xi \in \mathbb{R}^n$ and $s \neq 0$

$$0 \leq \left| \frac{B_{a,p}(s, \xi, y)}{S_{\varphi,a}(s, \xi)} \right| \leq \frac{B_{a,p}(s, \xi, 0)}{S_{\varphi,a}(s, \xi)} \leq 1. \quad (13)$$

Further, we have shown in [11, Theorem 6] that there exists a constant $C > 0$ such that for all $|s| > 0$ and for all $\xi \in \mathbb{R}^n$ and for all $0 < a \leq 1$

$$|B_{a,p}(s, \xi, y)| \leq \sum_{m \in \mathbb{Z}^n} \frac{1}{\left(s^2 + |\xi - 2\pi a^{-1}m|^2\right)^p} \leq \frac{C}{s^{2p}} (1 + |s|)^n. \quad (14)$$

We now extend Theorem 2 in the following way:

Theorem 3 *Let $a > 0$ and suppose that $f \in B_{2p-n-2}(\mathbb{R}^n) \cap C(\mathbb{R}^n)$. Then*

$$L_{p,a,f}(t, y) = \frac{1}{2\pi} \int_{\mathbb{R}^n} \int_{\mathbb{R}} e^{its} e^{i\langle y, \xi \rangle} \widehat{f}(\xi) \frac{B_{a,p}(s, \xi, y)}{S_{\varphi,a}(s, \xi)} ds d\xi. \quad (15)$$

Proof. Let us denote the expression on the right hand side in (15) by $S_{p,a,f}$. The assumption $f \in B_{2p-n-2}(\mathbb{R}^n)$ implies that $\widehat{f}$ is integrable. Formula (13) and (14) show that

$$g_y(s, \xi) := \widehat{f}(\xi) \frac{B_{a,p}(s, \xi, y)}{S_{\varphi,a}(s, \xi)}$$

is an integrable function. Hence $S_{p,a,f}$ is well-defined, and it is a continuous and bounded function, in particular it is a tempered distribution. Obviously, the integrand is differentiable with respect to the variable t and using (14) is easy to see that $t \mapsto S_{p,a,f}(t, y)$ is differentiable up to the order $2p - n - 2$. Moreover it can be shown by using similar estimates as in (14) that the function $y \mapsto B_{a,p}(s, \xi, y)$ is $2p - n - 2$ differentiable with

$$\frac{d^\beta}{dy^\beta} B_{a,p}(s, \xi, y) = \sum_{m \in \mathbb{Z}^n} \frac{\left(-\frac{2\pi}{a}\right)^{|\beta|} m^\beta}{\left(s^2 + |\xi - 2\pi a^{-1}m|^2\right)^p} e^{-\frac{2\pi}{a}i\langle y, m \rangle}.$$

It follows that $S_{p,a,f} \in C^{2p-n-2}(\mathbb{R}^{n+1})$. Now we want to show that $S_{p,a,f}$ is polyharmonic of order p on $\mathbb{R}^{n+1} \setminus \Gamma_a$. Since $\widehat{f}$ is integrable there exists a sequence of functions $\varphi_k \in C^\infty(\mathbb{R}^n)$ with compact support such that $\int \left| \widehat{f}(\xi) - \varphi_k(\xi) \right| d\xi \rightarrow 0$ for $k \rightarrow \infty$. Let us define f_k by Fourier inversion of φ_k , i.e. by putting

$$f_k(x) = \frac{1}{(2\pi)^n} \int e^{i\langle x, \xi \rangle} \varphi_k(\xi) d\xi. \quad (16)$$

Clearly, then $\widehat{f}_k = \varphi_k$. It follows that for $x = (t, y)$

$$|S_{p,a,f}(x) - S_{p,a,f_k}(x)| \leq \frac{1}{2\pi} \int_{\mathbb{R}^n} \int_{\mathbb{R}} \left| \widehat{f}(\xi) - \varphi_k(\xi) \right| \frac{B_{a,p}(s, \xi, 0)}{S_{\varphi,a}(s, \xi)} ds d\xi \quad (17)$$

and by (13) and (14) it follows that the right hand side tends to zero: thus we have uniform convergence on the left hand side. Since $\varphi_k \in C^\infty(\mathbb{R}^n)$ has compact support it follows that each φ_k satisfies (10). Hence $S_{p,a,f_k} \in SH_p(\mathbb{R}^{n+1}, \Gamma_a)$ by Theorem 2. In particular, S_{p,a,f_k} is polyharmonic of order p on $\mathbb{R}^{n+1} \setminus \Gamma_a$. As $S_{p,a,f}$ is the uniform limit of S_{p,a,f_k} we conclude that $S_{p,a,f}$ is polyharmonic of order p on $\mathbb{R}^n \setminus \Gamma_a$. As pointed out in [15, p. 147], $S_{p,a,f}$ with the proven properties is an element of $SH_p(\mathbb{R}^{n+1}, \Gamma_a)$.

Since $\int \left| \widehat{f}(\xi) - \varphi_k(\xi) \right| d\xi \rightarrow 0$ we conclude from (16) that $f_k(x)$ converges to $\frac{1}{(2\pi)^n} \int e^{i\langle x, \xi \rangle} \widehat{f}(\xi) d\xi$. Since f is continuous the Fourier inversion formula shows that $f_k(x)$ converges to $f(x)$. By Theorem 2 S_{p,a,f_k} interpolates f_k at the lattice points $\gamma \in \Gamma_a$, hence we conclude by (17) that $S_{p,a,f}(\gamma) = \lim_{k \rightarrow \infty} f_k(\gamma) = f(\gamma)$. The proof is complete. ■

3 The main result

First, let us recall Theorem 6 from [11].

Theorem 4 Suppose that $f \in B_{2p}(\mathbb{R}^n) \cap C(\mathbb{R}^n)$ satisfies the decay condition (10). Then the polyharmonic spline $L_{p,a,f}$ of order p for the grid Γ_a defined in (9) converges pointwise for $a \rightarrow 0$ to the function defined by

$$L_{p,f}(t, y) = \frac{1}{2\pi} \int_{\mathbb{R}^n} \int_{\mathbb{R}} e^{i\langle y, \xi \rangle} e^{its} \widehat{f}(\xi) \frac{1}{(s^2 + |\xi|^2)^p S_p(s, \xi)} ds d\xi, \quad (18)$$

where

$$S_p(s, \xi) := \sum_{k \in \mathbb{Z}} \frac{1}{((s + 2\pi k)^2 + |\xi|^2)^p}. \quad (19)$$

Moreover the function $L_{p,f}$ is a polyspline.

W. Madych and S. Nelson have shown in [15] the fundamental fact that the function $(s, \xi) \mapsto 1/S_{\varphi,a}(s, \xi)$ defined in (7) has an analytic extension to some strip in $\mathbb{C}^{n+1}$. We need an analogous result for the function (19). Since this function is not periodic in ξ the argument is more involved. The proof of the following three theorems will be given in the Appendix.

Theorem 5 There exists $\varepsilon > 0$ such that for each $\xi \in \mathbb{R}^n$ the function $s \mapsto 1/S_p(s, \xi)$ can be extended analytically to the strip $\{z \in \mathbb{C} : |\operatorname{Im} z| < \varepsilon\}$.

Theorem 6 There exists $\varepsilon > 0$ such that for all $z = s + i\eta$ with $|s| \leq \pi$ and $|\eta| \leq \varepsilon$ and for all $\xi \in \mathbb{R}^n$

$$3 \cdot |S_p(z, \xi)| \geq \sum_{k \in \mathbb{Z}} \frac{1}{|(z + 2\pi k)^2 + |\xi|^2|^p}, \quad (20)$$

where the function S_p is defined in (19).

Theorem 7 There exists $\varepsilon > 0$ such that for all $z = s + i\eta$ with $|s| \leq \pi$ and $|\eta| \leq \varepsilon$ and for all $\xi \in \mathbb{R}^n$ and for all a with $0 < a \leq 1$

$$3 |S_{\varphi,a}(z, \xi)| \geq \sum_{k \in \mathbb{Z}} \sum_{m \in \mathbb{Z}^n} \frac{1}{|(z + 2\pi k)^2 + |\xi - 2\pi a^{-1}m|^2|^p}, \quad (21)$$

where the function $S_{\varphi,a}$ is defined in (7). In particular, for all $0 < a \leq 1$ and $|s| \leq \pi$ and $|\eta| \leq \varepsilon$ and all $\xi \in \mathbb{R}^n$

$$\frac{1}{|S_{\varphi,a}(s + i\eta, \xi)|} \leq 3 |(s + i\eta)^2 + |\xi|^2|^p. \quad (22)$$

Now we are able to state our second result, containing the rate of the convergence established in Theorem 4:

Theorem 8 There exists a constant $\eta > 0$ such that for every function $f \in B_{2p}(\mathbb{R}^n) \cap C(\mathbb{R}^n)$ the following estimate holds for all $x = (t, y) \in \mathbb{R} \times \mathbb{R}^n$

$$|L_{p,a,f}(x) - L_{p,f}(x)| \leq 6e^{-\eta t} \int_{\mathbb{R}^n} |\widehat{f}(\xi)| \int_{-\pi}^{\pi} \sum_{\substack{m \in \mathbb{Z}^n \\ m \neq 0}} \frac{|S_p(s + i\eta, \xi + 2\pi m a^{-1})|}{|S_{\varphi,a}(s + i\eta, \xi)|} ds d\xi.$$

Proof. It is easy to see from (11) and (18) that

$$L_{p,a,f}(t, y) - L_f(t, y) = \frac{1}{2\pi} \int_{\mathbb{R}^n} e^{i\langle \xi, y \rangle} \widehat{f}(\xi) A(\xi, y, t) ds d\xi,$$

where $A(\xi, y, t) := \int_{-\infty}^{\infty} e^{its} R(s, \xi) ds$ and

$$R(s, \xi) := \frac{B_{a,p}(s, \xi, y)}{S_{\varphi,a}(s, \xi)} - \frac{1}{\left(s^2 + |\xi|^2\right)^p S_p(s, \xi)}.$$

By Theorem 5, for each $\xi \in \mathbb{R}^n$ the function $s \mapsto R(s, \xi)$ has an analytic extension for $z \in \mathbb{C}$ with $|\operatorname{Im} z| < \varepsilon$. Hence, by Cauchy's theorem,

$$A(\xi, y, t) = \int_{-\infty}^{\infty} e^{it(s+i\eta)} R(s+i\eta, \xi) ds.$$

In order to estimate this integral we define

$$B_{a,p}^{\times}(s, \xi, y) := \sum_{m \in \mathbb{Z}^n, m \neq 0} \frac{e^{-\frac{2\pi}{a}i\langle y, m \rangle}}{\left(s^2 + |\xi + 2\pi a^{-1}m|^2\right)^p}. \quad (23)$$

Clearly, $B_{a,p}(s, \xi, y) = \left(s^2 + |\xi|^2\right)^{-p} + B_{a,p}^{\times}(s, \xi, y)$ and a simple calculation shows that $R(s, \xi) = R_1(s, \xi) + R_2(s, \xi)$ where

$$\begin{aligned} R_1(s, \xi) &= \frac{B_{a,p}^{\times}(s, \xi, y)}{S_{\varphi,a}(s, \xi)}, \\ R_2(s, \xi) &= \frac{S_p(s, \xi) - S_{\varphi,a}(s, \xi)}{S_{\varphi,a}(s, \xi) \left(s^2 + |\xi|^2\right)^p S_p(s, \xi)}. \end{aligned}$$

Using the simple identity $\int_{-\infty}^{\infty} g(s) ds = \sum_{k=-\infty}^{\infty} \int_{-\pi}^{\pi} g(s+2\pi k) ds$ and the periodicity of $S_{\varphi,a}(s, \xi)$ with respect to s we obtain

$$\int_{-\infty}^{\infty} e^{it(s+i\eta)} R_1(s+i\eta, \xi) ds = e^{-\eta t} \sum_{k=-\infty}^{\infty} \int_{-\pi}^{\pi} e^{i(s+2\pi k)t} \frac{B_{a,p}^{\times}(s+2\pi k+i\eta, \xi, y)}{S_{\varphi,a}(s+i\eta, \xi)} ds.$$

Recall now the definition of $B_{a,p}^{\times}$ in (23) and apply the triangle inequality and use then the non-trivial inequality (20) valid for $|s| \leq \pi$:

$$\begin{aligned} & \sum_{k=-\infty}^{\infty} |B_{a,p}^{\times}(s+2\pi k+i\eta, \xi, y)| \\ & \leq \sum_{m \in \mathbb{Z}^n, m \neq 0} \sum_{k=-\infty}^{\infty} \frac{1}{\left|(s+2\pi k+i\eta)^2 + |\xi + 2\pi a^{-1}m|^2\right|^p} \\ & \leq 3 \sum_{m \in \mathbb{Z}^n, m \neq 0} |S_p(s+i\eta, \xi + 2\pi a^{-1}m)|. \end{aligned}$$

Thus $\left| \int_{-\infty}^{\infty} e^{it(s+i\eta)} R_1(s+i\eta, \xi) ds \right|$ is bounded by the constant

$$M := 3e^{-\eta t} \sum_{m \in \mathbb{Z}^n, m \neq 0} \int_{-\pi}^{\pi} \frac{|S_p(s+i\eta, \xi + 2\pi a^{-1}m)|}{|S_{\varphi,a}(s+i\eta, \xi)|} ds. \quad (24)$$

Similarly, if we define the function

$$d(s, \xi) := \frac{S_p(s, \xi) - S_{\varphi,a}(s, \xi)}{S_{\varphi,a}(s, \xi) S_p(s, \xi)},$$

which is periodic in s , we obtain

$$\int_{-\infty}^{\infty} e^{it(s+i\eta)} R_2(s+i\eta, \xi) ds = e^{-\eta t} \sum_{k=-\infty}^{\infty} \int_{-\pi}^{\pi} \frac{e^{i(s+2\pi k)t} d(s+i\eta, \xi)}{\left((s+2\pi k+i\eta)^2 + |\xi|^2\right)^p} ds.$$

Using again (20) we obtain

$$\begin{aligned} \left| \int_{-\infty}^{\infty} e^{it(s+i\eta)} R_2(s+i\eta, \xi) ds \right| &\leq 3e^{-\eta t} \int_{-\pi}^{\pi} |d(s+i\eta, \xi)| \cdot |S_p(s+i\eta, \xi)| ds \\ &= 3e^{-\eta t} \int_{-\pi}^{\pi} \left| \frac{S_p(s+i\eta, \xi) - S_{\varphi,a}(s+i\eta, \xi)}{S_{\varphi,a}(s+i\eta, \xi)} \right| ds. \end{aligned}$$

The estimate

$$|S_p(s+i\eta, \xi) - S_{\varphi,a}(s+i\eta, \xi)| \leq \sum_{m \in \mathbb{Z}^n, m \neq 0} |S_p(s+i\eta, \xi + 2\pi a^{-1}m)|$$

shows that

$$\left| \int_{-\infty}^{\infty} e^{it(s+i\eta)} R_2(s+i\eta, \xi) ds \right| \leq M$$

where M is defined in (24). The proof is complete. ■

We need the following

Proposition 9 *For all $s \in \mathbb{R}$ and $\sigma > 0$ the following inequality holds*

$$A := \sum_{k=-\infty}^{\infty} \frac{1}{\left|(s+2\pi k)^2 + \sigma^2\right|^p} \leq \frac{1+\sigma}{\sigma^{2p}}.$$

Proof. From the proof of Theorem 6 in [11] applied to the case $n = 1$ and $a = 1$ one obtains that

$$A \leq \frac{1}{\sigma^{2p}} + \frac{2}{\sigma^{2p}} \sum_{k=1}^{\infty} \frac{1}{\left(1 + \frac{\pi^2 k^2}{\sigma^2}\right)^p}.$$

The latter sum can be estimated by $\frac{\sigma}{\pi} \int_0^{\infty} \frac{1}{(1+y^2)^p} dy$. By an obvious estimate the last integral attains maximum for $p = 1$; for $p = 1$ it is a table integral equal to $\frac{\pi}{2}$. This ends the proof. ■

In the next Theorem we refine the estimate in Theorem 8 by estimating the right-hand side.

Theorem 10 *There exist constants $\eta > 0$ and $C > 0$ such that for every function $f \in B_{2p}(\mathbb{R}^n) \cap C(\mathbb{R}^n)$ the estimate*

$$|L_{p,a,f}(t, y) - L_f(t, y)| \leq a^{2p-1} C e^{-\eta t} \cdot \int |\widehat{f}(\xi)| \left(1 + |\xi|^2\right)^p d\xi$$

holds for all $t \in \mathbb{R}$ and $y \in \mathbb{R}^n$ and for all $0 < a \leq 1$.

Proof. Define the set $I_a := \{\xi \in \mathbb{R}^n : |\xi| \leq \pi a^{-1}\}$.

1. We are going to estimate at first the error

$$E_1(I_a) := \frac{1}{2\pi} \int_{I_a} |\widehat{f}(\xi)| \left| \int_{-\pi}^{\pi} \sum_{m \in \mathbb{Z}^n, m \neq 0} \frac{|S_p(s + i\eta, \xi + 2\pi m a^{-1})|}{|S_{\varphi,a}(s + i\eta, \xi)|} ds d\xi \right|.$$

Since $|z| \geq |\operatorname{Re} z|$ and $\operatorname{Re}(s + 2\pi k + i\eta)^2 = (s + 2\pi k)^2 - \eta^2$ we have

$$\left| (s + i\eta + 2\pi k)^2 + |\xi - 2\pi a^{-1}m|^2 \right| \geq \left| (s + 2\pi k)^2 + |\xi - 2\pi a^{-1}m|^2 - \eta^2 \right|.$$

Thus we can estimate

$$|S_p(s + i\eta, \xi + 2\pi m a^{-1})| \leq \sum_{k=-\infty}^{\infty} \frac{1}{\left| (s + 2\pi k)^2 + |\xi - 2\pi a^{-1}m|^2 - \eta^2 \right|^p}.$$

For $\xi \in I_a$ we have $|\xi - 2\pi m a^{-1}| \geq |2\pi m a^{-1}| - |\xi| \geq |\pi m a^{-1}|$ for every $m \neq 0$.

Further $\eta \leq 1$ and $a \leq 1$ imply $\frac{1}{2} |\pi m a^{-1}|^2 \geq \eta^2$ for $m \neq 0$, and therefore

$$\sigma^2 := |\xi - 2\pi a^{-1}m|^2 - \eta^2 \geq |\pi m a^{-1}|^2 - \eta^2 \geq \frac{1}{2} |\pi m a^{-1}|^2. \quad (25)$$

Hence Proposition 9 and (25) yield

$$\sum_{k=-\infty}^{\infty} \frac{1}{\left| (s + 2\pi k)^2 + \sigma^2 \right|^p} \leq \frac{1 + |\sigma|}{\sigma^{2p}} \leq 2^p \frac{1 + \sqrt{|\xi - 2\pi a^{-1}m|^2 - \eta^2}}{|\pi m a^{-1}|^{2p}}.$$

Since $1 \leq a^{-1}$ and $|\xi| \leq \pi a^{-1}$ for $\xi \in I_a$ we obtain

$$1 + \sqrt{|\xi - 2\pi a^{-1}m|^2 - \eta^2} \leq 1 + |\xi - 2\pi a^{-1}m| \leq a^{-1} (1 + \pi + 2\pi |m|).$$

Thus we proved $|S_p(s + i\eta, \xi + 2\pi m a^{-1})| \leq 2^p a^{2p-1} (1 + \pi + 2\pi |m|) / |\pi m|^{2p}$. Further we know by (22)

$$\frac{1}{|S_{\varphi,a}(s + i\eta, \xi)|} \leq 3 \left| (s + i\eta)^2 + |\xi|^2 \right|^p \leq 3 \left(2\pi^2 + |\xi|^2 \right)^p.$$

Thus we obtain the upper bound

$$E_1(I_a) \leq 3a^{2p-1} 2^p \int_{I_a} |\widehat{f}(\xi)| \left(2\pi^2 + |\xi|^2 \right)^p d\xi \sum_{m \in \mathbb{Z}^n, m \neq 0} \frac{1 + \pi + 2\pi |m|}{|\pi m|^{2p}}.$$

By enlarging the domain of integration to $\mathbb{R}^n$ we obtain the desired estimate.

2. Now let us discuss the integral

$$E_2(a) := \frac{1}{2\pi} \int_{\mathbb{R}^n \setminus I_a} |\widehat{f}(\xi)| \int_{-\pi}^{\pi} \sum_{m \in \mathbb{Z}^n, m \neq 0} \frac{|S_p(s + i\eta, \xi + 2\pi m a^{-1})|}{|S_{\varphi,a}(s + i\eta, \xi)|} ds d\xi.$$

From (21) and the definition of S_p and $S_{\varphi,a}$ follows immediately

$$\sum_{m \in \mathbb{Z}^n, m \neq 0} |S_p(s + i\eta, \xi + 2\pi m a^{-1})| \leq 3 |S_{\varphi,a}(s + i\eta, \xi)|$$

for all $s \in \mathbb{R}$, $\xi \in \mathbb{R}^n$ and $0 < a \leq 1$. Hence, $E_2(a) \leq 3 \int_{\mathbb{R}^n \setminus I_a} |\widehat{f}(\xi)| d\xi$. For $\xi \notin I_a$ we have the trivial estimate $1 + |\xi|^2 \geq a^{-2}$, hence

$$\int_{\mathbb{R}^n \setminus I_a} |\widehat{f}(\xi)| d\xi = \int_{\mathbb{R}^n \setminus I_a} \frac{|\widehat{f}(\xi)| (1 + |\xi|^2)^p}{(1 + |\xi|^2)^p} d\xi \leq a^{2p} \int_{\mathbb{R}^n \setminus I_a} |\widehat{f}(\xi)| (1 + |\xi|^2)^p d\xi.$$

The proof is complete. ■

4 Proof of the main Theorem 1

Proof of Theorem 1: Let $\eta > 0$ and $C > 0$ as in Theorem 10, and let $\sigma > 0$ as in the assumption of Theorem 1. By a classical result, see e.g. Schoenberg [19], or [18], or [8, p. 297, Lemma 15.3], there exists a constant $D(\eta, \sigma) > 0$ and a constant $R > 0$ such that for all $t \in \mathbb{R}$ with $|t| \geq R$ the following inequality holds:

$$\sum_{j=-\infty}^{\infty} |j|^\sigma e^{-\eta|t-j|} \leq D(\eta, \sigma) |t|^\sigma.$$

Since the function $F(t) := \sum_{j=-\infty}^{\infty} |j|^\sigma e^{-\eta|t-j|}$ is bounded on $[-R, R]$ we can find a constant $D_2 = D_2(\eta, \sigma)$ such that

$$\sum_{j=-\infty}^{\infty} |j|^\sigma e^{-\eta|t-j|} \leq D_2 (1 + |t|^\sigma) \quad (26)$$

for all $t \in \mathbb{R}$. The constant $c > 0$ in Theorem 1 will be defined as $c := C \cdot D_2$. Let now $d_j \in B_{2p}(\mathbb{R}^n) \cap C(\mathbb{R}^n)$, $j \in \mathbb{Z}$, satisfying $\|d_j\|_{2p} \leq D(1 + |j|^\sigma)$ for all $j \in \mathbb{Z}$. For each $j \in \mathbb{Z}$ let L_{p,d_j} be defined by (18) where we substitute f by d_j . We define S by the Lagrange scheme $S(t, y) = \sum_{j=-\infty}^{\infty} L_{p,d_j}(t - j, y)$ and recall from [2] (for the case $p = 2$) or from [3] for general $p \in \mathbb{N}$, that S is indeed a polyspline on strips of order p . Similarly, we can write $I_a(t, y) = \sum_{j=-\infty}^{\infty} L_{p,a,d_j}(t - j, y)$.

By Theorem 10 it follows that

$$\begin{aligned} |S(t, y) - I_a(t, y)| &\leq \sum_{j=-\infty}^{\infty} |L_{p, d_j}(t - j, y) - L_{p, a, d_j}(t - j, y)| \\ &\leq \sum_{j=-\infty}^{\infty} a^{2p-1} C e^{-\eta|t-j|} \cdot \int \left| \widehat{d_j}(\xi) \right| \left(1 + |\xi|^2\right)^p d\xi. \end{aligned}$$

Using (5) and (26) we arrive at

$$\begin{aligned} |S(t, y) - I_a(t, y)| &\leq a^{2p-1} C D \sum_{j=-\infty}^{\infty} e^{-\eta|t-j|} (1 + |j|^\sigma) \\ &\leq a^{2p-1} C D D_2 (1 + |t|^\sigma). \end{aligned}$$

This ends the proof. ■

5 Appendix: Proof of the Theorems 5, 6 and 7

The function $S_p(s, \xi)$ in (19) is defined for all pairs $(s, \xi) \in \mathbb{R}^{n+1} \setminus \Sigma$ where the set Σ is given by $\Sigma := \{(s, \xi) : s \in 2\pi\mathbb{Z}, |\xi| = 0\}$.

Proof of Theorem 6. Define a function q_ξ by $q_\xi(z) := z^2 + |\xi|^2$. For $z = s + i\eta$ we have

$$q_\xi(s + 2\pi k + i\eta) = (s + 2\pi k)^2 - \eta^2 + |\xi|^2 + 2i\eta(s + 2\pi k). \quad (27)$$

Thus $\operatorname{Re} q_\xi(z + 2\pi k) = (s + 2\pi k)^2 - \eta^2 + |\xi|^2$ which implies $\operatorname{Re} q_\xi(z + 2\pi k) \geq 4\pi^2 k^2 - 2\pi^2 |k| - \pi^2 \geq \pi^2 k^2$ for $|\eta| \leq \pi$ and $|s| \leq \pi$ and $k \in \mathbb{Z}, k \neq 0$. Hence, we have proved for $k \in \mathbb{Z}, k \neq 0$

$$|q_\xi(z + 2\pi k)| \geq \operatorname{Re} q_\xi(z + 2\pi k) \geq \pi^2 k^2. \quad (28)$$

It follows that $q_\xi(z + 2\pi k) \neq 0$ for $k \neq 0$, $|s| \leq \pi$ and $|\eta| \leq \pi$, hence $z \mapsto q_\xi^{-p}(z + 2\pi k)$ is well defined and analytic for $k \neq 0$, $|s| < \pi$ and $|\eta| < \pi$. For each $\xi \in \mathbb{R}^n$ define

$$S_p^\times(z, \xi) := \sum_{k \in \mathbb{Z} \setminus \{0\}} \frac{1}{q_\xi^p(z + 2\pi k)}. \quad (29)$$

For $|s| \leq \pi$ and $|\eta| \leq \pi$, we obtain by (28) the inequality

$$|S_p^\times(z, \xi)| \leq \sum_{k \in \mathbb{Z} \setminus \{0\}} \frac{1}{|q_\xi^p(z + 2\pi k)|} \leq \sum_{k \in \mathbb{Z} \setminus \{0\}} \frac{1}{\pi^{2p} k^{2p}} =: M < \infty. \quad (30)$$

We want to show (20). By multiplying (20) with $q_\xi(z)^p$ it suffices to show the following statement: there exists $\varepsilon > 0$ with $\varepsilon < \pi$ such that for $z = s + i\eta$ with $|s| \leq \pi$ and $|\eta| \leq \varepsilon$, and for all $\xi \in \mathbb{R}^n$ holds

$$3 \left| 1 + q_\xi^p(z) S_p^\times(z, \xi) \right| \geq 1 + \sum_{k \in \mathbb{Z} \setminus \{0\}} \frac{|q_\xi(z)|^p}{|q_\xi(z + 2\pi k)|^p}. \quad (31)$$

The constant $\varepsilon > 0$ for the estimate (31) will be chosen in the following way: first take δ with $1 > \delta > 0$ so small that $(2\delta^2 + \delta)^p \leq \min\{\frac{1}{2}, \frac{1}{2M}\}$, where we have defined M in (30). Then choose $\varepsilon > 0$ so small that

$$\varepsilon \leq \frac{\delta}{2\pi}, \text{ and } \arctan\left(\frac{2\varepsilon\pi}{\delta^2}\right) \leq \frac{\pi}{4p}. \quad (32)$$

Now let us prove the estimate. We consider two cases.

1. In the first case we assume $s^2 + |\xi|^2 \leq \delta^2$. Since $|\eta| \leq \varepsilon$, formula (27) for $k = 0$ implies $|q_\xi(z)| \leq s^2 + |\xi|^2 + \varepsilon^2 + 2\pi\varepsilon$. Using (32) we arrive at $|q_\xi(z)| \leq 2\delta^2 + \delta$, hence

$$|q_\xi(z)|^p \leq (2\delta^2 + \delta)^p \leq \frac{1}{2M}. \quad (33)$$

Then (30) and (33) yield

$$\left|1 + q_\xi^p(z) S_p^\times(z, \xi)\right| \geq 1 - \left|q_\xi^p(z) S_p^\times(z, \xi)\right| \geq 1 - \left|q_\xi^p(z)\right| M \geq \frac{1}{2}. \quad (34)$$

Hence, for $s^2 + |\xi|^2 \leq \delta^2$ holds

$$3 \left|1 + q_\xi^p(z) S_p^\times(z, \xi)\right| \geq \frac{3}{2}. \quad (35)$$

On the other hand, (33) and the definition of M in (30) give

$$1 + \sum_{k \in \mathbb{Z} \setminus \{0\}} \frac{|q_\xi(z)|^p}{|q_\xi(z + 2\pi k)|^p} \leq 1 + \frac{1}{2M} \sum_{k \in \mathbb{Z} \setminus \{0\}} \frac{1}{|q_\xi(z + 2\pi k)|^p} \leq \frac{3}{2}. \quad (36)$$

Clearly (35) and (36) imply (31) for $s^2 + |\xi|^2 \leq \delta^2$.

2. Assume now as the second (and more complicated) case that $s^2 + |\xi|^2 \geq \delta^2$. The inequality $|w| \geq \operatorname{Re} w$ yields

$$\left|1 + q_\xi^p(z) S_p^\times(z, \xi)\right| \geq 1 + \sum_{k \in \mathbb{Z} \setminus \{0\}} \operatorname{Re} \frac{q_\xi^p(z)}{q_\xi^p(z + 2\pi k)}. \quad (37)$$

Roughly speaking, we shall show that each factor in the summands on the right-hand side of (37) has an argument which is small (using that η is small) so each summand will have an argument which will be less than $\pi/4$. More precisely, we want to show that for $z = s + i\eta$ with $|s| \leq \pi$, $|\eta| \leq \varepsilon$, and $k \in \mathbb{Z}$, the argument $\varphi(z, \xi, k)$ in

$$q_\xi(z + 2\pi k) = |q_\xi(z + 2\pi k)| \cdot e^{i\varphi(z, \xi, k)} \quad (38)$$

can be estimated by $|\varphi(z, \xi, k)| \leq \pi/4p$. Hence, $q_\xi^p(z + 2\pi k)$ has an argument with an absolute value lower or equal to $\pi/4$. Thus

$$\operatorname{Re} \frac{q_\xi^p(z)}{q_\xi^p(z + 2\pi k)} = \frac{|q_\xi^p(z)|}{|q_\xi^p(z + 2\pi k)|} \cos \sigma,$$

where $|\sigma| \leq \pi/4$. Since $\cos \sigma \geq 1/\sqrt{2}$ for $|\sigma| \leq \pi/4$, we obtain

$$\operatorname{Re} \frac{q_\xi^p(z)}{q_\xi^p(z+2\pi k)} \geq \frac{1}{\sqrt{2}} \frac{|q_\xi^p(z)|}{|q_\xi^p(z+2\pi k)|}. \quad (39)$$

Now (37) and (39) lead to inequality (31).

We still have to prove (38). For $k=0$ we shall need our assumption $s^2+|\xi|^2 \geq \delta^2$: the estimate $|q_\xi(s+i\eta)| \geq \operatorname{Re} q_\xi(s+i\eta) \geq s^2+|\xi|^2-\varepsilon^2 \geq \frac{1}{2}\delta^2$ holds since by (32) follows $\varepsilon \leq \frac{1}{2\pi}\delta < \frac{1}{2}\delta$. Further $|\operatorname{Im} q_\xi(s+i\eta)| = |\eta s| \leq \varepsilon\pi$ implies

$$\frac{|\operatorname{Im} q_\xi(s+i\eta)|}{\operatorname{Re} q_\xi(s+i\eta)} \leq \frac{\varepsilon 2\pi}{\delta^2}.$$

Our choice of $\varepsilon > 0$ in (32) implies that $|\arg q_\xi(s+i\eta)| \leq \arctan \frac{\varepsilon 2\pi}{\delta^2} \leq \pi/4p$, where $\arg z$ denotes the argument of the complex number z .

For $k \neq 0$ the inequality (28) yields $\operatorname{Re} q_\xi(s+2\pi k+i\eta) \geq \pi^2 k^2$, and clearly $\operatorname{Im} q_\xi(s+2\pi k+i\eta) = \eta(s+2\pi k)$. Hence, for $k \in \mathbb{Z} \setminus \{0\}$

$$\frac{|\operatorname{Im} q_\xi(s+2\pi k+i\eta)|}{\operatorname{Re} q_\xi(s+2\pi k+i\eta)} \leq \frac{\varepsilon(\pi+2\pi|k|)}{\pi^2 k^2} = \varepsilon \frac{|k|+1}{k^2} \leq 2\varepsilon.$$

Since $2\varepsilon \leq \frac{\varepsilon 2\pi}{\delta^2}$ we have $\arctan 2\varepsilon \leq \pi/4p$. Hence $|\arg q_\xi(s+2\pi k+i\eta)| \leq \pi/4p$ for all $k \in \mathbb{Z}, k \neq 0$. This ends the proof. ■

Proof of Theorem 5: It is easy to see that $q_\xi(z+2\pi k) \neq 0$ for all $z = s+i\eta$ with $|s| \leq \frac{3}{2}\pi$ and $|\eta| \leq \pi$. Thus $z \mapsto S_p^\times(z, \xi)$ defined in (29) is an analytic function for $z = s+i\eta$ with $|s| < \frac{3}{2}\pi$ and $|\eta| < \pi$. It follows from equation (21) that $1 + q_\xi^p(z) S_{p,0}^\times(z, \xi) \neq 0$ for all $z = s+i\eta$ with $|s| \leq \pi$ and $|\eta| \leq \varepsilon$. Hence by continuity there exists δ_ξ such that $1 + q_\xi^p(z) S_{p,0}^\times(z, \xi) \neq 0$ for all $z = s+i\eta$ with $|s| < \pi + \delta_\xi$ and $|\eta| \leq \varepsilon$. Hence

$$\frac{1}{S_p(z, \xi)} = \frac{q_\xi^p(z)}{[1 + q_\xi^p(z) S_p^\times(z, \xi)]}$$

is an analytic extension of the function $1/S_p(s, \xi)$ for $z = s+i\eta$ with $|s| < \pi + \delta_\xi$ and $|\eta| < \varepsilon$. Since $S_p(z, \xi) = S_p(z+2\pi k, \xi)$ for all $k \in \mathbb{Z}$ it follows that $1/S_p(z, \xi)$ is analytic for $|\operatorname{Im} z| < \varepsilon$. The proof is complete. ■

Proof of Theorem 7: Let us write $\xi = (\xi_1, \dots, \xi_n)$ and $m = (m_1, \dots, m_n)$. Since $\xi \mapsto S_{\varphi,a}(s+i\eta, \xi)$ is $2\pi a^{-1}\mathbb{Z}^n$ -periodic we may and do assume that $|\xi_j| \leq \pi a^{-1}$ for $j = 1, \dots, n$. It is easy to see that $|\xi_j - 2\pi a^{-1}m_j| \geq \pi a^{-1}|m_j|$ for $j = 1, \dots, n$ and all $m \in \mathbb{Z}^n$. It follows that

$$|\xi - 2\pi a^{-1}m|^2 \geq \pi^2 a^{-2} |m|^2 \geq \pi^2 |m|^2. \quad (40)$$

Now we claim that

$$|(s+i\eta+2\pi k)^2 + |\xi - 2\pi a^{-1}m|^2| \geq \pi^2 k^2 + \frac{\pi^2}{2} |m|^2. \quad (41)$$

Indeed, $\left| (s + i\eta + 2\pi k)^2 + |\xi - 2\pi a^{-1}m|^2 \right| \geq \operatorname{Re} (s + i\eta + 2\pi k)^2 + |\xi - 2\pi a^{-1}m|^2$ and in case of $k \neq 0$ we have $\operatorname{Re} (s + i\eta + 2\pi k)^2 \geq \pi^2 k^2$ by (28). Hence (41) is proven for $k \neq 0$. If $k = 0$ we use that $|z| \geq \operatorname{Re} z$ and (40) in order to obtain for $|\eta| \leq \pi/\sqrt{2}$

$$\left| (s + i\eta)^2 + |\xi - 2\pi a^{-1}m|^2 \right| \geq \pi^2 |m|^2 + s^2 - \eta^2 \geq \pi^2 |m|^2 - \frac{1}{2}\pi^2.$$

Hence (41) is proven for the case $k = 0$ as well. We conclude that

$$S_{\varphi,a}^\times(s + i\eta, \xi) := \sum_{\substack{(k,m) \in \mathbb{Z}^{n+1} \\ (k,m) \neq 0}} \frac{1}{\left| (s + i\eta + 2\pi k)^2 + |\xi - 2\pi a^{-1}m|^2 \right|^p}$$

is bounded for all $|s| \leq \pi$, $|\eta| \leq \pi$, $0 < a \leq 1$, and $\xi \in \mathbb{R}^n$ since

$$\left| S_{\varphi,a}^\times(s + i\eta, \xi) \right| \leq \sum_{(k,m) \in \mathbb{Z}^{n+1}, (k,m) \neq 0} \frac{1}{\left(\pi^2 k^2 + \frac{\pi^2}{2} |m|^2 \right)^p} =: M. \quad (42)$$

Now it suffices to prove:

$$3 \left| 1 + q_\xi^p(z) S_{\varphi,a}^\times(z, \xi) \right| \geq 1 + \sum_{\substack{(k,m) \in \mathbb{Z}^{n+1} \\ (k,m) \neq 0}} \frac{\left| q_\xi^p(z) \right|^p}{\left| (z + 2\pi k)^2 + |\xi - 2\pi a^{-1}m|^2 \right|^p}. \quad (43)$$

Choose $1 > \delta > 0$ and $\varepsilon > 0$ as in the proof of Theorem 6 but with M now defined in (42), and conclude in the same fashion that (43) holds for the first case $s^2 + |\xi|^2 \leq \delta^2$.

In the second case $s^2 + |\xi|^2 \geq \delta^2$ we notice that this implies that $s^2 + |\xi - 2\pi a^{-1}m|^2 \geq \delta^2$ for all $m \in \mathbb{Z}^n$ since trivially $|\xi - 2\pi a^{-1}m| \geq \pi|m| \geq \delta$ for all $m \in \mathbb{Z}^n, m \neq 0$. Note that

$$(z + 2\pi k)^2 + |\xi - 2\pi a^{-1}m|^2 = q_{\xi - 2\pi a^{-1}m}(z + 2\pi k).$$

In the proof of Theorem 6 we have seen that

$$q_{\xi - 2\pi a^{-1}m}(z + 2\pi k) = \left| q_{\xi - 2\pi a^{-1}m}(z + 2\pi k) \right| e^{i\varphi(z, \xi - 2\pi a^{-1}m, k)}$$

where $|\varphi(z, \xi - 2\pi a^{-1}m, k)| \leq \pi/4p$. Now we argument in a way analogous to the proof of Theorem 6:

$$\begin{aligned} \left| 1 + q_\xi^p(z) S_{\varphi,a}^\times(z, \xi) \right| &\geq 1 + \sum_{\substack{(k,m) \in \mathbb{Z}^{n+1} \\ (k,m) \neq 0}} \operatorname{Re} \frac{\left| q_\xi^p(z) \right|^p}{\left| (z + 2\pi k)^2 + |\xi - 2\pi a^{-1}m|^2 \right|^p} \\ &\geq 1 + \sum_{\substack{(k,m) \in \mathbb{Z}^{n+1} \\ (k,m) \neq 0}} \frac{1}{\sqrt{2}} \frac{\left| q_\xi^p(z) \right|^p}{\left| (z + 2\pi k)^2 + |\xi - 2\pi a^{-1}m|^2 \right|^p} \end{aligned}$$

The proof is complete. ■

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