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Polyharmonicity and algebraic support of measures

Ognyan KOUNCHEV and Hermann RENDER
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Abstract. Our main result states that two signed measures $\mu$ and $\nu$ with bounded support contained in the zero set of a polynomial $P(x)$ are equal if they coincide on the subspace of all polynomials of polyharmonic degree $N_P$ where the natural number $N_P$ is explicitly computed by the properties of the polynomial $P(x)$. The method of proof depends on a definition of a multivariate Markov transform which is another major objective of the present paper. The classical notion of orthogonal polynomial of second kind is generalized to the multivariate setting: it is a polyharmonic function which has similar features to those in the one-dimensional case.

1. Introduction

Recall that a complex-valued function $f$ defined on a domain $G$ in the euclidean space $\mathbb{R}^n$ is polyharmonic of order $N$ if $f$ is $2N$-times continuously differentiable and

$$A^N f(x) = 0 \quad \text{for all } x \in G$$

where $A^N$ is the $N$-th iterate of the Laplace operator $A = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}$. For $N = 1$ this class of functions are just the harmonic functions, while for $N = 2$ the term biharmonic function is used which is important in elasticity theory. Fundamental work about polyharmonic functions is due to E. Almansi [2], M. Nicolesco (see e.g. [25]) and N. Aronszajn [3], and still this is an area of active research; see e.g. [7], [8], [9], [12], [17], [18], [23], [27], [28]. Polyharmonic functions are also important in applied mathematics, e.g. in approximation theory, radial basis functions and wavelet analysis; see e.g. [5], [19], [20], [21], [24].

In this paper we address the following question: Let $\mu$ and $\nu$ be signed measures with compact support. Suppose that there exists a polynomial $P(x)$ such that the supports of $\mu$ and $\nu$ are contained in the zero set of $P$. Under which conditions do $\mu$ and $\nu$ coincide? As motivating example consider the polynomial $P(x) = |x|^2 - 1$ where $|x| := r(x) := \sqrt{x_1^2 + \cdots + x_n^2}$ is the euclidean

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norm in $\mathbb{R}^n$. It is well known that two measures $\mu$ and $v$ with support in the unit sphere $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$ coincide if they are equal on the set of all harmonic polynomials. We shall show that two measures $\mu$ and $v$ with support in the set $K_P(R)$ (defined below in (2)), are equal if the moments $\mu(f)$ and $v(f)$ are equal for polyharmonic polynomials $f$ of a certain degree $N_P$ which depends on the polynomial $P$. In order to formulate this precisely, let us introduce the polyharmonic degree $d(f)$ defined by

$$d(f) := \min\{N \in \mathbb{N}_0 : A^{N+1}(f) = 0\}.$$  

In the appendix we shall compare properties of the polyharmonic degree and the total degree. Note that $f$ has polyharmonic degree $\leq N$ if and only if $f$ is of polyharmonic order $N + 1$.

Let us denote by $\mathcal{P}$ the set of all polynomials. One of the main results of this paper reads as follows:

**THEOREM 1.** Let

$$K_P(R) := \{x \in \mathbb{R}^n : P(x) = 0 \text{ and } |x| \leq R\}$$

for $R > 0$ and for a polynomial $P(x)$, and define

$$N_P = \sup\{d(P \cdot h) : h \text{ is a harmonic polynomial}\}.$$ 

Let $\mu$ and $\nu$ be signed measures with support contained in the set $K_P(R)$ for some $R > 0$. If $\int h \, d\mu = \int h \, dv$ for all polynomials $h$ in the subspace

$$U_{N_P} = \{Q \in \mathcal{P} : A^{N_P}Q = 0\}$$

then $\mu$ and $\nu$ are identical.

It is not difficult to see that $N_P$ is lower or equal to the total degree of the polynomial $P(x)$, see Corollary 20. In the appendix we shall give a procedure to determine the number $N_P$ explicitly.

An application of the Hahn-Banach theorem shows us the following consequence of Theorem 1: the space $U_{N_P}$ is dense in the space $C(K_P(R), \mathbb{C})$ of all continuous complex-valued functions on the compact space $K_P(R)$ endowed with the supremum norm, see Corollary 18. Let us emphasize that Theorem 1 is only a sufficient criterion, and does not always give the expected result: As illustrating examples consider the case of a sphere and an ellipsoid. In the first case, the defining polynomial $P(x) = |x|^2 - 1$ has the property that $N_P = 1$, so $U_{N_P}$ is equal to the space of all harmonic polynomials. In the case of an ellipsoid, $N_P$ is equal to 2, although it would be sufficient to know that the measures $\mu$ and $\nu$ are identical for harmonic polynomials. However, density results for solutions to $A^p h = 0$ in $C(K)$ for compact sets $K$ for $p > 1$ are much
more complicated and obtained with the techniques of Potential theory in the 1970s; see [13], [14] and the references therein. The following example shows that our approach delivers a nontrivial criterion for density which is not covered by the other approaches so far: take
\[ P(x) = \langle a, x \rangle (|x|^2 - 1) \]
where \( \langle a, x \rangle = a_1 x_1 + \cdots + a_n x_n \). Then \( N_F = 2 \), and we need now the space of all biharmonic polynomials to ensure that two measures \( \sigma \) and \( \nu \) are equal. Indeed, harmonic polynomials are not sufficient: take \( \sigma \) as the usual measure \( d\theta \) on the unit sphere \( S^{n-1} \) and \( \nu \) as the point evaluation in \( x = 0 \). Then \( \sigma \) and \( \nu \) coincide on the space of all harmonic polynomials and both measures have support in \( P(0) \). Clearly \( \sigma \) and \( \nu \) are different measures.

The proof of Theorem 1 will be a by-product of our investigation of the so-called multivariate Markov transform which we will introduce below and which we consider as a suitable generalization of the univariate Markov transform, an important tool in the classical moment problem and its applications to Spectral theory. Recall that the Markov transform\(^1\) of a finite measure \( \sigma \) with support in the interval \([-R, R]\) is defined on the upper half–plane by the formula
\[
\sigma(\zeta) := \int_{-R}^{R} \frac{1}{\zeta - x} \, d\sigma(x) \quad \text{for Im} \, \zeta > 0,
\]
see e.g. [1, Chapter 2], [26, Chapter 2.6]. Let us recall a central result called Markov’s theorem: the \( N \)-th Padé approximant \( \pi_N(\zeta) = Q_N(\zeta)/P_N(\zeta) \) of the asymptotic expansion of \( \sigma(\zeta) \) at infinity converges compactly in the upper half plane to \( \sigma(\zeta) \); here the polynomial \( P_N \) is the \( N \)-th orthogonal polynomial with respect to the measure \( \sigma \) and \( Q_N \) is the orthogonal polynomial of the second kind with respect to the measure \( \sigma \) given through the formula
\[
Q_N(\zeta) = \int \frac{P_N(\zeta) - P_N(x)}{\zeta - x} \, d\sigma(x).
\]
Further, to each \( \pi_N(\zeta) \) there corresponds a (non-negative) measure \( \sigma_N \) with support in the zeros of the nominator \( P_N \), thus leading to a proof of the famous Gauß quadrature formula.

Our definition of a multivariate Markov transform depends on the work of N. Aronszajn [3] on polyharmonic functions, and of L. K. Hua [15] about harmonic analysis on Lie groups; the definition is related to the Poisson formula for the ball \( B_R := \{ x \in \mathbb{R}^n : |x| < R \} \) which we recall now: Let \( R > 0 \)

\(^1\)In some recent works in Approximation theory, Potential theory, and Probability theory this function is called the Markov function of a measure, see e.g. [29] or [11]. On the other hand apparently Widder [32] was the first who has given the name Stieltjes transform to this function. If \( \mu \) has infinite support the transform is also called Stieltjes transform. This tradition has been followed by Akhiezer [1] and other Russian mathematicians.
and \( h \) be a function harmonic in the ball \( B_R \) and continuous on the closure \( \overline{B_R} \); then for any \( x \in \mathbb{R}^n \) with \( |x| < R \)

\[
h(x) = \frac{1}{\omega_n \int_{S^{n-1}}} \frac{(R^2 - |x|^2)^2}{r(R\theta - x)^n} h(R\theta) d\theta, \tag{6}
\]

where \( \omega_n \) denotes the area of \( S^{n-1} \), \( \theta \in S^{n-1} \), and \( r(x) \) is the euclidean norm of \( x \). Note that for fixed \( x \) with \( |x| < R \) the function \( \rho \mapsto r(\rho \theta - x) \) defined for \( \rho \in \mathbb{R} \) with \( |\rho| > R \) has an analytic continuation for \( \zeta \in \mathbb{C} \) with \( |\zeta| > R \), so we can write \( r(\zeta \theta - x) \) for \( \zeta \in \mathbb{C} \) with \( |\zeta| > R \). The following Cauchy type integral formula, proved in [3, p. 125], is important for our approach: for any polynomial \( u(x) \) and for any \( |x| < R \) the following identity holds

\[
u(x) = \frac{1}{2\pi i \omega_n} \int_{\Gamma_R} \int_{S^{n-1}} \frac{r^{n-1}}{r(\zeta \theta - x)} d\mu(\theta) d\zeta \tag{7}
\]

where the contour \( \Gamma_R(t) = R \cdot e^{it} \) for \( t \in [0, 2\pi] \). A similar result is also valid for holomorphic functions \( u \) defined on the so-called harmonicity hull of \( B_R \); we refer the reader to [3, p. 125] for details.

Assume now that \( \mu \) is a signed measure with support in the closed ball \( \{x \in \mathbb{R}^n : |x| \leq R \} \). The multivariate Markov transform \( \hat{\mu} \) of \( \mu \) is a function defined for all \( \theta \in S^{n-1} \) and all \( \zeta \in \mathbb{C} \) with \( |\zeta| > R \) by the formula

\[
\hat{\mu}(\zeta, \theta) = \frac{1}{\omega_n} \int_{\mathbb{R}^n} \frac{r^{n-1}}{r(\zeta \theta - x)} d\mu(x) \tag{8}
\]

Since \( \zeta \mapsto r(\zeta \theta - x) \) has no zeros for \( |\zeta| > R \) the function \( \zeta \mapsto \hat{\mu}(\zeta, \theta) \) is defined for all \( |\zeta| > R \). In the following Section we shall show that the multivariate Markov transform \( \hat{\mu} \) determines the measure \( \mu \) uniquely, cf. Theorem 3.

Our second main innovation is the introduction of the notion of the function \( Q_P(\zeta, \theta) \) of the second kind with respect to a given polynomial \( P(x) \) which is the multivariate analogue of (5), defined by

\[
Q_P(\zeta, \theta) = \int_{\mathbb{R}^n} \frac{P(\zeta \theta) - P(x)}{r(\zeta \theta - x)^n} r^{n-1} d\mu(x) \tag{9}
\]

for all \( |\zeta| > R, \theta \in S^{n-1} \). Let us emphasize that \( Q_P \) is in general not a polynomial. However, we shall show the surprising and interesting result that the function \( \rho \theta \mapsto r^{-(n-1)} Q_P(\rho \theta) \) is a polyharmonic function of order \( \leq \deg P(x) \) where \( \deg \) denotes the total degree of a polynomial.

One further main result of the paper, Theorem 13, is concerned with measures \( \mu \) having their supports in algebraic sets: Let us assume that the measure \( \mu \) has support in \( K_P(R) \). Then the Markov transform \( \hat{\mu} \) has the representation

...
\[ \hat{\mu}(\zeta, \theta) = \frac{Q_P(\zeta, \theta)}{P(\zeta \theta)} \quad \text{for } |\zeta| > R, \]  

where \( Q_P \) is the function of second kind with respect to \( P(x) \). The reverse statement holds as well, i.e. if the measure \( \mu \) with \( \text{supp}(\mu) \subset B_R \) satisfies (10) for some polynomial \( P \) where \( Q_P \) is defined by (9), then \( \text{supp}(\mu) \subset K_P(R) \). By means of these characterizations we can deduce our main result Theorem 1.

Finally let us recall some terminology from measure theory: a signed measure on \( \mathbb{R}^d \) is a set function on the Borel \( \sigma \)-algebra on \( \mathbb{R}^d \) which takes real values and is \( \sigma \)-additive. By the Jordan decomposition \[6, p. 125]\, a signed measure \( m \) is the difference of two non-negative finite measures, say \( m = m_+ - m_- \). The support of a non-negative measure \( m \) on \( \mathbb{R}^d \) is defined as the complement of the largest open set \( U \) such that \( m(U) = 0 \). The support of a signed measure \( \sigma \) is defined as the support of the total variation \( |\sigma| = \sigma_+ + \sigma_- \) (see \[6, p. 226\]). Recall that in general, the supports of \( \sigma_+ \) and \( \sigma_- \) are not disjoint (cf. exercise 2 in \[6, p. 231\]). Note that if a signed measure \( \mu \) has compact support then all polynomials are integrable with respect to \( \mu^+, \mu^- \), and \( |\mu| \).

2. The multivariate Markov transform

Recall that the univariate Markov transform has, for \( |\zeta| > R \), the asymptotic expansion

\[ \hat{\sigma}(\zeta) = \sum_{k=0}^{\infty} \frac{1}{\zeta^{k+1}} \int t^k \, d\sigma(t). \]  

Let \( \Gamma_R \) denote the contour in \( \mathbb{C} \) defined by \( \Gamma_R(t) = R \cdot e^{it} \) for \( t \in [0, 2\pi] \). By means of standard facts from complex analysis the following identity may be proved:

\[ \frac{1}{2\pi i} \int_{\Gamma_R} p(\zeta) \hat{\sigma}(\zeta) d\zeta = \int_{\mathbb{R}^d} p(x) d\sigma(x) \]  

for all polynomials \( p \) and any \( R_1 > R \).

In this Section we want to show that similar results hold for the multivariate Markov transform \( \hat{\mu} \); in particular the following is the analogue of formula (12) in the multivariate case:

**Proposition 2.** Let \( \mu \) be a signed measure over \( \mathbb{R}^d \) with support in \( B_R \) and let \( R_1 > R \). Then for every polynomial \( P(x) \)

\[ M_\mu(P) := \frac{1}{2\pi i} \int_{\Gamma_{R_1}} \int_{\mathbb{R}^d} P(\zeta \theta) \hat{\mu}(\zeta, \theta) d\zeta d\theta = \int_{\mathbb{R}^d} P(x) d\mu(x). \]
Proof. Replace $\hat{\mu}(\zeta, \theta)$ in (13) by (8) and interchange integration. Then

$$M_\mu(P) = \int_{\mathbb{R}^n} \frac{1}{2\pi i \omega_0} \int_{\Gamma_{R_1}} \int_{S^{n-1}} P(\zeta \theta) \frac{\zeta^{n-1}}{r(\zeta \theta - x)^n} d\zeta d\theta d\mu(x).$$

(14)

According to (7) we obtain $M_\mu(P) = \int P(x) d\mu(x)$. ■

Theorem 3. Let $\mu, \nu$ be finite signed measures over $\mathbb{R}^n$ with compact support. If the multivariate Markov transforms of $\mu$ and $\nu$ coincide for large $|\zeta|$, i.e., if there exists $R > 0$ such that $\hat{\mu}(\zeta, \theta) = \hat{\nu}(\zeta, \theta)$ for all $|\zeta| > R$ and for all $\theta \in S^{n-1}$, then $\mu$ and $\nu$ are identical.

Proof. Since the multivariate Markov transforms coincide for large $|\zeta|$ it is clear that the functionals $M_\mu$ and $M_\nu$ in (13) are identical by taking the radius $R_1$ of the path $\Gamma_{R_1}$ large enough. Then Proposition 2 shows that $\int P(x) d\mu(x) = \int P(x) d\nu(x)$ for all polynomials $P(x)$. Further we apply a standard argument: since $\mu$ and $\nu$ have compact supports we may apply the Stone–Weierstrass theorem according to which the polynomials are dense in the space $C(\text{supp}(\mu) \cup \text{supp}(\nu))$ which implies that $\mu = \nu$. ■

Next we want to determine the asymptotic expansion of the multivariate Markov transform and we need some notations from harmonic analysis; for a detailed account we refer to [4] or [30]. Recall that a function $Y : S^{n-1} \to \mathbb{C}$ is called a spherical harmonic of degree $k \in \mathbb{N}_0$ if there exists a homogeneous harmonic polynomial $P(x)$ of degree $k$ (in general, with complex coefficients\(^2\)) such that $P(\theta) = Y(\theta)$ for all $\theta \in S^{n-1}$. Throughout the paper we assume that $Y_{k,m}(x), \ m = 1, \ldots, a_k$, is a basis of the set of all harmonic homogeneous polynomials of degree $k$ which are orthonormal with respect to scalar product

$$\langle f, g \rangle_{S^{n-1}} := \int_{S^{n-1}} f(\theta) \overline{g(\theta)} d\theta.$$  

For a continuous function $f : S^{n-1} \to \mathbb{C}$ we define the Laplace-Fourier series by

$$f(\theta) = \sum_{k=0}^{\infty} \sum_{m=1}^{a_k} f_{k,m} Y_{k,m}(\theta)$$

and $f_{k,m} = \int_{S^{n-1}} f(\theta) Y_{k,m}(\theta) d\theta$ are the Laplace-Fourier coefficients of $f$.

Using the Gauss decomposition of a polynomial (see Theorem 5.5 in [4]) it is easy to see that the system

$$|x|^{2t} Y_{k,m}(x), \quad t, k \in \mathbb{N}_0, \ m = 1, \ldots, a_k$$

is a basis of the set of all polynomials. The numbers

\(^2\)One may restrict the attention to real valued spherical harmonics and this does not change the results essentially.
are sometimes called the \textit{distributed moments}, see [16]. For a treatment and formulation of the \textit{multivariate moment problem} we refer to [10], see also [31].

**Theorem 4.** Let $\mu$ be a signed measure over $\mathbb{R}^n$ with support in the closed ball $B_R$. Then for all $|\zeta| > R$ and for all $\theta \in S^{n-1}$ the following relation holds

$$
\check{\mu}(\zeta, \theta) = \sum_{t=0}^{\infty} \sum_{k=0}^{\infty} \sum_{m=1}^{a_k} \frac{a_k}{\zeta^{2t+k+1}} \int_{\mathbb{R}^n} |x|^{2t} \check{Y}_{k,m}(x) d\mu(x).
$$

**Proof.** A zonal harmonic of degree $k$ with pole $\theta \in S^{n-1}$ is the unique spherical harmonic $Z_\theta^{(k)}$ of degree $k$ such that for all spherical harmonics $Y$ of degree $k$ the relation $Y(\theta) = \int_{\mathbb{S}^{n-1}} Z_\theta^{(k)}(\eta) \check{Y}(\eta) d\eta$ holds. Let $p_n(\theta, x) = \frac{1}{\omega_n} \frac{1-|x|^2}{|x-\theta|^n}$ be the Poisson kernel for $0 \leq |x| < 1 = |\theta|$. Theorem 2.10 in [30, p. 145] gives

$$
p_n(\theta, x) = \sum_{k=0}^{\infty} \sum_{m=1}^{a_k} |x|^k \check{Y}_{k,m}(x') \theta^k \check{Y}_{k,m}(\theta).
$$

for $|x| < 1$. Let $R$ be as in the theorem, and replace now $x$ in (17) by $x/\rho$, $\rho \in \mathbb{R}$ such that $|x| < R < \rho$; one obtains that

$$
\frac{1}{\omega_n} \frac{\rho^{n-2}(\rho^2 - |x|^2)}{r(\rho - x)^n} = \sum_{k=0}^{\infty} \sum_{m=1}^{a_k} \frac{1}{\rho^k} \check{Y}_{k,m}(x) \check{Y}_{k,m}(\theta).
$$

The real variable $\rho$ can now be replaced by a complex variable $\zeta$ with $|\zeta| > R$. We multiply by $\zeta(\zeta^2 - |x|^2)^{-1}$, and integrate over the closed ball $B_R$ with respect to $\mu$. This gives

$$
\check{\mu}(\zeta, \theta) = \sum_{k=0}^{\infty} \sum_{m=1}^{a_k} \check{Y}_{k,m}(\theta) \zeta^{-k+1} \int_{\mathbb{R}^n} \check{Y}_{k,m}(x) \left( \frac{\zeta^2}{|x|^2} \right) d\mu(x),
$$

and we have determined the Laplace-Fourier series of $\theta \mapsto \check{\mu}(\zeta, \theta)$. Since $|\zeta| > R \geq |x|$ we can expand $1/(1 - x^2/\zeta^2)$ in a geometric series and we obtain

$$
\check{\mu}(\zeta, \theta) = \sum_{k=0}^{\infty} \sum_{m=1}^{a_k} \frac{\zeta^{k+1}}{\zeta^{2}} \check{Y}_{k,m}(\theta) \left( \frac{\sum_{t=0}^{\infty} |x|^{2t}}{\zeta^{2t}} \right) d\mu(x).
$$

After interchanging summation and integration the claim is obvious. ■
3. The function of the second kind

In the following we want to give a multivariate analogue of the polynomial of second kind. It turns out that in the multivariate case the corresponding definition does not lead to a polynomial but to a polyharmonic function $Q_P(\zeta, \theta)$ which is defined only for all $|\zeta| > R, \theta \in S^{n-1}$.

**Definition 5.** Let $P(x)$ be a polynomial and $\mu$ be a non-negative measure with support in $B_R$. Then the function $Q_P(\zeta, \theta)$ of the second kind is defined by

$$Q_P(\zeta, \theta) = \frac{1}{\omega_n} \int_{\mathbb{R}^n} \frac{P(\zeta \theta) - P(x)}{r(\zeta \theta - x)^n} \zeta^{n-1} d\mu(x)$$

for all $|\zeta| > R, \theta \in S^{n-1}$. Similarly we define the function $R_P(\zeta, \theta)$ by

$$R_P(\zeta, \theta) = \frac{1}{\omega_n} \int_{\mathbb{R}^n} \frac{P(x)}{r(\zeta \theta - x)^n} \zeta^{n-1} d\mu(x)$$

for all $|\zeta| > R, \theta \in S^{n-1}$.

The last definitions immediately give the identity

$$P(\zeta \theta) \hat{\mu}(\zeta, \theta) = Q_P(\zeta, \theta) + R_P(\zeta, \theta).$$

**Theorem 6.** Let $P(x)$ be a polynomial, $\mu$ be a signed measure with support in $B_R$ and $Q_P(\zeta, \theta)$ the function of the second kind. Then for any $R_1 > R$ and for each polynomial $h(x)$

$$\frac{1}{2\pi i} \int_{\gamma_{R_1}} \int_{S^{n-1}} h(\zeta \theta) Q_P(\zeta, \theta) d\zeta d\theta = 0.$$  \hspace{1cm} (24)

**Proof.** Let us denote the integral in (24) by $I(h)$. By (23) we obtain that $I(h) = I_1(h) - I_2(h)$ where

$$I_1(h) = \frac{1}{2\pi i} \int_{\gamma_{R_1}} \int_{S^{n-1}} h(\zeta \theta) P(\zeta \theta) \hat{\mu}(\zeta, \theta) d\zeta d\theta,$$ \hspace{1cm} (25)

$$I_2(h) = \frac{1}{2\pi i \omega_n} \int_{\gamma_{R_1}} \int_{S^{n-1}} h(\zeta \theta) \int_{\mathbb{R}^n} \frac{P(x)}{r(\zeta \theta - x)^n} \zeta^{n-1} d\mu(x) d\zeta d\theta.$$ \hspace{1cm} (26)

Proposition 2 yields $I_1(h) = \int_{\mathbb{R}^n} h(x) P(x) d\mu(x)$. Change the integration order in (26) and use formula (7). Then we obtain $I_2(h) = I_1(h)$, therefore $I(h) = 0$ which was our claim.  

A similar argument to that in the proof of formula (16) proves the following:
Theorem 7. The function \( R_P(\zeta, \theta) \) has the asymptotic expansion

\[
\sum_{i=0}^{\infty} \sum_{k=0}^{\infty} \sum_{m=1}^{\infty} \frac{Y_{k,m}(\theta)}{\zeta^{2i+k+1}} \int_{\mathbb{R}^n} P(x)|x|^{2i} Y_{k,m}(x) d\mu(x).
\]

(27)

Note that the map \( \zeta \mapsto R_P(\zeta, \theta) \) for \( |\zeta| > R \) and \( \theta \in S^{n-1} \) is holomorphic in the complex variable \( \zeta \). So we can consider the Laurent series of the function \( \zeta \mapsto R_P(\zeta, \theta) \) and we write for \( |\zeta| > R \) and fixed \( \theta \in S^{n-1} \)

\[
R_P(\zeta, \theta) = \sum_{s=0}^{\infty} r_s[\theta](\theta) \frac{1}{\zeta^{s+1}}.
\]

(28)

From (27), by putting \( s = 2t + k \), it follows that

\[
r_s[\theta](\theta) = \sum_{i=0}^{[s/2]} \sum_{m=1}^{a_{s-2t}} Y_{s-2t,m}(\theta) \int_{\mathbb{R}^n} P(x)|x|^{2i} Y_{s-2t,m}(x) d\mu(x).
\]

(29)

Hence the coefficient function \( r_s[\theta] \) is a sum of spherical harmonics with degree \( \leq s \).

We can now formulate a characterization of orthogonality in asymptotic analysis:

**Theorem 8.** Let \( \mu \) be a signed measure with compact support and \( P(x) \) be a polynomial. Then \( P \) is orthogonal to all polynomials of degree \( \leq M \) with respect to \( \mu \) if and only if

\[
r_0[P] = \cdots = r_{M-1}[P] = 0
\]

where \( r_s[P] \) are the functions defined in (28)–(29).

**Proof.** From (29) we see that \( r_0[P] = \cdots = r_{M-1}[P] = 0 \) if and only for all \( s = 0, \ldots, M - 1 \)

\[
\int_{\mathbb{R}^n} P(x)|x|^{2i} Y_{s-2t,m}(x) d\mu(x) = 0.
\]

But the polynomials \( |x|^{2i} Y_{s-2t,m}(x) \) with \( s = 0, \ldots, M - 1, \ t = 0, \ldots, [s/2] \), \( m = 1, \ldots, a_{s-2t} \), span up the space of polynomials of degree \( \leq M - 1 \). ☐

The next theorem, interesting in its own right, is not needed later, and therefore the proof will be omitted.

**Theorem 9.** Let \( \mu \) be a signed measure with compact support and let \( P(x) \) be a polynomial of degree \( 2N \). If \( P \) is orthogonal to all polynomials of degree \( \leq 2N \) and polyharmonic degree \( < N \) then \( r_0[P] = \cdots = r_{2N-1}[P] = 0 \) and \( r_{2N}[P](\theta) \) is constant.
4. Polyharmonicity of the function of second kind

In this Section we want to show that the function \( Q_P(\zeta, \theta) \) of the second kind, multiplied by \( \zeta^{-(n-1)} \), is a polyharmonic function.

Recall that we have defined \( N_P = \sup \{ d(P \cdot h) : h \text{ harmonic polynomial} \} \) for a polynomial \( P(x) \). In the Appendix we will show that \( N_P \leq \deg P(x) \) and an explicit determination of \( N_P \) will be given there as well.

**Proposition 10.** Let \( Y_{k,m}, m = 1, \ldots, a_k \), be an orthonormal basis of the space of all homogeneous harmonic polynomials. Then

\[
N_P = \sup_{k \in \mathbb{N_0}, m=1,\ldots,a_k} d(P(x)Y_{k,m}(x)).
\]

**Proof.** Let us denote the right hand side by \( M_P \). Then the inequality \( M_P \leq N_P \) is trivial. For the converse let \( h(x) \) be a harmonic polynomial and write \( h(x) = \sum_{k=0}^{N} \sum_{m=1}^{a_k} \lambda_{k,m} Y_{k,m}(x) \). Then

\[
d(P \cdot h) \leq \sup_{k \in \mathbb{N_0}, m=1,\ldots,a_k} d(P(x)Y_{k,m}(x)) \leq M_P.
\]

Note that \( N_P = \sup_{k \in \mathbb{N_0}, m=1,\ldots,a_k} d(P(x)Y_{k,m}(x)) \) since \( Y_{k,m}, m = 1, \ldots, a_k \) is an orthonormal basis as well. Now we determine the asymptotic expansion of the function of the second kind:

**Theorem 11.** Let \( P(x) \) be a polynomial and \( \mu \) be a signed measure with support in \( \overline{B_R} \). Then \( \theta \mapsto Q_P(\zeta, \theta) \), the function of the second kind, possesses a Laplace-Fourier series of the form

\[
Q_P(\zeta, \theta) = \sum_{k=0}^{\infty} \sum_{m=1}^{a_k} \frac{1}{r^{k-1}} p_{k,m}(\zeta^2) Y_{k,m}(\theta)^{(31)}
\]

where \( p_{k,m}(t) \) are univariate polynomials of degree strictly smaller than \( N_{k,m} := d(P(x)Y_{k,m}(x)) \). The function \( Q_P(\zeta, \theta) \) of the second kind depends on those distributed moments

\[
\int_{\mathbb{R}^n} h(x)|x|^2d\mu(x)
\]

where \( t \leq \sup_{k \in \mathbb{N_0}} \deg p_{k,m} \) and \( h(x) \) is a harmonic polynomial.

**Proof.** For each fixed \( \zeta \) with \( |\zeta| > R \) the function \( \theta \mapsto Q_P(\zeta, \theta) \) possesses a Laplace-Fourier expansion, say

\[
Q_P(\zeta, \theta) = \sum_{k=0}^{\infty} \sum_{m=1}^{a_k} e_{km}(\zeta) Y_{k,m}(\theta).
\]
Recall that \( Q_P(\zeta, \theta) = P(\zeta \theta) \hat{\mu}(\zeta, \theta) - R_P(\zeta, \theta) \), see (23). Formula (27) easily yields the Laplace-Fourier expansion of \( \theta \mapsto R_P(\zeta, \theta) \): in (27) one has only to compute the sum over the variable \( t \) obtaining

\[
R_P(\zeta, \theta) = \sum_{k=0}^{\infty} \sum_{m=1}^{\partial_k} Y_{k,m}(\theta) \frac{1}{\zeta^{k-m}} \int_{\mathbb{R}^n} \frac{P(x) Y_{k,m}(x)}{x^2 - |x|^2} \, d\mu(x). \tag{33}
\]

The Laplace-Fourier coefficients of \( \theta \mapsto P(\zeta \theta) \hat{\mu}(\zeta, \theta) \) are given through

\[
f_{k,m}(\zeta) := \int_{S^{n-1}} P(\zeta \theta) \hat{\mu}(\zeta, \theta) Y_{k,m}(\theta) d\theta. \tag{34}
\]

Let us write \( P(x) Y_{k,m}(x) \) in the Gauß decomposition, see Theorem 5.5 in [4], in the form

\[
P(x) Y_{k,m}(x) = \sum_{j=0}^{N_{k,m}} h_{j,k,m}(x) |x|^{2j}, \tag{35}
\]

where \( h_{j,k,m} \) are harmonic polynomials and \( N_{k,m} \) is the polyharmonic degree of \( P(x) Y_{k,m}(x) \). Then (34) and (35) yield

\[
f_{k,m}(\zeta) = \frac{1}{\zeta^k} \int_{S^{n-1}} P(\zeta \theta) \hat{\mu}(\zeta, \theta) Y_{k,m}(\theta) d\theta
\]

\[
= \frac{1}{\zeta^k} \sum_{j=0}^{N_{k,m}} \zeta^{2j} \int_{S^{n-1}} h_{j,k,m}(\zeta \theta) \hat{\mu}(\zeta, \theta) d\theta
\]

\[
= \frac{1}{\zeta^k} \sum_{j=0}^{N_{k,m}} \zeta^{2j} \int_{\mathbb{R}^n} \int_{S^{n-1}} h_{j,k,m}(\zeta \theta) \frac{1}{\omega_n} \frac{\zeta^{n-1}}{r(\zeta \theta - x)^n} \, d\theta d\mu(x).
\]

Since \( h_{j,k,m} \) is a harmonic polynomial the Poisson formula shows that for real \( \zeta > R \) holds

\[
h_{j,k,m}(x) = \frac{1}{\omega_n} \int_{S^{n-1}} h_{j,k,m}(\zeta \theta) \frac{\zeta^{n-2}(\zeta^2 - |x|^2)}{r(\zeta \theta - x)^n} \, d\theta.
\]

Since the integrand is holomorphic in \( \zeta \) this holds for all complex values \( \zeta \) with \( |\zeta| > R \) as well. Thus

\[
f_{k,m}(\zeta) = \frac{1}{\zeta^k} \sum_{j=0}^{N_{k,m}} \zeta^{2j} \int_{\mathbb{R}^n} \frac{\zeta}{\zeta^2 - |x|^2} h_{j,k,m}(x) d\mu(x) \tag{36}
\]

are the Laplace Fourier coefficients of \( \theta \mapsto P(\zeta \theta) \hat{\mu}(\zeta, \theta) \).
Replace now \( P(x) \bar{Y}_{k,m}(x) \) in (33) by the right hand side of (35) and take the difference of the Laplace-Fourier coefficients we computed so far. Then the Laplace-Fourier coefficients of \( Q_P(\zeta, \theta) \) are given by

\[
e_{k,m}(\zeta) = \frac{1}{\zeta^{k-1}} \sum_{j=0}^{N_{k,m}} \int_{\mathbb{R}^n} \frac{1}{\zeta^2 - |x|^2} h_{j,k,m}(x)(\zeta^{2j} - |x|^{2j}) d\mu(x).
\]

Note that for \( j = 0 \) the summand is just zero. For \( j \geq 1 \) we have

\[
\frac{\zeta^{2j} - |x|^{2j}}{\zeta^2 - |x|^2} = |x|^{2(j-1)} + |x|^{2(j-1)} \zeta^2 + \ldots + \zeta^{2(j-1)}.
\]

We conclude that \( \zeta \mapsto \zeta^{k-1} e_{k,m}(\zeta) = P_{k,m}(\zeta^2) \) is a polynomial in \( \zeta^2 \) of degree at most \( N_{k,m} - 1 \). It follows that \( e_{k,m}(\zeta) \) can be computed if we know all moments of the form (32) where \( t \leq \deg p_{k,m} \) and \( h(x) \) is a harmonic polynomial. The proof is complete. \( \blacksquare \)

From this we have the following interesting consequence:

**Corollary 12.** Let \( P(x) \) be a polynomial, \( \mu \) be a signed measure with support in \( \overline{B}_R \) and \( Q_P(\zeta, \theta) \) be the corresponding function of the second kind. Then the function \( r \mapsto r^{-(n-1)} Q_P(r, \theta) \) defined for \( r > R \) and \( \theta \in S^{n-1} \), is a polyharmonic function of polyharmonic degree \( < N_P \) where \( N_P \) is defined in (3).

**Proof.** By the last theorem the function \( \theta \mapsto r^{-(n-1)} Q_P(r, \theta) \) has the following Laplace-Fourier expansion

\[
f(r\theta) := r^{-(n-1)} Q_P(r, \theta) = \sum_{k=0}^{\infty} \sum_{m=1}^{a_k} \frac{1}{r^{n+k-2}} p_{k,m}(r^2) Y_{k,m}(\theta).
\]

Let us define the differential operator

\[
L_{(k)} := \frac{d^2}{dr^2} + \frac{n-1}{r} \frac{d}{dr} - \frac{k(k+n-2)}{r^2}.
\]  \hspace{1cm} (37)

It is known that a function \( g(r\theta) \) is a solution of \( \Delta^p g(x) = 0 \) if and only if the coefficient functions \( g_{k,m}(r) \) of its Laplace-Fourier expansion are solutions of the equation \( [L_{(k)}]^p g_{k,m}(r) = 0 \); an elaboration of these classical results can be found in [19]. Further the polynomials \( r^j \) with \( j = -k - n + 2, -k - n + 4, \ldots, -k - n + 2p \) are solutions of this equation. It follows that

\[
f_{k,m}(r) = \frac{1}{r^{n+k-2}} P_{k,m}(r^2)
\]

are solutions of the equation \( [L_{(k)}]^p g_{k,m}(r) = 0 \) when \( p \geq N_k \). The proof is complete. \( \blacksquare \)
5. Measures with algebraic support

A measure $\mu$ over $\mathbb{R}^n$ has algebraic support if the support of the measure is contained in an algebraic set, i.e. if the support of $\mu$ is contained in $P^{-1}(0)$ for some polynomial $P(x)$. Further we say that $\mu$ has finite support if the support has only finitely many elements. The following gives a characterization of algebraic support of a measure in terms of the Markov function:

**Theorem 13.** Let $\mu$ be a measure with support in $\overline{B_R}$ and let $P(x)$ be a polynomial. Then $\mu$ has support in $P^{-1}(0)$ if and only if

$$P(\zeta \theta) \mu(\zeta, \theta) = Q_P(\zeta, \theta) \quad \text{for all } \theta \in S^{n-1}, |\zeta| > R,$$

where $Q_P(\zeta, \theta)$ is the function of the second kind.

**Proof.** If $\mu$ has support in $P^{-1}(0)$ it follows that the function $R_P(\zeta, \theta)$ is equal to zero and (38) is evident by (23). For the converse assume that $P(\zeta \theta) \mu(\zeta, \theta) = Q_P(\zeta, \theta)$. Define the polynomial $P^*$ by $P^*(x) := \tilde{P}(x)$ for $x \in \mathbb{R}^n$. By Proposition 2 and Theorem 6

$$\int_{\Gamma_R} \int_{S^{n-1}} h(\zeta \theta) P^*(\zeta \theta) P(\zeta \theta) \mu(\zeta, \theta) d\zeta d\theta = 0$$

for any polynomial $h(x)$. Since the polynomials are dense it follows that $P^* P d\mu$ is the zero measure. Let $\mu = \mu^+ - \mu^-$ be the Jordan decomposition. It follows that $P^* P d\mu^+$ and $P^* P d\mu^-$ are zero measures, and it is easy to see that this implies that the support of $\mu^+$ and $\mu^-$ is contained in $P^{-1}(0)$. Thus $\mu$ has support in $P^{-1}(0)$. ■

Let $U$ be an open non-empty subset of the complex plane $\mathbb{C}$ and $f$ be a function defined on $U \times S^{n-1}$. We say that $f$ is pointwise rational if there exists a polynomial $P(x)$ in $n$ variables such that for each fixed $\theta \in S^{n-1}$ the function $\zeta \mapsto P(\zeta \theta) f(\zeta, \theta)$ is a polynomial in the variable $\zeta$.

**Proposition 14.** Let $\mu$ be a signed measure with bounded support and suppose that the Markov function $\mu(\zeta, \theta)$ is pointwise rational. Then $\mu$ has algebraic support.

**Proof.** Let $P(x)$ be a polynomial such that the map $\zeta \mapsto P(\zeta \theta) \mu(\zeta, \theta)$ is a polynomial in the variable $\zeta$. Then the integral over $\Gamma_R$ in (39) is already zero and as in the last proof we obtain that $\mu$ has support in $P^{-1}(0)$. ■

The converse of the last proposition is not true as the following result with $\sigma$ equal to the Lebesgue measure on the unit interval shows:
Proposition 15. Let $\sigma$ be a measure over $\mathbb{R}$ with compact support, $\delta_0$ the Dirac measure over $\mathbb{R}$ at the point 0 and let $\mu = \sigma \otimes \delta_0$. Then for $|\zeta| > R$ the multivariate Markov transform is given by

$$
\sigma \otimes \delta_0(\zeta, e^{it}) = \frac{1}{\omega_2} \sum_{l=0}^{\infty} \int x^l \ d\sigma(x) \frac{\sin(l+1)t}{\sin t} \frac{1}{\zeta^{l+1}}.
$$

(40)

The measure $\mu$ has algebraic support. Its multivariate Markov transform $\sigma \otimes \delta_0$ is pointwise rational if and only if the measure $\sigma$ has finite support.

**Proof.** Let $\theta = e^{it}$ with $t \in \mathbb{R}$. It is straightforward to verify that for $|\zeta| > R$ holds

$$
\sigma \otimes \delta_0(\zeta, \theta) = \frac{1}{\omega_2} \int_{\mathbb{R}^2} \frac{\zeta}{r(\zeta \theta - (x, y))^2} \ d(\sigma \otimes \delta_0) = \frac{1}{\omega_2} \int \frac{\zeta}{\zeta^2 - 2\zeta x \cos t + x^2} \ d\sigma.
$$

Note that

$$
\frac{2i\zeta \sin t}{\zeta^2 - 2\zeta x \cos t + x^2} = \frac{1}{\zeta \theta - x} - \frac{1}{\zeta \theta - x}.
$$

Define for the measure $\sigma$ the one-dimensional Markov transform by $\tilde{\sigma}(\zeta) = \int \frac{1}{\zeta x} \ d\sigma(x)$. Then $2i\omega_2 \sin t \cdot \sigma \otimes \delta_0(\zeta, \theta) = \tilde{\sigma}(\zeta \theta) - \tilde{\sigma}(\zeta \theta)$ and the asymptotic expansion of $\tilde{\sigma}$ leads to (40).

Assume now that $\sigma \otimes \delta_0(\zeta, \theta)$ is pointwise rational. Then for $t = \pi/2$ the function

$$
\sigma \otimes \delta_0(\zeta, \pi/2) = \frac{1}{\omega_2} \sum_{k=0}^{\infty} \int x^{2k} \ d\sigma(x) (-1)^k \frac{1}{\zeta^{2k+1}} = \frac{1}{\omega_2} \int \frac{1}{\zeta^2 + x^2} \ d\sigma(x)
$$

must be a rational functional in $\zeta$. As it is known from univariate Padé approximation this implies that $\sigma$ must have finite support, [26, chapter 2, section 3, Theorem 3.1]. Conversely, if a measure $\mu$ over $\mathbb{R}^n$ has finite support, and the dimension $n$ is even then it is easy to see that $\zeta \mu(\zeta, \theta)$ is a quotient of two polynomials, in particular it is pointwise rational.

We finish this section with the following example:

Example 16. Let $\mu$ be the Lebesgue measure on the unit circle $S^1$. Since the measure is rotation-invariant it follows that $\mu(\zeta, \theta) = \frac{\zeta}{\zeta - 1}$. Hence the multivariate Markov transform $\zeta \mu(\zeta, \theta)$ is pointwise rational but $\mu$ does not have finite support.
6. Proof of Theorem 1

Proof. In Theorem 11 we have seen that \( Q_{\mu,P} \) and \( Q_{\nu,P} \) only depend on the moments \( c_{t,k,m} \) defined in (15) where \( t < N_P \). It follows that \( Q_{\mu,P} = Q_{\nu,P} \). By Theorem 13 \( P(\zeta)\hat{\mu}(\zeta, \theta) = Q_{\mu,P}(\zeta, \theta) \) and \( P(\zeta)\hat{\nu}(\zeta, \theta) = Q_{\nu,P}(\zeta, \theta) \) for all large \( \zeta \) and for all \( \theta \in S^{n-1} \), therefore \( P(\zeta)\hat{\mu}(\zeta, \theta) = P(\zeta)\hat{\nu}(\zeta, \theta) \). We want to conclude that \( \hat{\mu}(\zeta, \theta) = \hat{\nu}(\zeta, \theta) \); in that case Theorem 3 yields \( \mu = \nu \). If \( P(\zeta) \) has no zeros for large \( \zeta \) it is clear that \( \hat{\mu}(\zeta, \theta) = \hat{\nu}(\zeta, \theta) \). In the general case, it suffices to show that \( A := \{(\zeta, \theta) \in C \times S^{n-1} : P(\zeta) = 0 \} \) is nowhere dense since then a continuity argument leads to \( \hat{\mu}(\zeta, \theta) = \hat{\nu}(\zeta, \theta) \). This fact will be proven in the next Proposition. ■

Proposition 17. The set \( A := \{(\zeta, \theta) \in C \times S^{n-1} : P(\zeta) = 0 \} \) is closed and has no interior point, i.e. \( A \) is nowhere dense in \( C \times S^{n-1} \).

Proof. Clearly \( A \) is closed. Suppose that there \( \theta_0 \in S^{n-1} \) and \( \zeta_0 \) such that \( P(\zeta) = 0 \) for all \( \zeta \) in a neighborhood \( U \) of \( \zeta_0 \) and for all \( \theta \) in a neighborhood \( V \) of \( \theta_0 \). For fixed \( \theta \in V \) it follows that \( \zeta \rightarrow P(\zeta) \) must be the zero polynomial since for all \( \zeta \in U \) (hence uncountably many \( \zeta \)) we have \( P(\zeta) = 0 \). It follows that \( P(\zeta) = 0 \) for all \( \zeta \in C \) and for all \( \theta \in V \). Hence \( P(x) = 0 \) for all \( x \) in an open set \( W \) of \( R^n \) and, by the properties of real analytic functions, we conclude that \( P \equiv 0 \).

Corollary 18. Let \( P(x) \) be a polynomial and \( N_P \) be given by (30). Then the space

\[
U_{N_P} := \{ Q \in P : A^{N_P} Q = 0 \}
\]

is dense in the space \( C(K_P(R), C) \) of all continuous complex-valued functions on \( K_P(R) \) endowed with the supremum norm.

Proof. Since \( U_{N_P} \) is closed under complex conjugation we may reduce the problem to the case of real-valued continuous functions. Suppose that \( U_{N_P} \) is not dense in \( C(K_P(R), R) \). By the Hahn-Banach theorem there exists a continuous non-trivial real-valued functional \( L \) which vanishes on \( U_{N_P} \). By Riesz’s Theorem there exists a signed measures \( \sigma \) representing the functional \( L \) with support in \( K_P \). By Theorem 1 (applied to \( \sigma \) and the zero measure) we conclude that \( \sigma = 0 \), a contradiction. ■

7. Appendix: The polyharmonic degree

We want to list some of the properties of the polyharmonic degree map. Note that the inequality \( d(P + Q) \leq \max\{d(P), d(Q)\} \) is trivial. In [3] the important equality
\[ d(Q \cdot |x|^2) = d(Q) + d(|x|^2) = d(Q) + 1. \] (41)

is proved for any polyharmonic function \( Q \) defined on a domain containing zero. The following inequality is implicitly contained in [3, Theorem 1.2, p. 31]. For completeness we give the short proof.

**Proposition 19.** Let \( f, g \) be harmonic. Then \( d(fg) \leq \min\{\deg f, \deg g\} \) and \( d(ff^*) = \deg f \).

**Proof.** Let \( Vf \) be the gradient of \( f \). Then \( A(fg) = (Af)g + 2\langle \nabla f, \nabla g \rangle + fAg \). If \( h \) and \( g \) are harmonic it is easy to show by induction that

\[ A^p(fg) = 2^p \sum_{i_1, \ldots, i_p=1}^n \left( \frac{\partial}{\partial x_{i_1}} \cdots \frac{\partial}{\partial x_{i_p}} f \right) \left( \frac{\partial}{\partial x_{i_1}} \cdots \frac{\partial}{\partial x_{i_p}} g \right). \] (42)

Suppose that \( s := \deg f \leq \deg g \). Then \( \frac{\partial^s}{\partial x^s} f = 0 \) for all \( \beta \in \mathbb{N}_0^n \) with \( |\beta| = s + 1 \). It follows from (42) that \( A^{s+1}(fg) = 0 \). Hence \( d(fg) \leq s \) and the first statement is proved. Clearly this implies also that \( d(ff^*) \leq \deg f \).

Suppose that \( A^{p+1}(ff^*) = 0 \) for some \( p \in \mathbb{N} \). Then \( \sum_{i_1, \ldots, i_p=1}^n \left| \frac{\partial}{\partial x_{i_1}} \cdots \frac{\partial}{\partial x_{i_p}} f \right|^2 = 0 \). It follows that \( \frac{\partial^\beta}{\partial x^\beta} f = 0 \) for all \( \beta \in \mathbb{N}_0^n \) with \( |\beta| = p + 1 \). Hence \( \deg f \leq p \), and we have proved that \( \deg f \leq d(ff^*) \). \( \blacksquare \)

Now we can prove the following:

**Corollary 20.** Let \( P(x) \) be a polynomial with the Gauß decomposition

\[ P(x) = h_0(x) + |x|^2 h_1(x) + \cdots + |x|^{2N} h_N(x). \] (43)

Then for \( N_P \) defined in (3) the following inequality holds:

\[ N_P \leq \max_{r=0, \ldots, N} \{r + \deg h_r\} \leq \deg P(x). \] (44)

**Proof.** Recall formula (30) for \( N_P \) and let \( Y_k \) be a harmonic homogeneous polynomial of degree \( k \). An iteration argument in (41) implies that \( d(|x|^{2r} h_r Y_k) = r + d(h_r Y_k) \). By Proposition 19 \( d(h_r Y_k) \leq \deg h_r \). Hence \( d(P \cdot Y_k) \leq \max_{r=0, \ldots, N} \{r + \deg h_r\} \), and this proves the first inequality. Further we know that \( \deg(|x|^{2r} h_r) = 2r + \deg h_r \leq \deg P \) for \( r = 0, \ldots, N \). Hence the second inequality is established. \( \blacksquare \)

In the following we want to give an explicit formula for \( N_P \). We need the following result which is interesting in its own right:

**Theorem 21.** Let \( Y_{k,m}(x) \) be an orthonormal basis of spherical harmonics with \( k \in \mathbb{N}_0 \) and \( m = 1, \ldots, a_k \). Then \( d(Y_{k,m}(x) Y_{k,m}(x)) = k \) if and only if \( m = m_1 \).
Proof. We start with a general remark: Let \( Y_k \) and \( Y_l \) be harmonic homogeneous polynomials of degree \( k \) and \( l \) respectively. Clearly \( Y_k(x)Y_l(x) \) is a homogeneous polynomial of degree \( k + l \). By Proposition 19 it has polyharmonic degree at most \( \min\{k, l\} \). By Gauß decomposition there exist harmonic homogeneous polynomials \( h_{k+l-2u} \), either \( h_{k+l-2u} \) is zero or of exact degree \( k + l - 2u \) for \( u = 0, \ldots, \min\{k, l\} \), such that

\[
Y_k(x)Y_l(x) = \sum_{u=0}^{\min\{k, l\}} |x|^{2u} h_{k+l-2u}(x). \tag{45}
\]

Now assume that \( Y_k(x) = Y_{k,m}(x) \) and \( Y_l(x) = Y_{k,m_1}(x) \). Let us consider the summand \( |x|^{2k} h_0(x) \) for \( u = k \). Then \( h_0 \) must have degree 0, hence it is a constant polynomial. Integrate equation (45) with respect to \( d\theta \). Since \( h_{2k-2u} \) is either 0 or of exact degree \( 2k - 2u > 0 \) for \( u = 0, \ldots, k - 1 \) the integral over the sphere of \( |x|^{2u} h_{k+l-2u}(x) \) will vanish. Then we obtain with the orthogonality relations for spherical harmonics

\[
\delta_{m,m_1} = \int_{S^{n-1}} h_0 \ d\theta = h_0 \omega_n.
\]

Hence for \( m \neq m_1 \) we see that the polyharmonic degree is less than \( k \), for \( m = m_1 \) it is exactly \( k \). The proof is finished. \( \blacksquare \)

Theorem 22. Let \( P(x) \) be a homogeneous polynomial of degree \( N \), say of the form

\[
P(x) = \sum_{t,k \in \mathbb{N}_0, 2t+k=N} \sum_{m=1}^{a_t} b_{t,k,m} |x|^{2t} Y_{k,m}(x).
\]

Let \( k_0 = k_0(P) \) be the largest natural number such that \( b_{t_0,k_0,m_0} \neq 0 \) for some \( m_0 \) and \( t_0 \) in the above sum. Then

\[
N_P = \frac{1}{2}(N + k_0(P)).
\]

Proof. Let \( k_0 \) be as specified in the theorem. Let \( k_1 \in \mathbb{N}_0 \) and \( m_1 \in \{1, \ldots, a_{k_1}\} \), then

\[
d(P(x)Y_{k_1,m_1}(x)) \leq \max \{ |x|^{2t} Y_{k,m_1}Y_{k_1,m_1}(x) \} \quad (46)
\]

where the maximum ranges over all indices \( t, k, m \) with \( b_{t,k,m} \neq 0 \). Using (41) and the inequality \( d(Y_{k,m}Y_{k_1,m_1}) \leq k \) in (46) we arrive at (note that \( 2t + k = N \))

\[
d(P(x)Y_{k_1,m_1}(x)) \leq \max \{t + k\} = \frac{1}{2} \max \{N + k\} \leq \frac{1}{2}(N + k_0),
\]
where the last inequality follows from the chooce of \(k_0\). Now (30) yields \(N_P \leq \frac{1}{2}(N + k_0)\). For the other direction it suffices to show that \(P(x) Y_{k_0,m_0}\) has polyharmonic degree \(\geq \frac{1}{2}(N + k_0)\). Clearly it suffices to show that there exists a polynomial \(R(x)\) of polyharmonic degree \(< \frac{1}{2}(N + k_0)\) such that

\[
P(x) Y_{k_0,m_0} = b_{t_0,k_0,m_0} x^{2t_0} Y_{k_0,m_0} + R(x)
\]

since (41) and Theorem 21 imply that \(b_{t_0,k_0,m_0} x^{2t_0} Y_{k_0,m_0} Y_{k_0,m_0}\) has polyharmonic degree \(t_0 + d(Y_{k_0,m_0} Y_{k_0,m_0}) = t_0 + k_0 = \frac{1}{2}(N + k_0)\)

using the fact that \(2t_0 + k_0 = N\). It remains to prove that \(R(x)\) has polyharmonic degree less than \(\frac{1}{2}(N + k_0)\). It suffices to show that for each non-zero summand \(b_{t,k,m} x^{2t} Y_{k,m} Y_{k_0,m_0}\) in \(R(x)\)

\[
d(b_{t,k,m} x^{2t} Y_{k,m} Y_{k_0,m_0}) = t + d(Y_{k,m} Y_{k_0,m_0}) < \frac{1}{2}(N + k_0).
\]

If \(k < k_0\) this is clear since \(d(Y_{k,m} Y_{k_0,m_0}) \leq k\) and \(t + k = \frac{1}{2}(N + k)\). If \(k = k_0\) we know that \(m \neq m_0\), and by Theorem 21 we have again strict inequality. By choice of \(k_0\) we always have \(k \leq k_0\), so the theorem is proved.

In the last theorem it is essential that the polynomial \(P(x)\) is homogeneous. If \(P(x)\) is arbitrary, we can write \(P(x) = \sum_{j=0}^{N} P_j(x)\) where \(P_j(x)\) are homogeneous polynomials. It is not very difficult to see that

\[
d(P \cdot Y_{k,m}) = \max_{j=0,\ldots,N} d(P_j \cdot Y_{k,m}),
\]

see e.g. the proof of Theorem 1.27 in [4]. Hence \(N_P\) is the maximum of \(N_{P_j}\) for \(j = 0,\ldots,N\).

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Ognyan Kounchev

Institute of Mathematics and Informatics
Bulgarian Academy of Sciences
8 Acad. G. Bonchev Str.
1113 Sofia, Bulgaria

e-mail: kounchev@math.bas.bg, kounchev@math.uni-duisburg.de

Hermann Render

Departamento de Matemáticas y Computación
Universidad de la Rioja
Edificio Vives, Luis de Ulloa, s/n.
26004 Logroño, Spain

e-mail: render@gmx.de; hermann.render@dmc.unirioja.es