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The approximation order of polysplines

O. Kounchev and H. Render

March 1, 2014

Abstract

We show that the scaling spaces defined by the polysplines of order $p$ provide approximation order $2p$. For that purpose we refine the results on one-dimensional approximation order by $L$--splines obtained in [2].

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1 Introduction

In the last decade the approximation order of shift-invariant subspaces of the space $L^2(\mathbb{R}^n)$ of all square-integrable functions on the euclidean space $\mathbb{R}^n$ has been investigated extensively, e.g. in the survey paper [10] approximately 100 references are given. The problem can be formulated in a rather general way: suppose that $(V_h)_{h \in I}$ is a family of subspaces of $L^2(\mathbb{R}^n)$ (not necessarily shift-invariant) where $I$ is subset of $(0, \infty)$ having 0 as an accumulation point. One has to estimate the rates of decay of the approximation error

$$E(f, V_h) := \inf \left\{ \| f - s \|_{L^2(\mathbb{R}^n)} : s \in V_h \right\}$$

(1)
for $h$ tending to $0$. If $W$ is a subspace of $L^2(\mathbb{R}^n)$ endowed with a norm $\|\cdot\|_W$ we say that $(V_h)_{h \in I}$ provides approximation order $m$ with respect to the norm $\|\cdot\|_W$ if there exists a constant $c_W$ such that for every $f \in W$ and for every $h \in I$

$$E(f, V_h) \leq c_W \cdot h^m \cdot \|f\|_W.$$  

(2)

Usually $W$ is the potential space $W^m_2(\mathbb{R}^n)$ for $m \in (0, \infty)$ defined as the subspace of those $f \in L^2(\mathbb{R}^n)$ such that

$$\|f\|_{W^m_2(\mathbb{R}^n)} := (2\pi)^{-\frac{n}{2}} \| (1 + |\xi|)^m \hat{f}(\xi) \|_{L^2(\mathbb{R}^n)} < \infty.$$  

(3)

In this note we want to prove that cardinal polysplines of order $p$ provide approximation order $2p$.

The motivation for the present work comes from the fact that polysplines are useful for solving multivariate interpolation problems [4], [5], [6] and they are of importance for the multivariate Wavelet Analysis, cf. the monograph [9]. Recall that a function $S : \mathbb{R}^n \setminus \{0\} \to \mathbb{C}$ is called a cardinal polyspline\footnote{The first author introduced in 1991 polysplines in a more general setting with arbitrary interfaces, see [3] and [9].} (on annuli) of order $p$ if $S$ is $(2p - 2)$-times continuously differentiable and the restriction of $S$ to each open annulus $\{x \in \mathbb{R}^n : e^l < |x| < e^{l+1}\}$ is a polyharmonic function\footnote{Recall that a function $f$ defined on an open set $U$ in the euclidean space $\mathbb{R}^n$ is polyharmonic of order $p$ if $f$ is $2p$-times continuously differentiable and $\Delta^p f(x) = 0$ for all $x \in U$ where $\Delta$ is the Laplace operator and $\Delta^p$ its $p$-th iterate.} of order $p$ for $l \in \mathbb{Z}$. The reason for calling such polysplines "cardinal" is found in Theorem 3 where it is seen that after expanding $S$ in a Fourier–Laplace series of spherical harmonics the coefficients $S_{k,l}(\log r)$ are cardinal $L$--splines in the usual sense of the word, cf. Micchelli’s paper of 1976 [8].

Introducing a parameter $h > 0$, by $P_h$ we denote the set of all functions $S : \mathbb{R}^n \setminus \{0\} \to \mathbb{C}$ which are $(2p - 2)$-times continuously differentiable and whose restriction to each open annulus $A_{h,l} := \{x \in \mathbb{R}^n : e^l < |x| < e^{h(l+1)}\}$ is a polyharmonic function of order $p$ for $l \in \mathbb{Z}$. Then the scaling spaces of polysplines of order $p$, shortly $PV_h$, are defined as the $L^2$--closure of $P_h \cap L^2(\mathbb{R}^n)$, $h > 0$.

The main result is the following:

**Theorem 1** The sequence $(PV_h)_{h > 0}$ provides approximation order $2p$ where $p$ denotes the order of the polysplines. More precisely, there exists a constant
C > 0 such that for all h with 0 < h < 1 and f ∈ L^2(\mathbb{R}^n) the following inequality holds

$$\inf \left\{ \| f - g \|_{L^2(\mathbb{R}^n)} : g \in PV_h \right\} \leq C \cdot h^{2p} \cdot \left( \int_{\mathbb{R}^n} |x|^{2p} \cdot |\Delta^p f (x)|^2 \, dx \right)^{\frac{1}{2}}.$$  

Note that in place of the norm (3) we have a semi–norm on the right–hand side which is zero on the polyharmonic functions of order p.

The paper is organized as follows: in the next Section we discuss the approximation order of cardinal L–splines by using important results from [2]. In Section 3 the main result will be proven.

2 Approximation order of cardinal L–splines

Let us recall Theorem 4.3 in the fundamental paper [2]: Suppose that for every h > 0, the space S_h is the L^2(\mathbb{R}^n)–closure of the linear space generated by the shifts φ_h (· − m), m ∈ \mathbb{Z}^n of the functions φ_h ∈ L^2(\mathbb{R}^n) (so S_h is the shift-invariant space generated by φ_h) and that V_h = \{ s (\bar{x}) : s ∈ S_h \}. Then the family (V_h)_{h ∈ I} provides approximation order m with respect to the norm \| · \|_{W^m_2(\mathbb{R}^n)} defined in (3) if and only if there exists D > 0 such that for all h ∈ I and for almost all x ∈ C := [−π, π]^n

$$|\Lambda_{φ_h} (x)| \leq D \cdot (h + |x|^m),$$

where

$$(\Lambda_{φ_h} (\xi))^2 := \frac{\sum_{\alpha ∈ \mathbb{Z}^n, \alpha \neq 0} |\widehat{φ_h} (\xi + 2\pi\alpha)|^2}{\sum_{\beta ∈ \mathbb{Z}^n} |\widehat{φ_h} (\xi + 2\pi\beta)|^2} \leq 1.$$  

We will need a refinement of that result. For our purposes it will be useful to consider instead of (3) different norms. In the following we replace the function (1 + |x|^m) by a measurable function Q(x) with the following properties: (i) the zero set Q^{-1}(0) of Q is a set of Lebesgue measure zero and (ii) there exists a constant D_1 > 0 such that

$$|Q (\frac{x}{h})| \geq D_1 \frac{1}{h^m} \quad \text{for all } x \notin C := [−π, π]^n.$$  

(5)

Suppose further that there exists a constant D_2 > 0 such that for all x ∈ C and for all 0 < h < 1

$$|\Lambda_{φ_h} (hx)| \leq h^m D_2 |Q (x)|.$$  

(6)
An analysis of the proof in [2] shows that then the following inequality holds (for us the constants $D_1$ and $D_2$ defined in the formula will be very important!)

$$E(f, V_h) \leq \left(D_2 (2\pi)^{\frac{n}{2}} + \frac{1}{D_1 (2\pi)^{\frac{n}{2}}}\right) \cdot h^m \cdot \left\| Q(\xi) \hat{f}(\xi) \right\|_{L_2(\mathbb{R}^n)}.$$  \hspace{1cm} (7)

Let us recall some facts about $L$-splines: Let $L$ be a linear differential operator with constant coefficients of order $N + 1$, say

$$L = M_\Lambda := \prod_{j=1}^{N+1} \left( \frac{d}{d\omega} - \lambda_j \right) \quad \text{where} \quad \Lambda := (\lambda_1, \ldots, \lambda_{N+1}). \hspace{1cm} (8)$$

Then a function $u : \mathbb{R} \to \mathbb{R}$ is called cardinal $L$-spline on the mesh $h\mathbb{Z}$ ($h > 0$) if $u$ is $(N - 1)$-times continuously differentiable and if for every $l \in \mathbb{Z}$ there exists $f_l \in U_L := \{ f \in C^\infty(\mathbb{R}) : Lf = 0 \}$ such that $u(t) = f_l(t)$ for all $t \in (lh, (l + 1)h)$. The set of all cardinal $L$-splines for the operator $L = M_\Lambda$ on $h\mathbb{Z}$ will be denoted by $S_{h\mathbb{Z}}(\Lambda)$. The scaling spaces $V_h(\Lambda)$ are defined by

$$V_h(\Lambda) = L^2(\mathbb{R}) \text{-closure of } S_{h\mathbb{Z}}(\Lambda) \cap L^2(\mathbb{R}). \hspace{1cm} (9)$$

Let $Q_\Lambda$ be the basic spline which can be defined by its Fourier transform by

$$\hat{Q}_\Lambda(\xi) = \frac{\prod_{j=1}^{N+1} (e^{-\lambda_j} - e^{-i\xi})}{\prod_{j=1}^{N+1} (i\xi - \lambda_j)}. \hspace{1cm} (10)$$

**Theorem 2** Let $N \in \mathbb{N}$ be fixed. Then there exists a constant $D > 0$ such that for all $\Lambda = (\lambda_1, \ldots, \lambda_{N+1}) \in \mathbb{R}^{N+1}$ and for all $f \in L_2(\mathbb{R})$ the following inequality holds

$$E(f, V_h(\Lambda)) \leq h^{N+1} \cdot D \left\| P_\Lambda(\xi) \hat{f}(\xi) \right\|_{L_2(\mathbb{R})}, \hspace{1cm} (11)$$

where the polynomial $P_\Lambda(x) = \prod_{j=1}^{N+1} (ix - \lambda_j)$.

**Remark 3** Note that if we used the usual Sobolev norm (3) then we could not be able to obtain the sharp constant $D$ of inequality (11); the last is the main virtue of Theorem 2.
Proof. By the above we have to check (5) and (6). Note that for \( Q := P \) we have the estimate

\[
|P \left( \frac{x}{h} \right)|^2 = \prod_{j=1}^{N+1} \left( \left( \frac{x}{h} \right)^2 + \lambda_j^2 \right) \geq \pi^{2(N+1)} \frac{1}{h^{2(N+1)}} \tag{12}
\]

for all \(|x| \geq \pi\) and for all \( h > 0 \). Hence it suffices to show that

\[
|\Lambda_{\varphi_h} (h\xi)|^2 \leq h^{2(N+1)} |P \Lambda (\xi)|^2 \sum_{\alpha \in \mathbb{Z}, \alpha \not= 0} \frac{1}{(\pi |\alpha|)^{2(N+1)}} \tag{13}
\]

The trivial inequality \((\Lambda_{\varphi_h} (\xi))^2 \leq \frac{\sum_{\alpha \in \mathbb{Z}, \alpha \not= 0} |\varphi_h (\xi+2\pi\alpha)|^2}{|\varphi_h (\xi)|^2}\) and the estimate

\[
\left| \frac{\hat{\varphi}_h (\xi + 2\pi\alpha)}{|\varphi_h (\xi)|^2} \right|^2 = \left| \frac{\hat{Q}_{h\Lambda} (\xi + 2\pi\alpha)}{|\hat{Q}_{h\Lambda} (\xi)|^2} \right|^2 = \prod_{j=1}^{N+1} \left| \frac{i\xi - h\lambda_j}{i(\xi+2\pi\alpha) - h\lambda_j} \right|^2
\]

yields

\[
|\Lambda_{\varphi_h} (h\xi)|^2 \leq h^{2(N+1)} \prod_{j=1}^{N+1} (\xi^2 + \lambda_j^2) \sum_{\alpha \in \mathbb{Z}, \alpha \not= 0} \prod_{j=1}^{N+1} \frac{1}{(h\xi + 2\pi\alpha)^2 + h^2\lambda_j^2}.
\]

Since \((h\xi + 2\pi\alpha)^2 + h^2\lambda_j^2 \geq (h\xi + 2\pi\alpha)^2 \geq (2\pi|\alpha| - |h\xi|)^2\) we obtain for \(0 < h < 1\) and \(|\xi| \leq \pi\) the estimate \(2\pi|\alpha| - |h\xi| \geq \pi|\alpha|\) (since \(\alpha \not= 0\)) arriving at (13). ■

3 The approximation order of Polysplines

Let \( S^{n-1} = \{ x \in \mathbb{R}^n ; |x| = 1 \} \) be the unit sphere. Each \( x \in \mathbb{R}^n \) will be written in spherical coordinates \( x = r\theta \) with \( r \geq 0 \) and \( \theta \in S^{n-1} \). Recall that a function \( Y : S^{n-1} \to \mathbb{C} \) is a spherical harmonic of degree \( k \in \mathbb{N}_0 \) if there exists a homogeneous harmonic polynomial \( P(x) \) of degree \( k \) such that \( P(\theta) = Y(\theta) \) for all \( \theta \in S^{n-1} \). The set \( \mathcal{H}_k \) of all spherical harmonics of degree exactly \( k \) is a linear space of dimension \( a_k := \dim \mathcal{H}_k = \binom{n+k-1}{k} - \binom{n+k-3}{k-2} \). We denote by \( Y_{k,l} \) with \( l = 1, 2, ..., a_k \) a basis for \( \mathcal{H}_k \). For a detailed account we refer to Stein–Weiss [12].
Let $u : (R_1, R_2) \rightarrow \mathbb{C}$ be infinitely differentiable and $Y_k \in \mathfrak{S}_k$. Then it is well known that $\Delta (u (r) Y_k (\theta)) = Y_k (\theta) L_k (u (r))$ where we have put

$$L_k (u) = \frac{d^2}{dr^2} + \frac{n - 1}{r} \frac{d}{dr} - \frac{k (k + n - 2)}{r^2}. \quad (14)$$

By iteration we have $\Delta^p u = Y_k (\theta) \cdot [L_k (u)]^p u (r)$. Let us put for convenience

$$\Lambda_+ (k, p) := \{k, k + 2, \ldots, k + 2p - 2\},$$

$$\Lambda_- (k, p) := \{-k - n + 2, -k - n + 4, \ldots, -k - n + 2p\}.$$  

The space of solutions of the equation $L_k^p (f (r)) = 0$ which are $C^\infty$ for $r > 0$ is generated by a simple basis: for $j \in \Lambda_+ (k, p) \cup \Lambda_- (k, p)$ the function $r^j$ is clearly a solution, while for $j \in \Lambda_+ (k, p) \cap \Lambda_- (k, p)$ we obtain a second solution $r^j \log r$. It will be convenient to make a transform of the variable $r$ to $v = \log r$. Then a solution of the form $r^j$ will be transformed to $e^{jv}$ and a solution of the form $r^j \log r$ is transformed to $ve^{jv}$. We see immediately that all solutions to the equation $L_k^p (f (r)) = 0$ are transformed to solutions of the equation $M_{\Lambda (k)} g (v) = 0$ where $M_{\Lambda (k)}$ is defined by (8) with respect to the vector

$$\Lambda := (k + 2, \ldots, k + 2p - 1, -k - n + 2, \ldots, -k - n + 2p). \quad (15)$$

The dependence on the parameter $p$ and $n$ will be suppressed throughout the paper.

A proof of the following can be found in [6], [9, Theorem 9.7].

**Theorem 4** Let $S : \mathbb{R}^n \setminus \{0\} \rightarrow \mathbb{R}$ be a polyspline of order $p$. Then the Laplace-Fourier coefficient $S_{k,l} : \mathbb{R} \rightarrow \mathbb{R}$ defined by

$$S_{k,l} (v) := \int_{S^{n-1}} S (e^v \theta) Y_{k,l} (\theta) d\theta \quad (16)$$

is a cardinal $L$-spline with respect to the linear differential operator $M_{\Lambda (k)}$.

We want to characterize the $L^2 (\mathbb{R}^n)$-closure $PV_h$. It is a temptation to assume that for $S \in PV_h$ the Fourier-Laplacian coefficient defined through formula (16) will be in $V_h (\Lambda_k)$, i.e. in the closure of $S_{h,2} (\Lambda_k) \cap L_2 (\mathbb{R})$. This is not true since the transformation rule will give us an additional weight for $f \in L_2 (\mathbb{R}^n)$:

$$\int_{\mathbb{R}^n} |f (x)|^2 dx = \int_0^\infty \int_{S^{n-1}} |f (r\theta)|^2 r^{n-1} d\theta dr. \quad (17)$$

Fortunately, this problem can be easily solved, see e.g. [7].
Theorem 5 Define $\overline{\Lambda_k} = \left( \frac{n}{2}, \ldots, \frac{n}{2} \right) + \Lambda_k$. Then for each $k \in \mathbb{N}_0$, $l = 1, \ldots, a_\mathcal{K}$, the following map, defined on $P_h \cap L^2(\mathbb{R}^n)$ by
\[
S \mapsto S_{k,l}(v) := e^{\frac{n}{2}v} \int_{S^{n-1}} S(e^\theta)^T Y_{k,l}(\theta) \, d\theta,
\]
maps onto $S_{hZ}(\overline{\Lambda_k}) \cap L^2(\mathbb{R}, dv)$, and by continuity it can be extended to a map from $PV_h$ onto $V_h(\overline{\Lambda_k})$. Further, $PV_h$ is isomorphic to $V_h := \bigoplus_{k \in \mathbb{N}_0, l=1, \ldots, a_\mathcal{K}} V_h(\overline{\Lambda_k})$.

Proof of Theorem 1. Let $f \in L^2(\mathbb{R}^n)$ and $g \in PV_h$. Then by the transformation rule (17)
\[
\|f - g\|_{L^2(\mathbb{R}^n)}^2 = \int_0^\infty \int_{S^{n-1}} |f(r\theta) - g(r\theta)|^2 r^{n-1} \, d\theta \, dr.
\]
Let $f_{k,l}$ and $g_{k,l}$ be the Laplace Fourier coefficients of $f$ and $g$ respectively as defined in (16). Note that $v \mapsto g_{k,l}(e^v) := e^{\frac{n}{2}v} f_{k,l}(e^v)$ is in $V_h(\overline{\Lambda_k})$. Since $Y_{k,l}(\theta)$ constitutes an orthonormal basis we obtain
\[
\|f - g\|_{L^2(\mathbb{R}^n)}^2 = \sum_{k=0}^\infty \sum_{l=1}^{a_\mathcal{K}} \int_{-\infty}^{\infty} |f_{k,l}(e^v) - g_{k,l}(e^v)|^2 e^{nv} \, dv.
\]
Minimizing the expression $g \mapsto \|f - g\|_{L^2(\mathbb{R}^n)}^2$ for $g \in PV_h$ is equivalent to minimizing the expression
\[
\int_{-\infty}^{\infty} \left| e^{\frac{n}{2}v} f_{k,l}(e^v) - g_{k,l}(e^v) \right|^2 \, dv
\]
for each $k \in \mathbb{N}$, $l = 1, \ldots, a_\mathcal{K}$, where $g_{k,l} \in V_h(\overline{\Lambda_k})$. Theorem 2 applied to $\Lambda = \overline{\Lambda_k}$ (hence $N + 1 = 2p$) shows that there exists a constant $C_p > 0$ which only depends on $p$ (and not on the values $\lambda_j$ in $\overline{\Lambda_k}$) such that
\[
E \left( e^{\frac{n}{2}v} f_{k,l}(e^v), V_h(\overline{\Lambda_k}) \right) \leq h^{2p} \cdot C_p \left\| P_{\overline{\Lambda_k}} \cdot e^{\frac{n}{2}v} f_{k,l}(e^v) \right\|_{L^2(\mathbb{R})}.
\]
Put $G_{k,l}(v) := e^{\frac{n}{2}v} f_{k,l}(e^v)$. A simple computation (using Parseval’s identity and the fact that differentiation becomes multiplication via Fourier transform) shows that
\[
\frac{1}{2\pi} \left\| P_{\overline{\Lambda_k}} \cdot \overline{G_{k,l}} \right\|_{L^2(\mathbb{R})}^2 = \int_{-\infty}^{\infty} |M_{\overline{\Lambda_k}} G_{k,l}(v)|^2 \, dv.
\]
A calculation shows that $M_{\kappa_{k}}(e^{\frac{2}{n}v}f_{k,l}(e^{v})) = e^{\frac{2}{n}v}M_{\lambda_{k}}(f(e^{v}))$. Then (20) and (21) yield
\[
E(f, PV_{h})^{2} \leq h^{4p} \cdot 2\pi C_{p}^{2} \sum_{k=0}^{\infty} \sum_{l=1}^{a_{k}} \| e^{\frac{2}{n}v}M_{\lambda_{k}}(f(e^{v})) \|_{L_{2}(\mathbb{R}^{n})}^{2}.
\]

The next theorem applied to the case $p = q$ finishes the proof. ■

**Theorem 6** Let $p, q \in \mathbb{N}_{0}$ and define $\| f(x) \|_{q,p}^{2} := \int |x|^{2q} \cdot \Delta^{p} f(x) \|^{2} dx$ for $f \in L_{2}(\mathbb{R}^{n})$. Then
\[
\| f(x) \|_{q,p}^{2} = \sum_{k=0}^{\infty} \sum_{l=1}^{a_{k}} \int |e^{v(2q-2p+\frac{2}{n})}M_{\lambda_{k}}(f_{k,l}(e^{v}))|^{2} dv
\]
where $f_{k,l}(r)$ are the Laplace-Fourier coefficients of $f$ defined as in equality (16).

**Proof.** Assume that $f(r\theta) = f_{k,l}(r)Y_{k,l}(\theta)$. Since $\Delta^{p} f(x) = L_{k}^{p} f_{k,l}(r)Y_{k,l}(\theta)$ we obtain
\[
\| f(x) \|_{q,p}^{2} = \int_{0}^{\infty} \int_{S^{n-1}} \left| r^{2q}L_{k}^{p} Y_{k,l}(\theta) \right|^{2} r^{n-1} dr d\theta.
\]
The integration over $\theta$ only gives a factor 1. Now we change the variable $r = e^{v}$ and apply the identity $(L_{k}^{p} f_{k,l})(e^{v}) = e^{-2vp}M_{\lambda_{k}}(f_{k,l}(e^{v}))$, see e.g. Theorem 10.34 in [9]. Then
\[
\| f(x) \|_{q,p}^{2} = \int |e^{2qv}e^{-2vp}M_{\lambda_{k}}(f_{k,l}(e^{v}))|^{2} e^{nv} dv.
\]
Finally we see that for arbitrary $f \in L_{2}(\mathbb{R}^{n})$ the result follows via the orthogonal decomposition of $f$ in spherical harmonics. ■

**References**


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