The approximation order of polysplines

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March 1, 2014

Abstract

We show that the scaling spaces defined by the polysplines of order $p$ provide approximation order $2p$. For that purpose we refine the results on one-dimensional approximation order by $L$-splines obtained in [2].

AMS (MOS) Classification. Primary 41A15; Secondary 35J40, 31B30.

Key words and phrases. cardinal splines, cardinal $L$-splines, polysplines, approximation order of splines, polyharmonic functions, cardinal polysplines.

Acknowledgement. The first author was partially sponsored by the Fulbright Program during his stay at the University of Wisconsin–Madison. The second author thanks the Alexander von Humboldt–Foundation for supporting him in the framework of the Feodor–Lynen–program.

1 Introduction

In the last decade the approximation order of shift-invariant subspaces of the space $L^2(\mathbb{R}^n)$ of all square-integrable functions on the euclidean space $\mathbb{R}^n$ has been investigated extensively, e.g. in the survey paper [10] approximately 100 references are given. The problem can be formulated in a rather general way: suppose that $(V_h)_{h \in I}$ is a family of subspaces of $L^2(\mathbb{R}^n)$ (not necessarily shift-invariant) where $I$ is subset of $(0, \infty)$ having 0 as an accumulation point. One has to estimate the rates of decay of the approximation error

$$E(f, V_h) := \inf \left\{ \| f - s \|_{L^2(\mathbb{R}^n)} : s \in V_h \right\}$$

(1)
for \( h \) tending to 0. If \( W \) is a subspace of \( L^2(\mathbb{R}^n) \) endowed with a norm \( \| \cdot \|_W \) we say that \( (V_h)_{h \in I} \) provides approximation order \( m \) with respect to the norm \( \| \cdot \|_W \) if there exists a constant \( c_W \) such that for every \( f \in W \) and for every \( h \in I \)
\[
E(f, V_h) \leq c_W \cdot h^m \cdot \| f \|_W .
\]
(2)

Usually \( W \) is the potential space \( W^m_2(\mathbb{R}^n) \) for \( m \in (0, \infty) \) defined as the subspace of those \( f \in L^2(\mathbb{R}^n) \) such that
\[
\| f \|_{W^m_2(\mathbb{R}^n)} := (2\pi)^{-\frac{n}{2}} \left\| (1 + |\xi|^m \hat{f}(\xi)) \right\|_{L^2(\mathbb{R}^n)} < \infty.
\]
(3)

In this note we want to prove that cardinal polysplines of order \( p \) provide approximation order \( 2p \).

The motivation for the present work comes from the fact that polysplines are useful for solving multivariate interpolation problems [4], [5], [6] and they are of importance for the multivariate Wavelet Analysis, cf. the monograph [9]. Recall that a function \( S : \mathbb{R}^n \setminus \{0\} \to \mathbb{C} \) is called a cardinal polyspline\(^1\) (on annuli) of order \( p \) if \( S \) is \((2p - 2)\)-times continuously differentiable and the restriction of \( S \) to each open annulus \( \{x \in \mathbb{R}^n : e^l < |x| < e^{l+1}\} \) is a polyharmonic function\(^2\) of order \( p \) for \( l \in \mathbb{Z} \). The reason for calling such polysplines "cardinal" is found in Theorem 3 where it is seen that after expanding \( S \) in a Fourier–Laplace series of spherical harmonics the coefficients \( S_{k,l}(\log r) \) are cardinal \( L \)-splines in the usual sense of the word, cf. Micchelli’s paper of 1976 [8].

Introducing a parameter \( h > 0 \), by \( P_h \) we denote the set of all functions \( S : \mathbb{R}^n \setminus \{0\} \to \mathbb{C} \) which are \((2p - 2)\)-times continuously differentiable and whose restriction to each open annulus \( A_{h,l} := \{x \in \mathbb{R}^n : e^{hl} < |x| < e^{h(l+1)}\} \) is a polyharmonic function of order \( p \) for \( l \in \mathbb{Z} \). Then the scaling spaces of polysplines of order \( p \), shortly \( PV_h \), are defined as the \( L^2 \)-closure of \( P_h \cap L^2(\mathbb{R}^n), h > 0 \).

The main result is the following:

**Theorem 1** The sequence \( (PV_h)_{h>0} \) provides approximation order \( 2p \) where \( p \) denotes the order of the polysplines. More precisely, there exists a constant

\(^1\)The first author introduced in 1991 polysplines in a more general setting with arbitrary interfaces, see [3] and [9].

\(^2\)Recall that a function \( f \) defined on an open set \( U \) in the euclidean space \( \mathbb{R}^n \) is polyharmonic of order \( p \) if \( f \) is \( 2p \)-times continuously differentiable and \( \Delta^p f(x) = 0 \) for all \( x \in U \) where \( \Delta \) is the Laplace operator and \( \Delta^p \) its \( p \)-th iterate.
\( C > 0 \) such that for all \( h \) with \( 0 < h < 1 \) and \( f \in L^2(\mathbb{R}^n) \) the following inequality holds

\[
\inf \left\{ \| f - g \|_{L^2(\mathbb{R}^n)} : g \in PV_h \right\} \leq C \cdot h^{2p} \cdot \left( \int_{\mathbb{R}^n} |x|^{2p} \cdot \Delta^p f(x) \, dx \right)^{1/2}.
\]

Note that in place of the norm (3) we have a semi–norm on the right–hand side which is zero on the polyharmonic functions of order \( p \).

The paper is organized as follows: in the next Section we discuss the approximation order of cardinal \( L \)–splines by using important results from [2]. In Section 3 the main result will be proven.

2 Approximation order of cardinal \( L \)–splines

Let us recall Theorem 4.3 in the fundamental paper [2]: Suppose that for every \( h > 0 \), the space \( S_h \) is the \( L^2(\mathbb{R}^n) \)–closure of the linear space generated by the shifts \( \varphi_h (\cdot - m), m \in \mathbb{Z}^n \) of the functions \( \varphi_h \in L^2(\mathbb{R}^n) \) (so \( S_h \) is the shift–invariant space generated by \( \varphi_h \)) and that \( V_h = \{ s(\xi) : s \in S_h \} \). Then the family \((V_h)_{h \in I}\) provides approximation order \( m \) with respect to the norm \( \| \cdot \|_{W^m_2(\mathbb{R}^n)} \) defined in (3) if and only if there exists \( D > 0 \) such that for all \( h \in I \) and for almost all \( x \in C := [-\pi, \pi]^n \)

\[
|\Lambda_{\varphi_h}(x)| \leq D \cdot (h + |x|^m),
\]

where

\[
(\Lambda_{\varphi_h}(\xi))^2 := \sum_{\alpha \in \mathbb{Z}^n, \alpha \neq 0} |\hat{\varphi_h}(\xi + 2\pi\alpha)|^2 \\
\sum_{\beta \in \mathbb{Z}^n} |\hat{\varphi_h}(\xi + 2\pi\beta)|^2 \leq 1.
\]

We will need a refinement of that result. For our purposes it will be useful to consider instead of (3) different norms. In the following we replace the function \((1 + |x|^m)\) by a measurable function \(Q(x)\) with the following properties: (i) the zero set \(Q^{-1}(0)\) of \(Q\) is a set of Lebesgue measure zero and (ii) there exists a constant \( D_1 > 0 \) such that

\[
|Q\left(\frac{x}{h}\right)| \geq D_1 \frac{1}{h^m} \quad \text{for all } x \notin C := [-\pi, \pi]^n.
\]

Suppose further that there exists a constant \( D_2 > 0 \) such that for all \( x \in C \) and for all \( 0 < h < 1 \)

\[
|\Lambda_{\varphi_h}(hx)| \leq h^m D_2 |Q(x)|.
\]
An analysis of the proof in [2] shows that then the following inequality holds (for us the constants $D_1$ and $D_2$ defined in the formula will be very important!)

$$E(f, V_h) \leq \left( D_2 (2\pi)^2 + \frac{1}{D_1 (2\pi)^2} \right) \cdot h^m \cdot \left\| Q(\xi) \hat{f}(\xi) \right\|_{L_2(\mathbb{R}^n)}. \quad (7)$$

Let us recall some facts about $L$-splines: Let $L$ be a linear differential operator with constant coefficients of order $N + 1$, say

$$L = M_\Lambda := \prod_{j=1}^{N+1} \left( \frac{d}{dv} - \lambda_j \right) \quad \text{where } \Lambda := (\lambda_1, ..., \lambda_{N+1}). \quad (8)$$

Then a function $u : \mathbb{R} \to \mathbb{R}$ is called cardinal $L$-spline on the mesh $h\mathbb{Z}$ ($h > 0$) if $u$ is $(N-1)$-times continuously differentiable and if for every $l \in \mathbb{Z}$ there exists $f_l \in U_L := \{ f \in C^\infty(\mathbb{R}) : Lf = 0 \}$ such that $u(t) = f_l(t)$ for all $t \in (lh, (l+1)h)$. The set of all cardinal $L$-splines for the operator $L = M_\Lambda$ on $h\mathbb{Z}$ will be denoted by $S_{h\mathbb{Z}}(\Lambda)$. The scaling spaces $V_h(\Lambda)$ are defined by

$$V_h(\Lambda) = L^2(\mathbb{R}) \text{-closure of } S_{h\mathbb{Z}}(\Lambda) \cap L^2(\mathbb{R}). \quad (9)$$

Let $Q_\Lambda$ be the basic spline which can be defined by its Fourier transform by

$$\widehat{Q_\Lambda}(\xi) = \frac{\prod_{j=1}^{N+1} (e^{-\lambda_j} - e^{-i\xi})}{\prod_{j=1}^{N+1} (i\xi - \lambda_j)}. \quad (10)$$

**Theorem 2** Let $N \in \mathbb{N}$ be fixed. Then there exists a constant $D > 0$ such that for all $\Lambda = (\lambda_1, ..., \lambda_{N+1}) \in \mathbb{R}^{N+1}$ and for all $f \in L_2(\mathbb{R})$ the following inequality holds

$$E(f, V_h(\Lambda)) \leq h^{N+1} \cdot D \left\| P_\Lambda(\xi) \hat{f}(\xi) \right\|_{L_2(\mathbb{R})}; \quad (11)$$

where the polynomial $P_\Lambda(x) = \prod_{j=1}^{N+1} (ix - \lambda_j)$.

**Remark 3** Note that if we used the usual Sobolev norm (3) then we could not be able to obtain the sharp constant $D$ of inequality (11); the last is the main virtue of Theorem 2.
Proof. By the above we have to check (5) and (6). Note that for $Q := P$ we have the estimate
\begin{equation}
\left| P_A \left( \frac{x}{h} \right) \right|^2 = \prod_{j=1}^{N+1} \left( \frac{x}{h} \right)^2 + \lambda_j^2 \geq \pi^{2(N+1)} \frac{1}{h^{2(N+1)}} \end{equation}
for all $|x| \geq \pi$ and for all $h > 0$. Hence it suffices to show that
\begin{equation}
|A_{\varphi_h} (h \xi)|^2 \leq h^{2(N+1)} |P_A (\xi)|^2 \sum_{\alpha \in \mathbb{Z}, \alpha \neq 0} \frac{1}{(\pi |\alpha|)^2(N+1)} \end{equation}
The trivial inequality $(A_{\varphi_h} (\xi))^2 \leq \frac{\sum_{\alpha \in \mathbb{Z}, \alpha \neq 0} |Q_{hA} (\xi + 2\pi \alpha)|^2}{|\varphi_h (\xi)|^2}$ and the estimate
\begin{equation}
\frac{|Q_{hA} (\xi + 2\pi \alpha)|^2}{|Q_{hA} (\xi)|^2} = \prod_{j=1}^{N+1} \left| \frac{i \xi - h \lambda_j}{i (\xi + 2\pi \alpha) - h \lambda_j} \right|^2 \end{equation}
yields
\begin{equation}
|A_{\varphi_h} (h \xi)|^2 \leq h^{2(N+1)} \prod_{j=1}^{N+1} (\xi^2 + \lambda_j^2) \sum_{\alpha \in \mathbb{Z}, \alpha \neq 0} \prod_{j=1}^{N+1} \frac{1}{(h \xi + 2\pi \alpha)^2 + h^2 \lambda_j^2}. \end{equation}
Since $(h \xi + 2\pi \alpha)^2 + h^2 \lambda_j^2 \geq (h \xi + 2\pi \alpha)^2 \geq (2\pi |\alpha| - |h \xi|)^2$ we obtain for $0 < h < 1$ and $|\xi| \leq \pi$ the estimate $2\pi |\alpha| - |h \xi| \geq \pi |\alpha|$ (since $\alpha \neq 0$) arriving at (13). $\blacksquare$

3 The approximation order of Polysplines

Let $S^{n-1} = \{ x \in \mathbb{R}^n; |x| = 1 \}$ be the unit sphere. Each $x \in \mathbb{R}^n$ will be written in spherical coordinates $x = r \theta$ with $r \geq 0$ and $\theta \in S^{n-1}$. Recall that a function $Y : S^{n-1} \to \mathbb{C}$ is a spherical harmonic of degree $k \in \mathbb{N}_0$ if there exists a homogeneous harmonic polynomial $P (x)$ of degree $k$ such that $P (\theta) = Y (\theta)$ for all $\theta \in S^{n-1}$. The set $\mathcal{H}_k$ of all spherical harmonics of degree exactly $k$ is a linear space of dimension $a_k := \dim \mathcal{H}_k = \binom{n+k-1}{k} - \binom{n+k-3}{k-2}$. We denote by $Y_{k,l}$ with $l = 1, 2, ..., a_k$ a basis for $\mathcal{H}_k$. For a detailed account we refer to Stein–Weiss [12].
Let \( u : (R_1, R_2) \rightarrow \mathbb{C} \) be infinitely differentiable and \( Y_k \in \mathcal{G}_k \). Then it is well known that \( \Delta (u (r) Y_k (\theta)) = Y_k (\theta) L_{(k)} u (r) \) where we have put

\[
L_{(k)} = \frac{d^2}{dr^2} + \frac{n - 1}{r} \frac{d}{dr} - \frac{k (k + n - 2)}{r^2}.
\]

By iteration we have \( \Delta^p u = Y_k (\theta) \cdot [L_{(k)}]^p u (r) \). Let us put for convenience

\[
\Lambda_+ (k, p) := \{ k, k + 2, \ldots, k + 2p - 2 \},
\]

\[
\Lambda_- (k, p) := \{ -k - n + 2, -k - n + 4, \ldots, -k - n + 2p \}.
\]

The space of solutions of the equation \( L_{(k)}^p f (r) = 0 \) which are \( C^\infty \) for \( r > 0 \) is generated by a simple basis: for \( j \in \Lambda_+ (k, p) \cup \Lambda_- (k, p) \) the function \( r^j \) is clearly a solution, while for \( j \in \Lambda_+ (k, p) \cap \Lambda_- (k, p) \) we obtain a second solution \( r^j \log r \). It will be convenient to make a transform of the variable \( r \) to \( v = \log r \). Then a solution of the form \( r^j \) will be transformed to \( e^{jv} \) and a solution of the form \( r^j \log r \) is transformed to \( ve^{jv} \). We see immediately that all solutions to the equation \( L_{(k)}^p f (r) = 0 \) are transformed to solutions of the equation \( M_{(k)} g (v) = 0 \) where \( M_{(k)} \) is defined by (8) with respect to the vector

\[
\Lambda_k := (k, k + 2, \ldots, k + 2 (p - 1), - (k + n) + 2, \ldots, - (k + n) + 2p).
\]

The dependence on the parameter \( p \) and \( n \) will be suppressed throughout the paper.

A proof of the following can be found in [6], [9, Theorem 9.7].

**Theorem 4** Let \( S : \mathbb{R}^n \setminus \{ 0 \} \rightarrow \mathbb{R} \) be a polyspline of order \( p \). Then the Laplace-Fourier coefficient \( S_{k,l} : \mathbb{R} \rightarrow \mathbb{R} \) defined by

\[
S_{k,l} (v) := \int_{S_n^{n-1}} S (e^v \theta) Y_{k,l} (\theta) d\theta
\]

is a cardinal \( L \)-spline with respect to the linear differential operator \( M_{(k)} \).

We want to characterize the \( L^2 (\mathbb{R}^n) \)-closure \( PV_h \). It is a temptation to assume that for \( S \in PV_h \) the Fourier-Laplace coefficient defined through formula (16) will be in \( V_h (\Lambda_k) \), i.e. in the closure of \( S_h (\Lambda_k) \cap L_2 (\mathbb{R}) \). This is not true since the transformation rule will give us an additional weight for \( f \in L_2 (\mathbb{R}^n) \):

\[
\int_{\mathbb{R}^n} \| f (x) \|^2 dx = \int_0^\infty \int_{S_n^{n-1}} |f (r \theta)|^2 r^{n-1} d\theta dr.
\]

Fortunately, this problem can be easily solved, see e.g. [7].
Theorem 5 Define \( \overline{\Lambda}_k = \left( \frac{n}{2}, \ldots, \frac{n}{2} \right) + \Lambda_k \). Then for each \( k \in \mathbb{N}_0, l = 1, \ldots, a_k \), the following map, defined on \( P_h \cap L^2(\mathbb{R}^n) \) by

\[
S \mapsto S_{k,l}(v) := e^{\frac{n}{2}v} \int_{S^{n-1}} S(e^{\theta}v) Y_{k,l}(\theta) d\theta,
\]

maps onto \( S_{h,\mathbb{Z}}(\overline{\Lambda}_k) \cap L^2(\mathbb{R}, dv) \), and by continuity it can be extended to a map from \( PV_h \) onto \( V_h(\overline{\Lambda}_k) \). Further, \( PV_h \) is isomorphic to

\[
V_h := \bigoplus_{k \in \mathbb{N}_0, l = 1, \ldots, a_k} V_h(\overline{\Lambda}_k).
\]

Proof of Theorem 1. Let \( f \in L^2(\mathbb{R}^n) \) and \( g \in PV_h \). Then by the transformation rule (17)

\[
\|f - g\|_{L^2(\mathbb{R}^n)}^2 = \int_0^\infty \int_{S^{n-1}} |f(r\theta) - g(r\theta)|^2 r^{n-1} d\theta dr.
\]

Let \( f_{k,l} \) and \( g_{k,l} \) be the Laplace Fourier coefficients of \( f \) and \( g \) respectively as defined in (16). Note that \( v \mapsto g_{k,l}(e^v) := e^{\frac{n}{2}v} g_{k,l}(e^v) \) is in \( V_h(\overline{\Lambda}_k) \). Since \( Y_{k,l}(\theta) \) constitutes an orthonormal basis we obtain

\[
\|f - g\|_{L^2(\mathbb{R}^n)}^2 = \sum_{k=0}^\infty \sum_{l=1}^{a_k} \int_{-\infty}^{\infty} |f_{k,l}(e^v) - g_{k,l}(e^v)|^2 e^{nv} dv.
\]

Minimizing the expression \( g \mapsto \|f - g\|_{L^2(\mathbb{R}^n)}^2 \) for \( g \in PV_h \) is equivalent to minimizing the expression

\[
\int_{-\infty}^{\infty} |e^{\frac{n}{2}v} f_{k,l}(e^v) - \overline{g_{k,l}(e^v)}|^2 dv
\]

for each \( k \in \mathbb{N}, l = 1, \ldots, a_k \), where \( \overline{g_{k,l}} \in V_h(\overline{\Lambda}_k) \). Theorem 2 applied to \( \Lambda = \overline{\Lambda}_k \) (hence \( N + 1 = 2p \)) shows that there exists a constant \( C_p > 0 \) which only depends on \( p \) (and not on the values \( \lambda_j \) in \( \overline{\Lambda}_k \)) such that

\[
E \left( e^{\frac{n}{2}v} f_{k,l}(e^v), V_h(\overline{\Lambda}_k) \right) \leq \hbar^{2p} \cdot C_p \left\| P_{\overline{\Lambda}_k} \cdot e^{\frac{n}{2}v} f_{k,l}(e^v) \right\|_{L^2(\mathbb{R})}.
\]

Put \( G_{k,l}(v) := e^{\frac{n}{2}v} f_{k,l}(e^v) \). A simple computation (using Parseval’s identity and the fact that differentiation becomes multiplication via Fourier transform) shows that

\[
\frac{1}{2\pi} \left\| P_{\overline{\Lambda}_k} \cdot \overline{G_{k,l}} \right\|_{L^2(\mathbb{R})}^2 = \int_{-\infty}^{\infty} |M_{\overline{\Lambda}_k} G_{k,l}(v)|^2 dv.
\]
A calculation shows that \( M_{X_k} (e^{\frac{2}{n}f_{k,l}(e^v)}) = e^{\frac{2}{n}v}M_{\Lambda_k} (f(e^v)) \). Then (20) and (21) yield

\[
E(f, PV_h)^2 \leq h^4p \cdot 2\pi C_p^2 \sum_{k=0}^\infty \sum_{l=1}^{a_k} \left\| e^{\frac{2}{n}v}M_{\Lambda_k} (f(e^v)) \right\|_{L_2(\mathbb{R}^n)}^2.
\]

The next theorem applied to the case \( p = q \) finishes the proof. ■

**Theorem 6** Let \( p, q \in \mathbb{N}_0 \) and define \( \| f(x) \|_{q,p}^2 := \int \| x \|^{2q} \cdot \Delta^p f(x) \|^2 \, dx \) for \( f \in L_2(\mathbb{R}^n) \). Then

\[
\| f(x) \|_{q,p}^2 = \sum_{k=0}^\infty \sum_{l=1}^{a_k} \int |e^{v(2q-2p+\frac{2}{n})}M_{\Lambda_k} (f_{k,l}(e^v))|^2 \, dv
\]

where \( f_{k,l}(r) \) are the Laplace-Fourier coefficients of \( f \) defined as in equality (16).

**Proof.** Assume that \( f(r\theta) = f_{k,\ell}(r)Y_{k,\ell}(\theta) \). Since \( \Delta^p f(x) = L_k^p f_{k,\ell}(r)Y_{k,\ell}(\theta) \) we obtain

\[
\| f(x) \|_{q,p}^2 = \int_0^\infty \int_{\mathbb{S}^{n-1}} \left| r^{2q}L_{(k)}^p f_{k,\ell}(r)Y_{k,\ell}(\theta) \right|^2 r^{n-1} \, d\theta \, dr.
\]

The integration over \( \theta \) only gives a factor 1. Now we change the variable \( r = e^v \) and apply the identity \((L_k^p f_{k,\ell})(e^v) = e^{-2vp}M_{\Lambda_k} (f_{k,\ell}(e^v))\), see e.g. Theorem 10.34 in [9]. Then

\[
\| f(x) \|_{q,p}^2 = \int |e^{2qv}e^{-2vp}M_{\Lambda_k} (f_{k,\ell}(e^v))|^2 e^{nv} \, dv.
\]

Finally we see that for arbitrary \( f \in L^2(\mathbb{R}^n) \) the result follows via the orthogonal decomposition of \( f \) in spherical harmonics. ■

**References**


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