<table>
<thead>
<tr>
<th>Title</th>
<th>An irregular lattice method for elastic wave propagation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Authors(s)</td>
<td>O'Brien, G. S.; Bean, Christopher J.</td>
</tr>
<tr>
<td>Publication date</td>
<td>2011-10</td>
</tr>
<tr>
<td>Publisher</td>
<td>Oxford University Press</td>
</tr>
<tr>
<td>Item record/more information</td>
<td><a href="http://hdl.handle.net/10197/5660">http://hdl.handle.net/10197/5660</a></td>
</tr>
<tr>
<td>Publisher's statement</td>
<td>This article has been accepted for publication in Geophysical Journal International ©: 2011 the authors, the Royal Astronomical Society. Published by Oxford University Press on behalf of the Royal Astronomical Society. All rights reserved.</td>
</tr>
<tr>
<td>Publisher's version (DOI)</td>
<td>10.1111/j.1365-246X.2011.05229.x</td>
</tr>
</tbody>
</table>
An irregular lattice method for elastic wave propagation

Gareth S. O’Brien and Christopher J. Bean

School of Geological Sciences, University College Dublin, Belfield, Dublin 4, Ireland. E-mail: gareth.obrien@ucd.ie

Received 2011 September 9; in original form 2011 April 8

SUMMARY

Lattice methods are a class of numerical scheme which represent a medium as a connection of interacting nodes or particles. In the case of modelling seismic wave propagation, the interaction term is determined from Hooke’s Law including a bond-bending term. This approach has been shown to model isotropic seismic wave propagation in an elastic or viscoelastic medium by selecting the appropriate underlying lattice structure. To predetermine the material constants, this methodology has been restricted to regular grids, hexagonal or square in 2-D or cubic in 3-D. Here, we present a method for isotropic elastic wave propagation where we can remove this lattice restriction. The methodology is outlined and a relationship between the elastic material properties and an irregular lattice geometry are derived. The numerical method is compared with an analytical solution for wave propagation in an infinite homogeneous body along with comparing the method with a numerical solution for a layered elastic medium. The dispersion properties of this method are derived from a plane wave analysis showing the scheme is more dispersive than a regular lattice method. Therefore, the computational costs of using an irregular lattice are higher. However, by removing the regular lattice structure the anisotropic nature of fracture propagation in such methods can be removed.

Key words: Numerical solutions; Computational seismology; Wave propagation.

1 INTRODUCTION

The physical theory behind seismic wave propagation is well understood and seismic waves can be modelled as waves propagating through geological materials with different rheologies, such as elastic, viscoelastic or poroelastic media, see Aki & Richards (2002). The wave equation can be solved in a variety of ways, for example, analytical solutions for simple models (e.g. Carcione 2007), ray tracing approximations (e.g. Brokešová 2006), finite-difference methods (e.g. Virieux 1986) and spectral element methods (e.g. Komatitsch & Vilotte 1998). However, because the Earth is heterogeneous at all scales, analytical solutions to the elastic, viscoelastic and poroelastic wave equations are very limited in their applicability to geological problems and numerical solutions are becoming common tools used by seismologist to study the Earth’s interior.

Seismic wave propagation is governed by a wave equation. This wave equation is a continuum equation in which the variables representing material properties are continuous within the medium. These variables are averaged over a representative elemental volume (REV). The REV is defined such that regardless where in space the volume is placed the averages obtained are statistically meaningful. The averages obtained must be independent of the REV size and the REV must be smaller than the overall size of the system. Once average variables are defined, the equations of state are derived from conservation laws and momentum and force balance equations. The most common techniques for solving seismic wave propagation utilise the continuum approach. An alternative approach to continuum mechanics is to use a discrete method.

These discrete methods do not solve continuum equations by any approximate numerical method. Instead, the aim is to replicate the underlying physics at a ‘microscopic scale’ employing discrete micromechanical interaction rules between discrete material particles or nodes. In most cases, by assuming some form of linearization or approximate global behaviour, a macroscopic continuum equation can be derived from the local interactions. It should be stressed that the schemes are not solving the macroscopic continuum equation but this macroscopic behaviour arises through the local mesoscopic interactions. A class of discrete methods using particle–particle mechanical interactions (e.g. Hooke’s law) are capable of modelling seismic waves propagating in a heterogeneous Earth model. These discrete particle–particle methods, also called elastic lattice methods (ELM), originate from solid-state physics models of crystalline materials (Hoover et al. 1974). As these schemes solely rely on particle–particle mechanical interactions, complex boundary conditions can be readily implemented which makes them ideally suited to geological processes. One of the first discrete particle methods was developed by Cundall & Strack (1979) for granular material and is a mesh-free method. These mesh-free or irregular lattices have been primarily used to study static deformation and fracturing. They are not suitable for wave propagation as the methods have to be calibrated so the P-wave velocity and Poisson’s ratio can be found. The P-wave velocity may also be heterogeneous and anisotropic due to the irregular lattice.

Regular lattices have been used to overcome the difficulties in deriving material constants and have been used to model seismic waves by several authors but they have lost the advantage of a
A lattice method consists of particles (filled circles) connected by springs shown as solid lines. The particle here is connected to 96 other particles with the outer bound of the interaction zone shown by the solid line. By increasing the number of neighbours irregular lattices such as the distribution shown here can be used to model isotropic elastic wave propagation.

where $K_{in}$ is the elastic spring constant between nodes $i$ and $n$, $G_{nim}$ is the bond-bending constant and $\theta_{nim}$ is the angle between particles $nim$ with node $i$ being the apex of the angle. $r_{im}$ is the vector connecting nodes $x_i$ and $x_n$ with a superscript * representing the undistorted lattice. The energy density $\Phi$ is the total energy stored in each spring divided by the total volume and can be written as

$$\Phi = \frac{1}{4} \sum_{i=1}^{N} K_{in} (|r_{in}|-|r_{in}^o|)^2$$

$$+ \frac{1}{2} \sum_{nim} G_{nim} (\cos \theta_{nim} - \cos \theta_{nim}^o)^2. \quad (2)$$

To derive the macroscopic properties of the system, we assume the deformation is small and hence restricts the analysis to linear elasticity. However, this assumption only applies when deriving the macroscopic behaviour as the equations of motion of the particles are not restricted to linear elasticity. Linearising eq. (2) yields

$$\Phi = \frac{1}{4} \sum_{i=1}^{N} K_{in} (u_{in} \cdot r_{in})^2$$

$$+ \frac{1}{2} \sum_{nim} G_{nim} \frac{(u_{im} \cdot s_{im})}{|r_{in}|^2} \left( \frac{x_{in}}{|r_{in}|} - \frac{y_{in}}{|r_{in}|} \right)^2, \quad (3)$$

where the displacement $u_{in}$ is the displacement vector ($u_{i} - u_{n}$) and

$$s_{im} \equiv (s_{i}, s_{n}) = \left( \frac{x_{in}}{|r_{in}|}, -\frac{y_{in}}{|r_{in}|} \right). \quad (4)$$

Substituting the $x$- and $y$-displacement given by

$$u_{in} = x_{in} \frac{\partial u}{\partial x} + y_{in} \frac{\partial u}{\partial y} \quad (5)$$

$$v_{in} = x_{in} \frac{\partial v}{\partial x} + y_{in} \frac{\partial v}{\partial y} \quad (6)$$
into the linear elastic density (eq. 3) gives

$$\Phi = \frac{1}{4} \sum_{n} \frac{K_{n}}{|r_{n}|^{2}} \left( \frac{x_{n}^{2}}{a_{n}} \frac{\partial u}{\partial x} + \frac{y_{n}^{2}}{a_{n}} \frac{\partial v}{\partial y} \right)^{2} + \frac{1}{2} \sum_{n,m} G_{n,m} \left( a_{n,m} \frac{\partial u}{\partial x} + b_{n,m} \frac{\partial u}{\partial y} = c_{n,m} \frac{\partial v}{\partial x} - a_{n,m} \frac{\partial v}{\partial y} \right)^{2},$$

where

$$a_{n,m} = \frac{x_{n} y_{m}}{|r_{n}|}, \quad b_{n,m} = \frac{y_{n} y_{m}^{2}}{|r_{n}|}, \quad c_{n,m} = \frac{x_{n}^{2} y_{m}}{|r_{n}|}. \quad (7)$$

The elastic energy density of an isotropic elastic continuum (Landau & Lifshitz 1986) can be written in terms of the Lamé constants as follows:

$$\Phi = \left( \frac{\lambda}{2} + \mu \right) \left( \frac{\partial u}{\partial x} \right)^{2} + \left( \frac{\lambda}{2} + \mu \right) \left( \frac{\partial v}{\partial y} \right)^{2} + \lambda \left( \frac{\partial u}{\partial x} \right) \left( \frac{\partial v}{\partial y} \right) \mu \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)^{2}. \quad (9)$$

Expanding eq. (7) and comparing terms with the continuum energy density yields, the set of 10 equations relating the lattice geometry, number of neighbours and the spring and bond bending constants to the macroscopic material properties.

$$\sum_{n} K_{n} x_{n}^{2} y_{n}^{2} \frac{1}{|r_{n}|^{2}} + \sum_{n,m} G_{n,m} a_{n,m}^{2} \frac{1}{|r_{n}|^{2}} = \frac{\lambda}{2} + \mu,$$

$$\sum_{n} K_{n} x_{n}^{2} y_{n}^{2} \frac{1}{|r_{n}|^{2}} + \sum_{n,m} G_{n,m} b_{n,m}^{2} \frac{1}{|r_{n}|^{2}} = \frac{\lambda}{2} + \mu,$$

$$\sum_{n} K_{n} x_{n}^{2} y_{n}^{2} \frac{1}{|r_{n}|^{2}} + \sum_{n,m} G_{n,m} c_{n,m}^{2} \frac{1}{|r_{n}|^{2}} = \mu \frac{1}{2},$$

$$\sum_{n} K_{n} x_{n}^{2} y_{n}^{2} \frac{1}{|r_{n}|^{2}} + \sum_{n,m} G_{n,m} a_{n,m} b_{n,m} \frac{1}{|r_{n}|^{2}} = 0,$$

$$\sum_{n} K_{n} x_{n}^{2} y_{n}^{2} \frac{1}{|r_{n}|^{2}} + \sum_{n,m} G_{n,m} a_{n,m} c_{n,m} \frac{1}{|r_{n}|^{2}} = 0,$$

$$\sum_{n} K_{n} x_{n}^{2} y_{n}^{2} \frac{1}{|r_{n}|^{2}} + \sum_{n,m} G_{n,m} b_{n,m} c_{n,m} \frac{1}{|r_{n}|^{2}} = 0.$$

Therefore, if the lattice method is to model an isotropic elastic continuum in the limit of linear elasticity, the set of 10 equations in (10) above must be satisfied. Several different strategies can be employed to solve this system of equations. By using a regular lattice, the equations can easily be solved for a fixed set of spring and bond-bending constants. A solution of this type for a hexagonal and square lattice is given in Monette & Anderson (1992) and a solution for a cubic lattice is given in O’Brien & Bean (2004). Another possible strategy is to remove the restriction on the regular lattice. This then requires that each spring and bond-bending constant becomes a free parameter and the set of equations for every node must be solved for every $K_{n}$ and $G_{n,m}$. This leads to an inversion problem that can be intractable. We will not address this strategy but highlight two problems. First, it may be that no solution may exist and secondly and more likely, because $K_{n}$ is a free parameter the solution will involve negative spring constants which are physically unrealistic.

The approach set out later is to keep the advantage of a mesh-free grid while not having to solve the inversion problem. To do this we assume the spring constants $K$ are fixed for every lattice link. For the system of equations to be consistent under this assumption, eq. (11) must hold.

$$\sum_{n} x_{n}^{2} \frac{1}{|r_{n}|^{2}} = \sum_{n} y_{n}^{2} \frac{1}{|r_{n}|^{2}} = \frac{1}{3} \sum_{n} x_{n}^{2} y_{n}^{2} \frac{1}{|r_{n}|^{2}},$$

$$\sum_{n,m} a_{n,m}^{2} = \sum_{n,m} b_{n,m}^{2} = \sum_{n,m} c_{n,m}^{2} = -\sum_{n,m} b_{n,m} c_{n,m},$$

$$\sum_{n} x_{n}^{2} y_{n}^{2} \frac{1}{|r_{n}|^{2}} = \sum_{n} x_{n}^{2} y_{n}^{2} \frac{1}{|r_{n}|^{2}} = 0, \quad (11)$$

Consider a circle with radius $R$ centred on the origin (Fig. 1). If we randomly populate the circle with particles and calculate the sums in eq. (11) using the positions, we can see that for a large enough sample the equations in (11) hold. To show this is true, as the number of particles increases we can replace the sums with the integral equations. The first line of equations in (11) becomes

$$\sum_{n} x_{n}^{2} \frac{1}{|r_{n}|^{2}} \rightarrow \int_{0}^{R} \frac{r^{2}}{2} \cos^{2}(\theta) d\theta = \frac{R^{2}}{8},$$

$$\sum_{n} y_{n}^{2} \frac{1}{|r_{n}|^{2}} \rightarrow \int_{0}^{R} \frac{r^{2}}{2} \sin^{2}(\theta) d\theta = \frac{R^{2}}{8},$$

$$\sum_{n} x_{n}^{2} y_{n}^{2} \frac{1}{|r_{n}|^{2}} \rightarrow \int_{0}^{R} \frac{r^{2}}{2} \cos^{2}(\theta) \sin^{2}(\theta) d\theta = \frac{R^{2}}{24},$$

$$\sum_{n} x_{n}^{2} y_{n}^{2} \frac{1}{|r_{n}|^{2}} \rightarrow \int_{0}^{R} \frac{r^{2}}{2} \cos(\theta) \sin(\theta) d\theta = 0,$$

$$\sum_{n} x_{n}^{2} y_{n}^{2} \frac{1}{|r_{n}|^{2}} \rightarrow \int_{0}^{R} \frac{r^{2}}{2} \cos(\theta) \sin(\theta) d\theta = 0,$$

$$p(r) = R \quad p(\theta) = 2 \pi. \quad (12)$$

where the Cartesian coordinates have been transformed to polar coordinates. The probability density functions $p(r)$ and $p(\theta)$ are for a uniform distribution. From the equations it can be clearly seen that as the number of neighbours tends to infinity the required relationships hold. For the second set of equations in the set (11), a similar derivation applies.

$$\sum_{n} a_{n,m} \rightarrow \frac{1}{p(r_{n})} \frac{1}{p(\theta_{n})} \frac{1}{p(\theta_{n})} \int_{0}^{R} \int_{0}^{R} \int_{0}^{2\pi} F(\theta_{m}, \theta_{n}) d\theta_{m} = \frac{3}{16}. \quad (13)$$
\[ F(\theta_a, \theta_b) = (1 - \cos(\theta_a)\cos(\theta_b) - \sin(\theta_a)\sin(\theta_b)) \]
\[ (\cos(\theta_a)\cos(\theta_b) + 1 + \sin(\theta_a)\sin(\theta_b)) \]
\[ (\cos(\theta_a)\sin(\theta_b) - \cos(\theta_b)\sin(\theta_a))^2. \]  
(14)

Similarly, the same transformation to an integral equation can be performed for \( b_{\text{anim}}^m \), \( c_{\text{anim}}^m \), and \( a_{\text{anim}}b_{\text{anim}} \), \( a_{\text{anim}}c_{\text{anim}} \), and \( b_{\text{anim}}c_{\text{anim}} \) which satisfy the remaining equations. So by assuming the number of neighbours distributed randomly inside a circle of radius \( R \) tends to infinity, the elastic lattice method tends to an isotropic elastic continuum. A large number of particles are required to satisfy the integral solutions, however, the absolute value of the integral solution is not required only the relative values between the set of equations. These relative values are attained with enough accuracy to illustrate why the system tends to an elastic continuum.

The velocity-Verlet integration scheme is symplectic and time reversible, which implies good stability. For numerical stability, the Courant condition applies,
\[ \Delta t < \frac{\Delta x}{V_{\text{max}}}, \]  
(18)

where \( \Delta t \) is the time step, \( \Delta x \) is the minimum distance between two nodes and \( V_{\text{max}} \) is the maximum wave velocity. Heterogeneity can be incorporated into the model by changing the elastic spring constants on each spring. Topography is introduced by simply removing any particles above the required free surface. This imposes an implicit vacuum free-surface boundary condition. Previous lattice methods for wave propagation have solely relied on regular lattices to solve eq. (10). By arbitrarily increasing the number of neighbours and keeping the spring constant \( K \) and the bond-bending constant \( G \) fixed, the restriction on the regular lattice is removed. This is true if and only if the set of equations in eq. (11) hold. Whether this set of equations hold will depend on including a large enough number of neighbours over a large enough circle of influence.

### 3 Dispersion Analysis

Before we test the method outlined above against analytical and numerical solutions we first derive the dispersion relationship using von Neumann analysis. The velocity-Verlet scheme for the particle motion is given in eqs (19)–(22).

\[ u^{i+\Delta t} = u^i + \Delta t v^i + \frac{\Delta t^2 F^i}{2\rho}. \]  
(19)

\[ v^{i+\Delta t/2} = v^i + \frac{\Delta t^2 F^{i}}{2\rho}. \]  
(20)

\[ \mathbf{F}^{i+\Delta t} = \sum_{i=1}^{N} K_{\text{in}} (|r_{in}^i| - |r_{in}^0|) \cdot r_{in}^i + \sum_{n,m} G_{n,m} \left( \frac{u_{n}^i}{|r_{in}^i|^2} \right). \]  
(21)

\[ v^{i+\Delta t} = v^{i+\Delta t/2} + \frac{\Delta t^2 F_{\perp}^{i+\Delta t}}{2\rho}. \]  
(22)

The scheme is second order in time and can be approximated by substituting eqs (20) and (22) into eq. (19).

\[ u^{i+\Delta t} = 2u^i - u^{i-\Delta t} + \frac{\Delta t^2 F_{\perp}^i}{\rho}. \]  
(23)

Letting the displacement be a plane wave
\[ u = (u_0 e^{-i(\omega t + k_{\parallel} x)} , 0) \]  
(24)

where \( \omega \) is the angular frequency, \( k_{\parallel} \) is the wavenumber and \( v_0 \) is the wave amplitude. Substituting eq. (24) into eq. (23) and using eq. (21) to calculate the forces gives

\[ -4\sin^2 \left( \frac{\omega \Delta t}{2} \right) = \frac{\Delta t^2}{\rho} \sum_{i=1}^{N} K_{\text{in}} \left( \sqrt{(x_{in} + u_0 e^{ik_{\parallel}x + id_{in}s} - u_0)^2 + y_{in}^2} - |r_{in}^0| \right) \cdot r_{in}^0 + \sum_{n,m} G_{n,m} \left( \frac{u_{n}^0 e^{ik_{\parallel}x} - u_{n}^i}{|r_{in}^i|^2} \right), \]  
(25)

which can be rearranged to give the dispersion equation.

\[ \omega(k) \]

\[ = 2 \sin^{-1} \left( \frac{\Delta t}{2\sqrt{\rho}} \right) \left( \delta - \sum_{i=1}^{N} K_{\text{in}} \left( \sqrt{(x_{in} + u_0 e^{ik_{\parallel}x + id_{in}s} - u_0)^2 + y_{in}^2} - |r_{in}^0| \right) \cdot r_{in}^0 + \sum_{n,m} G_{n,m} \left( \frac{u_{n}^0 e^{ik_{\parallel}x} - u_{n}^i}{|r_{in}^i|^2} \right) \right) \]  
(26)

The dispersion curve is shown in Fig. 2 for a lattice where the particles were distributed on a square lattice with a length of \( \Delta x \). The position of each particle was then randomly distributed about the original point inside a circle of radius \( \Delta x/2 \). This irregular lattice was then assigned a \( P \)-wave velocity of 3500 m s\(^{-1}\), a \( S \)-wave velocity of 2200 m s\(^{-1}\) and a material density 2500 kg m\(^{-3}\) with \( \Delta x = 10 \) m and \( \Delta t = 5 \times 4 \) s. This then implies the minimum and maximum spatial grid step was 5 and 15 m, respectively. Each particle was assigned 96 neighbours. The dispersion curve shows both the \( P \)- and \( S \)-wave. The straight line shows the non-dispersive waves. The dispersion curve shows that a dense lattice is required to sample small wavelengths adequately. The dispersion curve for the lattice method using a square grid (eight neighbours) is shown for comparison and is clearly less dispersive for a smaller number of grid points per wavelength. The effect of changing the number of neighbours is shown in Fig. 3. As the number of neighbours increases, the scheme tends to the correct \( P \)- and \( S \)-wave velocities, although the dispersion characteristics do not improve as the
number of neighbours continually increases. The misfit between the expected phase velocities and the lattice method for a small number of particles is related to eq. (11). For a small number of neighbours, these equations do not hold and therefore the scheme does not converge to an elastic continuum. The effect of changing the time step can be assessed by comparing Fig. 4(a) and (b). The irregular lattice was generated in a similar manner to the examples above with a P-wave velocity of 3500 m s\(^{-1}\), a S-wave velocity of 2000 m s\(^{-1}\) and a material density 2500 kg m\(^{-3}\) with \(\Delta x = 10\) m. In Fig. 4(a), the time step was set to \(5e^{-4}\) s and to \(1e^{-3}\) s in Fig. 4(b). A smaller time step improves the dispersion properties as is expected from the regular lattice dispersion characteristics.

In Fig. 4c, the lattice structure was generated by distorting a square lattice in the same manner as above but increasing the distortion to \(2\Delta x\). This leads to a more dispersive grid when comparing Fig. 4(a) and (c). This can be attributed to the fact that the maximum distance between particles has increased in certain zones, therefore to have the same dispersion characteristics larger wavelengths are required. In the final panel of Fig. 4, the S-wave velocity is increased to 2500 m s\(^{-1}\). Similar behaviour to Fig. 4(a) is observed. In summary, the scheme has the same dispersion behaviour as a regular lattice method but requires a larger number of grid points per minimum wavelength. The actual number of grid points needed per minimum wavelength will depend on the required accuracy for specific problems.

4 NUMERICAL EXAMPLES

In this section, we test the accuracy of the scheme against an analytical solution and against other numerical solutions.
Figure 4. Dispersion curves for an irregular lattice method. (A) P-wave velocity of 3500 m s$^{-1}$, an S-wave velocity of 2000 m s$^{-1}$ and a density 2500 kg m$^{-3}$ with $\Delta x = 10$ m and $\Delta t = 5e^{-4}$ s. The dispersion curve shows both the P and S waves and the straight line for non-dispersive waves. (B) The time step was set to $1e^{-3}$ s with all other properties the same as (A). (C) The lattice structure was generated by distorting a square lattice in the same manner as (A) but increasing the distortion to $2\Delta x$. (D) The S-wave velocity is increased to 2500 m s$^{-1}$.

4.1 Homogeneous medium

An irregular lattice was constructed by deforming a square lattice. Each square node was randomly redistributed inside a circle centred on the original node of radius $\Delta x/2$. The P- and S-wave velocities were then set to 3500 and 2200 m s$^{-1}$, respectively. The density was set to 2500 kg m$^{-3}$ and the spatial and temporal grid steps were set to $\Delta x = 10$ m and $\Delta t = 5e^{-4}$ s. A 4 Hz Ricker wavelet function was used as a vertical force point source. Fig. 5 shows the vertical and horizontal wavefield after 1 s and the location of the

Figure 5. Vertical and horizontal velocity wavefields calculated after 1 s using the irregular lattice method in a homogeneous medium. The source was located at the centre of the model and the triangles show the location of the seismic traces in Figs 6 and 7.
Seismic recorders. The seismic traces calculated at these points were compared with an analytical solution taken from Carcione (2007) for an infinite homogenous viscoelastic medium. In this case, the Q factor was set to infinity. A comparison of the numerical and analytical solutions, along with the least squares misfit, is shown in Fig. 6. The least squares misfit is

\[
\text{Misfit} = \frac{\sum_{t=0}^{T} (S_{\text{elm}}(t) - S_{\text{ref}}(t))^2}{\sum_{t=0}^{T} S_{\text{ref}}(t)^2},
\]

(27)

where \(S_{\text{ref}}(t)\) is the reference seismogram and \(S_{\text{elm}}(t)\) is the random elastic lattice seismogram.

Three different cases were compared with the analytical solution, 64 nearest neighbours, 96 nearest neighbours and 128 nearest neighbours. We found that using less than approximately 50 neighbours gave unsatisfactory results. However, this number depends on the frequency, the lattice distortion and material properties. On close inspection of Fig. 5, some residual scatter can be observed trailing the main phases. This small scale scattering is related to the source input. By inputting the source over a finite region, this scatter can be eliminated. We have done this by inputting the source over a region with a Gaussian weight using the same initial conditions as shown in Fig. 5. The seismic traces are then compared with another numerical scheme (a square elastic lattice method; Fig. 7). The misfit between these two solutions decreases relative to the previous example as the small source fluctuations are removed. In Fig. 8, we have preformed the same simulation but in a material with a different Poisson’s ratio. This was done by changing the S-wave velocity to give a Poisson’s ratios of 0.17, 0.25 and 0.32. From the figure it can be seen that there is little difference in the misfit.
4.2 Heterogeneous medium

In this case we tested the random lattice method against a numerical solution for wave propagation in a layered elastic medium. The same lattice construction was used in this case as in the previous example along with the same spatiotemporal grid steps. The material properties in the top layer were set to 2000 m s\(^{-1}\), 1250 m s\(^{-1}\) and 2000 kg m\(^{-3}\) for the \(P\)-wave, \(S\)-wave and density, respectively. The material properties in the next layer down were set to 2500 m s\(^{-1}\), 1500 m s\(^{-1}\) and 2300 kg m\(^{-3}\) for the \(P\)-wave, \(S\)-wave and density, respectively. The material properties in the next layer were set to 3500 m s\(^{-1}\), 2000 m s\(^{-1}\) and 2400 kg m\(^{-3}\) for the \(P\)-wave, \(S\)-wave and density, respectively. The material properties in the next layer were set to 4000 m s\(^{-1}\), a \(S\)-wave velocity of 2250 m s\(^{-1}\) and a density of 2500 kg m\(^{-3}\). We have compared the lattice solution with a solution of a spectral element code (J.-P. Ampuero, SEM2DPACK: A spectral element method for 2-D wave propagation and earthquake source dynamics, version 2.3.3, 2008). Fig. 9 shows the horizontal wavefield for the random lattice numerical solutions along with the layers and recorder locations. We have plotted several seismic traces along with the difference between the spectral element solution and lattice solution in Fig. 10. From the traces and difference, we can see that the two schemes match closely.

5 DISCUSSION AND CONCLUSIONS

Discrete particle or lattice methods are a class of numerical methods which solve the equations of motion of particles or nodes which interact through local ‘microscopic’ rules. The global or mesoscopic behaviour of such a system can tend towards a continuum equation of state. We have presented a version of this method using an irregular lattice where a Hookean spring coupled with a bond-bending term is chosen as the local interaction rule. We have shown that by assuming linear elasticity, which underlies the elastic wave equation, the irregular lattice can behave as an isotropic elastic continuum. The system tends to an elastic continuum if a large enough number of neighbours are linked together. The geometry of the underlying lattice is then removed by these long-range interactions. We have compared the method against an analytical solution for an infinite homogeneous medium and against a numerical solution for a layered medium. In both cases, the results agree well. There are two costs involved in this approach; first, including more neighbours requires more computational memory and calculations. Secondly, the dispersion analysis shows the scheme is more dispersive than a regular lattice method. This then necessitates more grid points per minimum wavelength again increasing the computing costs. The advantages of taking this approach is that (i) the material properties of the medium can be determined directly as opposed to other discrete schemes where calibration tests must be performed, (ii) the wave propagation is isotropic and not predetermined from the lattice geometry. Anisotropic wave propagation can be included by appropriate choice of the constants in eq. (7). (iii) Structural heterogeneity can be included using irregular lattices, for example curved free surfaces or faults and fractures. Because the scheme does not solve the continuum equation, the method can be used to model both static and dynamic deformation with relative ease. We would envisage the method being used for this reason and for brittle failure as opposed to pure seismic propagation where more flexible less expensive methods exist. Several crack propagation studies have been performed using this type of methodology, for example Marder &
Lattice method for elastic wave propagation


ACKNOWLEDGMENTS

The authors thank that this work was funded by the Department of Communications, Energy and Natural Resources under the National Geoscience Programme 2007–2013. The SFI/HEA Irish Centre for High-End Computing (ICHEC) is acknowledged for the provision of computational facilities and support.

REFERENCES


