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Quasi-static deformations of biological soft tissue

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Abstract

Quasi-static motions are motions for which inertial effects can be neglected, to the first order of approximation. It is crucial to be able to identify the quasi-static regime in order to efficiently formulate constitutive models from standard material characterisation test data. A simple non-dimensionalisation of the equations of motion for continuous bodies yields non-dimensional parameters which indicate the balance between inertial and material effects. It will be shown that these parameters depend on whether the characterisation test is strain- or stress-controlled and on the constitutive model assumed. A rigorous definition of quasi-static behaviour for both strain- and stress-controlled experiments is obtained for elastic solids and a simple form of a viscoelastic solid. Adding a rate dependence to a constitutive model introduces internal time scales and this complicates the identification of the quasi-static regime. This is especially relevant for biological soft tissue as this tissue is typically modelled as being a non-linearly viscoelastic solid. The results obtained here are applied to some problems in cardiac mechanics and to data obtained from simple shear experiments on porcine brain tissue at high strain rates.

1 Introduction

Inertia is the tendency of a body to resist any change in its motion. A physical manifestation of inertia in material characterisation tests is easily observed: when an experimental rig reaches its controlled state, the specimen being tested continues to move in the direction of applied force or moment before its elasticity pulls the specimen back. This cycle of overshooting and pullback is ended by either internal or external friction. In solid mechanics, motions for which inertial effects are neglected are called *quasi-static* and the vast bulk of the analysis of solids adopts this assumption. However, there have been few attempts to justify this hypothesis in a sound and rigorous way. Even in the celebrated encyclopaedic treatises by Truesdell and Toupin [22], Truesdell and Noll [23] and Bell [3] the idea of quasi-static motions remains undefined, although used implicitly.

In fluid mechanics, steady flows for which inertial effects can be neglected have long been studied (see, for example, Batchelor [2]) and are usually called creeping flows. For Newtonian

fluids, creeping flows are characterized as having low values of the Reynolds number and this small parameter can be exploited analytically using perturbation theory in the steady case. The Stokes solutions for viscous flow around a circle, the corresponding paradox and the Oseen solution constitute a classical example of the effectiveness of perturbation methods in steady fluid mechanics (see Van Dyke [24] for further details).

In the classical material characterisation tests of solid bodies, it is important to know when the quasi-static assumption is justified. These tests are based on the underlying deformation being homogeneous. If the quasi-static assumption holds, then the equations of motion are satisfied identically, to the leading order of approximation. One can then use the test stress-strain data to motivate and guide the constitutive assumptions for the material being tested. This has been found to be an efficient way of arriving at a mathematical model.

Physical intuition alone is not sufficient to justify the quasi-static assumption. Ideally we need a reliable *dimensionless* indicator of quasi-static behaviour and we show that a simple non-dimensionalisation of the general equations of motion yields a set of dimensionless parameters which indicates the balance between inertial and material effects. Infinitesimal values of these parameters indicate that inertial effects may be safely ignored in analysis. These parameters can therefore be viewed as a generalisation of the concept of the Reynolds number and the analysis presented here may therefore be applicable not only to solid mechanics but also to non-Newtonian fluid mechanics, where the rigorous characterisation of creeping flows is of interest (see, for example, the discussions in Metzner and Reed [11] and in Delplace and Leuliet [4]).

It will be shown that the set of quasi-static parameters depends on the constitutive assumptions adopted for the solid body, with increasing difficulty in the identification of these parameters associated with increasing sophistication of the material model. In the hierarchy of material models for solids, elasticity is the simplest, with no associated internal time-scale, and will be considered here first. It will then be shown that when, for example, viscosity is taken into account, identification of the quasi-static regime becomes much more difficult. This is because in the elastic case we have to take into account only a characteristic *external* time, associated with the unsteady boundary conditions, whereas in the framework of viscoelastic materials various characteristic *internal* times, associated with the constitutive law, must also be introduced. In addition to the type of material being considered, the non-dimensional parameters also depend on whether the experiment is controlled with either stress or strain (see, for example, Läuger and Stettin [8] for a discussion of the differences between the two types of control in the present context). Therefore there will be two distinct sets of quasi-static parameters for each class of materials since each of these control environments has associated with it a distinct natural time-scale which must be accommodated in the non-dimensional formulation of the equations of motion.

It will be assumed that data is being gathered to characterise the material during the time interval $[0, t_s]$. It will also be shown that, even when quasi-static conditions hold, there is a time interval near $t = 0$ in which inertial effects are *not* negligible. An estimate of the length of this interval is obtained, based on the same non-dimensional analysis that yielded the quasi-static conditions. This has practical implications: no experimental data should be sampled from this time interval when conducting classical material characterisation tests. It should be noted however that, in addition to the standard characterisation tests, material characterisation tests have recently been developed based on the very inertial effects ignored in classical experiments. For example, Baravian and Quemada [1] used inertio-elastic oscillations to characterise complex fluids and Yao et al. [25] used these oscillations to investigate the non-linear rheology of fibrin networks.

The results obtained here for elastic materials are applied to some problems in cardiac mechanics since ventricular myocardium has typically been modelled as a hyperelastic material.

Specifically, it will be shown that the beating heart can be modelled within a quasi-static framework and that simple shear data for passive ventricular myocardium obtained by Dokos et al. [5] was obtained within a quasi-static framework, as claimed. Finally, some high strain-rate experiments on porcine brain tissue in simple shear are assessed to discover whether the high strain-rates adopted compromise the desired quasi-static environment. It will be shown that the simple viscoelastic model assumed for the brain tissue is not compatible with the quasi-static assumption for the given experiments, illustrating that the results obtained here could prove useful as a tool in selecting appropriate constitutive models.

2 Quasi-Static Deformations

Let B_0 denote the collection of particles comprising a continuous body under no applied forces, ∂B_0 its boundary and let $\boldsymbol{X}$ denote the position vector of a typical particle in this configuration. Assume now that forces are applied so that the relative position of the particles is disturbed. Let B denote the collection of deformed particles, with boundary ∂B, and denote the current position vector of a typical particle by $\boldsymbol{x}$. As the forces change, the relative position of the particles change and therefore

$$\boldsymbol{x} = \boldsymbol{x}\left(\boldsymbol{X}, t\right), \tag{1}$$

where it will be assumed that only a finite time interval, say $t \in [0, t_s]$, is of interest. At each $\boldsymbol{x} \in B, t \in [0, t_s]$, in the absence of body forces, the equations of motion,

$$\mathrm{div}\boldsymbol{T} = \rho \frac{\partial^2 \boldsymbol{x}}{\partial t^2}, \tag{2}$$

hold, where $\boldsymbol{T}$ is the Cauchy stress tensor and ρ is the density of the body in the current configuration.

Assume that the traction is specified on some part of the boundary of the deformed body, denoted by ∂B^T, so that at each $\boldsymbol{x} \in \partial B^T, t \in [0, t_s]$,

$$\boldsymbol{T}\boldsymbol{n} = \boldsymbol{t}_g, \tag{3}$$

where $\boldsymbol{n}$ is the normal to ∂B^T. Assume that the displacement $\boldsymbol{u} \equiv \boldsymbol{x} - \boldsymbol{X}$ is specified on the complement of ∂B^T, denoted by ∂B^D, so that at each $\boldsymbol{x} \in \partial B^D, t \in [0, t_s]$,

$$\boldsymbol{u} = \boldsymbol{u}_g. \tag{4}$$

In general, one must also consider the initial conditions $\boldsymbol{u}(\boldsymbol{x}, 0) = \boldsymbol{u}_0(\boldsymbol{x})$ and $\boldsymbol{u}_t(\boldsymbol{x}, 0) = \boldsymbol{u}_{t,0}(\boldsymbol{x})$. However in the standard material characterisation tests of interest here, these initial conditions are identically zero and this will be assumed in what follows.

Let $\|\cdot\|$ denote the *sup* norm[1]

$$S \equiv \|\boldsymbol{t}_g\|, \quad C \equiv \|\partial \boldsymbol{t}_g/\partial t\|, \quad U \equiv \|\boldsymbol{u}_g\|, \quad V \equiv \|\partial \boldsymbol{u}_g/\partial t\|. \tag{5}$$

Let L denote a typical length-scale of the body. Then a natural non-dimensionalisation of the following variables in the equations of motion is given by:

$$\hat{\boldsymbol{T}} \equiv \boldsymbol{T}/S, \quad \hat{\boldsymbol{x}} \equiv \boldsymbol{x}/L. \tag{6}$$

At this stage we avoid specifying a constitutive assumption for the material and therefore obtain the dimensionless form of the balance equations (2) using only the boundary conditions. When conducting experiments it is usual to distinguish between *soft* and *hard* devices: in a soft device the stress is controlled during the experiment whereas with a hard device we control the strain.

[1]For a *bounded* function $f(x,t)$ defined on a set $\mathcal{U} \times [0, t^+]$, define the sup norm as being the nonnegative number $\sup\{\sup\{|f(x,t)| : x \in \mathcal{U}\} : t \in [0, t^+]\}$.

2.1 Soft devices

If the applied traction is the controlled parameter in experiments, then the natural time-scale is how quickly the applied traction is being changed and therefore the time variable should be scaled as follows:

$$\hat{t} \equiv Ct/S. \tag{7}$$

Using an obvious notation for the divergence operator, the corresponding non-dimensional form of the equations of motion (2) is therefore

$$\hat{\mathrm{div}}\hat{\boldsymbol{T}} = \mathcal{Q}_s \frac{\partial \hat{\boldsymbol{x}}^2}{\partial \hat{t}^2}, \tag{8}$$

where the non-dimensional parameter $\mathcal{Q}_s$ is defined by

$$\mathcal{Q}_s \equiv \frac{\rho C^2 L^2}{S^3}. \tag{9}$$

For the important special case of a periodic applied traction with frequency ω, $C = S\omega$ and the quasi-static parameter for stress-controlled experiments takes the form

$$\mathcal{Q}_{sp} \equiv \frac{\rho \omega^2 L^2}{S}. \tag{10}$$

A necessary condition for quasi-static deformations for *all* continuous bodies is therefore that

$$\mathcal{Q}_s << 1. \tag{11}$$

It is worth noting that this condition is not a sufficient condition for quasi-static deformations to be a good approximation of the full equations of motion since the $\partial \hat{\boldsymbol{x}}^2/\partial \hat{t}^2$ term could be large. Thus quasi-static deformations should be properly considered within the context of *singular* perturbation theory. This consideration does not typically arise in fluid mechanics since only steady flows are usually of interest. In what follows, however, it will be assumed that the non-dimensional acceleration term is of $\mathcal{O}(1)$.

Exactly the same method illustrated above can be used if the first Piola-Kirchoff stress, $\boldsymbol{P}$, is used as the stress measure and the corresponding dimensionless parameters are obtained by replacing ρ by the density in the undeformed configuration, ρ_0, and by interpreting S, C as being the norms of the first Piola-Kirchoff traction vector and the rate of change of this traction vector respectively.

2.2 Hard devices

Now assume that strain is the controlled parameter. If one adopts $\gamma \equiv U/L$ as the strain measure, then a natural measure of the strain rate, $\dot{\gamma}$, is given by

$$\dot{\gamma} \equiv V/L. \tag{12}$$

Therefore, in this case, one should scale the time variable as

$$\bar{t} \equiv Vt/L = \dot{\gamma} t. \tag{13}$$

Non-dimensionalisation of the equations of motion then yields the following non-dimensional parameter

$$\mathcal{Q}_h \equiv \frac{\rho \dot{\gamma}^2 L^2}{S} = \frac{\rho V^2}{S}, \tag{14}$$

with infinitesimal values of this parameter being a necessary condition for all continuous bodies for quasi-static deformations.

2.3 Initial time interval

We have already remarked that a singular perturbation problem in the time variable is obtained if the condition (11) holds. There are therefore two distinct solutions to the equations of motion: there is the initial layer solution in some initial time interval within which the inertial effects must be considered [9] and there is the outer approximation for later times, which corresponds to the quasi-static regime. The usual method of investigating the boundary layer is to re-scale the time variable so that the magnitude of the singular term is no longer negligible. For both stress and strain controlled experiments, an appropriate re-scaling of the original time variable t is given by

$$\breve{t} = \mathcal{Q}_s^{-1/2}\hat{t} = \mathcal{Q}_h^{-1/2}\bar{t} = \frac{1}{L}\sqrt{\frac{S}{\rho}}t. \tag{15}$$

In the resulting equations of motion, there is now a balance between material and inertial effects and consequently inertial effects cannot be ignored. These re-balanced equations are valid for the range $\breve{t} \in [0, 1]$ or, alternatively, in terms of the 'real' time variable t, for the range

$$t \in \left[0, L\sqrt{\frac{\rho}{S}}\right], \tag{16}$$

where the time upper bound for which inertial effects are important is rate independent. This will be discussed later when considering experimental data obtained from the simple shearing of brain tissue. This 'dead' time interval can be further refined for specific materials and boundary conditions.

2.4 Constitutive assumptions

A constitutive equation is a relationship between stress and strain[2]. Postulating a constitutive assumption therefore inevitably links the the two dimensionless numbers we have just derived for stress- and strain-controlled experiments, with the precise relationship between the two numbers, of course, depending on the assumption made. To illustrate this, we consider a linear, elastic body for which

$$S = c_m\left(U/L\right), \quad C = c_m\left(V/L\right), \tag{17}$$

where c_m is a material constant depending on the type of experiment being performed and $\gamma \equiv U/L << 1$. The two quasi-static parameters then take the form

$$\mathcal{Q}_h^{lin} = \frac{\rho V^2 L}{c_m U}, \quad \mathcal{Q}_s^{lin} = \frac{\rho C^2 L^2}{S^3} = \frac{\rho V^2 L^3}{c_m U^3} = \frac{\mathcal{Q}_h^{lin}}{\gamma^2}, \tag{18}$$

revealing the precise relationship between the two dimensionless parameters. Note that $\mathcal{Q}_s^{lin} >> \mathcal{Q}_h^{lin}$, reflecting the fact that changes in the infinitesimal strain measure do not necessarily lead to infinitesimal percentage changes in the stress. A necessary condition for quasi-static infinitesimal strains of an elastic body in a strain-controlled environment is therefore that

$$\frac{\rho V^2}{c_m} << \gamma << 1, \tag{19}$$

whereas for stress-controlled experiments,

$$\frac{\rho V^2}{c_m} = \frac{\rho C^2 L^2}{c_m^3} << \gamma^3 << 1. \tag{20}$$

[2]More complex theories may involve *derivatives* of stress and strain in the constitutive assumption but these will not be considered here.

3 Viscoelastic materials

We will now consider the special class of viscoelastic materials, noting that the quasi-static assumption is needed to interpret two of the most important characterising experiments for these materials: the creep and recovery experiments [20].

Viscoelastic materials are typically defined in terms of their Cauchy stress response, $\boldsymbol{T}$. In a nonlinear setting the constitutive equation for such materials may be very complex, but here, to illustrate the central ideas, only materials whose Cauchy stress representation contains a term linear in the velocity gradient tensor, $\boldsymbol{D}$, and no other dependence on $\boldsymbol{D}$, will be considered, i.e., only those materials for which

$$\boldsymbol{T} = \boldsymbol{T}_e + \nu \boldsymbol{D}, \tag{21}$$

where $\boldsymbol{T}_e$ denotes the elastic response of the material. This class of materials obviously includes Newtonian fluids and some of the simpler types of viscoelastic materials. For these materials the viscosity parameter, ν, with physical units $Pa\,s = P$, is necessarily present in the constitutive law. An example of such materials is the classical linear Kelvin-Voigt material. Materials defined by this constitutive relation will be called Newtonian viscoelastic (NVE) materials. They are a very special case of materials of differential type [23].

There is an *internal* time-scale associated with these materials as a result of the presence of the viscosity parameter. This time-scale needs to be accounted for in the non-dimensionalisation process, together with the time-scale associated with the change in boundary conditions. An obvious non-dimensional time-scale associated with the viscosity is given by

$$\tilde{t} = St/\nu. \tag{22}$$

For purely elastic bodies, there is no associated internal time scale and consequently the $\mathcal{Q}_s$ and $\mathcal{Q}_h$ parameters obtained in the last section as being necessary conditions for quasi-static deformations for all continuous bodies will also be assumed sufficient conditions for quasi-static deformations for elastic bodies.

3.1 Soft devices

For stress-controlled experiments $t = t\left(\hat{t},\, \tilde{t}\,\right)$ and therefore

$$\frac{\partial \boldsymbol{x}}{\partial t} = \frac{\partial \boldsymbol{x}}{\partial \hat{t}}\frac{C}{S} + \frac{\partial \boldsymbol{x}}{\partial \tilde{t}}\frac{S}{\nu}, \quad \frac{\partial^2 \boldsymbol{x}}{\partial t^2} = \frac{\partial^2 \boldsymbol{x}}{\partial \hat{t}^2}\frac{C^2}{S^2} + 2\frac{\partial^2 \boldsymbol{x}}{\partial \hat{t}\partial \tilde{t}}\frac{C}{\nu} + \frac{\partial^2 \boldsymbol{x}}{\partial \tilde{t}^2}\frac{S^2}{\nu^2}. \tag{23}$$

The non-dimensional form of the equations of motion for stress-controlled experiments on viscoelastic materials therefore takes the form

$$\hat{\mathrm{div}}\hat{\boldsymbol{T}} = \mathcal{Q}_s\frac{\partial^2 \hat{\boldsymbol{x}}}{\partial \hat{t}^2} + 2\mathcal{Q}_{1v}\frac{\partial^2 \hat{\boldsymbol{x}}}{\partial \hat{t}\partial \tilde{t}} + \mathcal{Q}_{2v}\frac{\partial^2 \hat{\boldsymbol{x}}}{\partial \tilde{t}^2}, \tag{24}$$

where $\mathcal{Q}_s$ is defined in (9) and

$$\mathcal{Q}_{1v} \equiv \frac{\rho C L^2}{S\nu}, \quad \mathcal{Q}_{2v} \equiv \frac{\rho S L^2}{\nu^2}. \tag{25}$$

The three non-dimensional parameters in (24) are not independent since $\mathcal{Q}_{1v} = \sqrt{\mathcal{Q}_s\mathcal{Q}_{2v}}$. It follows therefore that inertial effects can be neglected if

$$max\{\mathcal{Q}_s,\, \mathcal{Q}_{2v}\} << 1. \tag{26}$$

The first of these parameters can be thought of as the *elastic* quasi-static parameter and $\mathcal{Q}_{2v}$ as the *viscous* parameter.

This criterion is reminiscent of that of Ferry [7] for the identification of the quasi-static regime for viscoelastic materials. Using physical intuition alone, Ferry [7] proposed that for quasi-static oscillatory motions of incompressible, viscoelastic materials the length-scale along which waves will propagate must be small compared to the wavelength of shear waves propagated through the material and that shear waves experience very little attenuation. Ferry's mathematical encapsulation of these ideas is that inertial forces will be small if, in the notation of Section 2,

$$max\left(\frac{\rho\omega^2 L^2}{G'}, \frac{\rho\omega^2 L^2}{G''},\right) << 1, \tag{27}$$

where ω is the oscillatory frequency in Hertz and G', G'' are the storage and loss moduli respectively. Although there are therefore some similarities between this criterion and (26), the differences are significant. Instead of using physical intutition, an analytical approach results in (26) and this criterion is completely general; (27) is only valid for oscillatory motions. Also, our approach determines the quasi-static regime for stress- and strain-controlled experiments separately; this is not reflected in Ferry's approach. Finally, and very importantly from an experimenter's point of view, the non-dimensional parameters derived here are more easily determined than those suggested by Ferry.

As noted previously, there is an initial time interval during which inertial effects *are* important. This interval must contain the time interval (16), associated with the $\mathcal{Q}_s$ parameter. To determine the time interval associated with $\mathcal{Q}_{2v}$, another rescaling of the time variables in (24) is needed. To this end, let

$$t^\star \equiv \mathcal{Q}_{2v}^{-1/2}\tilde{t} = \frac{1}{L}\sqrt{\frac{S}{\rho}}t, \tag{28}$$

which is exactly the same non-dimensional time variable associated with both the $\mathcal{Q}_s, \mathcal{Q}_h$ parameters obtained in (15). It follows that inertial effects are important for stress-controlled experiments on viscoelastic materials during the same time interval (16) as obtained previously. In addition to being independent of the stress rate, the time upper bound for inertial effects is therefore also independent of the viscosity for viscoelastic materials.

For Newtonian fluids, characterised by the constitutive relation

$$S = \nu\dot{\gamma}, \tag{29}$$

the quasi-static parameter $\mathcal{Q}_{2v}$ takes a special form. Substitution of (29) into $(25)_2$ yields

$$\mathcal{Q}_{2v} = \frac{\rho V L}{\nu} = \mathcal{R}, \tag{30}$$

where $\mathcal{R}$ is the classical Reynolds number. For oscillatory flows of frequency ω, the $\mathcal{Q}_s$ parameter defined in (10) becomes

$$\mathcal{Q}_{sp} = \frac{\rho V L}{\nu} \times \frac{\omega^2 L^2}{V^2} = \mathcal{R} \times \mathcal{S}_{num}^2.$$

where $\mathcal{S}_{num}$ is the classical Strouhal number. Thus for oscillatory stress-controlled experiments on Newtonian fluids, inertial effects can be neglected if

$$max\{\mathcal{R},\, \mathcal{R}\,\mathcal{S}_{num}^2\} << 1. \tag{31}$$

This result appears new and hasn't been obtained before because of the focus on steady flow problems, for which only the Reynolds number is necessary to determine the quasi-static regime.

3.2 Hard devices

For strain-controlled experiments,

$$\frac{\partial^2 \boldsymbol{x}}{\partial t^2} = \frac{\partial^2 \boldsymbol{x}}{\partial \bar{t}^2}\dot{\gamma}^2 + 2\frac{\partial^2 \boldsymbol{x}}{\partial \bar{t}\partial \tilde{t}}\frac{\dot{\gamma}S}{\nu} + \frac{\partial^2 \boldsymbol{x}}{\partial \tilde{t}^2}\frac{S^2}{\nu^2}. \tag{32}$$

The corresponding non-dimensional equations of motion are therefore given by

$$\hat{\mathrm{div}}\hat{\boldsymbol{T}} = \mathcal{Q}_h\frac{\partial^2 \hat{\boldsymbol{x}}}{\partial \bar{t}^2} + 2\mathcal{Q}_{3v}\frac{\partial^2 \hat{\boldsymbol{x}}}{\partial \bar{t}\partial \tilde{t}} + \mathcal{Q}_{2v}\frac{\partial^2 \hat{\boldsymbol{x}}}{\partial \tilde{t}^2}, \tag{33}$$

where $\mathcal{Q}_h$ is defined in (14), $\mathcal{Q}_{2v}$ is defined in $(25)_2$ and $\mathcal{Q}_{3v}$ is defined as

$$\mathcal{Q}_{3v} \equiv \frac{\rho\dot{\gamma}L^2}{\nu}. \tag{34}$$

As might be expected from the analysis for stress-controlled materials, these three parameters are not independent since $\mathcal{Q}_{3v} = \sqrt{\mathcal{Q}_h\mathcal{Q}_{2v}}$. Inertial effects can therefore be neglected for strain-controlled experiments if

$$max\{\mathcal{Q}_h,\ \mathcal{Q}_{2v}\} << 1. \tag{35}$$

These two parameters are identical if and only if (29) holds, i.e., if and only if the material is a Newtonian fluid. In fact for these fluids,

$$\mathcal{Q}_h = \mathcal{Q}_{2v} = \mathcal{R}. \tag{36}$$

Thus for Newtonian fluids, the quasi-static criterion (35) collapses to simply requiring that the Reynolds number be small.

Thus, in summary, the quasi-static regime is determined by $\mathcal{Q}_{2v}$ and by the appropriate elastic control parameter: $\mathcal{Q}_h$ for strain-controlled experiments and $\mathcal{Q}_s$ for stress experiments. The time interval for which inertial effects are important is given by (16). This time interval for which inertial effects are important is therefore common to both stress- and strain-controlled experiments for both elastic and NVE materials.

Clearly, if we consider more complex theories of viscoelasticity, new characteristic times are introduced. The analysis presented here, although dealing with materials with only one internal time measure, introduces the main ideas that are likely to be encountered for more complex constitutive models.

There is an immediate problem associated with determining the viscosity for an NVE material: to determine the viscosity, one usually conducts quasi-static experiments but to determine the quasi-static regime, one needs to know the viscosity. One solution to this conundrum is to determine the viscosity without relying on quasi-static conditions; for example, the techniques discussed in Baravian and Quemada [1] using instrumental inertia to determine material properties hold promise in this regard. In the absence of an accurate estimate for viscosity, the best approach seems to be to adopt the following procedure: determine the viscosity from experiments *assuming quasi-static conditions hold* and then validate this assumption using the derived value for the viscosity. If the criteria for quasi-static conditions are indeed found to be satisfied, one accepts the viscosity value as being reasonable and the experimental data can then to used to determine the elastic part of the stress response in (21). If the these criteria are not satisfied, one can reasonably conclude that the constitutive assumption (21) is not compatible with quasi-static conditions for the experimental data. This does not preclude the possibility that the same set of experimental data could be deemed to have been obtained under quasi-static conditions if another, and in the case of the NVE material a more complex, constitutive assumption is introduced. This procedure will be illustrated in the next section with a study of the effect of strain rate on the shear response of brain tissue.

There is also the practical problem of specifying an upper limit on the various non-dimensional parameters that is consistent with the requirement of being 'much smaller than one'. It seems reasonable to exclude parameters greater than or equal to 0.1, with parameters in the range $(0.01, 0.1)$ possibly being acceptable if all other parameters are in this range or smaller. The arbitrary standard proposed here is that the non-dimensional parameters should be less than than 0.01.

4 Investigating experimental data

4.1 Cardiac mechanics

Developing accurate mathematical models of the heart is a major focus of research in biomechanics because heart failure is a leading cause of death in developed countries. Although much effort has been devoted to the development of constitutive models of the heart walls that reflect both ventricular and myocardial structure and the development of finite element codes that incorporate these models (see, for example, the review articles of Hunter et al. [10] and Nash and Hunter [12]), little or no attention has been paid to the more basic question as to whether the inherently dynamic beating of the heart can be modelled as a quasi-static system. This will be addressed here using the analysis of the previous sections.

This analysis is appropriate because there seems to be a consensus (see the review articles mentioned above) that the walls of the heart should be modelled as being non-linearly elastic. The stress-controlled parameter is appropriate as the boundary conditions are given in terms of the blood pressure on the endocardium. Since the applied tractions are periodic, a value of the quasi-static parameter (10) will be calculated. Since the human body is primarily water, $\rho \approx 1000\, kgm^{-3}$. The frequency of a beating heart, ω, is $1\, Hz$. A review of the reported peak systolic pressures in the left ventricle in Section 7 of Nash and Hunter [12] suggests that $S \approx 20\, kPa$. Measurement of left ventricular wall thickness of 24 patients by echocardiography by Troy et al. [21] suggest that $L \approx 8\, mm$. Substitution of these values into (10) yields $\mathcal{Q}_{sp} = 3.2 \times 10^{-6} << 1$ and so the beating of the heart is quasi-static.

The development of constitutive models mentioned above is crucially dependent on appropriate and accurate experimental data. One important set of data for the simple shearing of passive ventricular myocardium was obtained by Dokos et al. [5]. Models that accurately predict the response of myocardium in shear are important because it has been hypothesised that the shearing of myocardial layers plays an important role in the mechanical functioning of the heart. Dokos et al. [5] claimed that the testing was performed under quasi-static conditions. The sinusoidal shear tests were performed with the shear rate as the controlling parameter. In a period of $30\, s$, the maximum displacement was 40% of the specimen thickness of $3\, mm\, (= L)$. This translates into a shear rate, $\dot{\gamma}$, of $0.16\, mm\, s^{-1}$. The myocardium was found to be anisotropic, with the smallest maximum shear stress exerted found to be $3kPa$. This will be taken as the S value. Substituting these values into (14) yields $\mathcal{Q}_h = 8 \times 10^{-14} << 1$ and therefore the testing was indeed quasi-static.

4.2 Brain tissue

There has been a lot of interest in the literature in determining the stress response of brain tissue in shear. The impetus for much of this work is the desire to obtain a predictive capability of the damage done to brain tissue due to impact loads [13, 14, 15]. Since brain tissue is typically modelled as being a viscoelastic, non-linear medium [16, 17, 18], much of the experimental effort has inevitably focused on observing the effect on the stress response of varying the strain-rate in simple shear experiments. There is the intuitive expectation that in order to predict the

damage of brain tissue due to impact, one would have to use models based on data obtained at high strain rates.

This is the motivation for our set of simple shear tests on brain tissue [19], whose results are summarized in Figure 1, with a typical experiment illustrated in Figure 2.

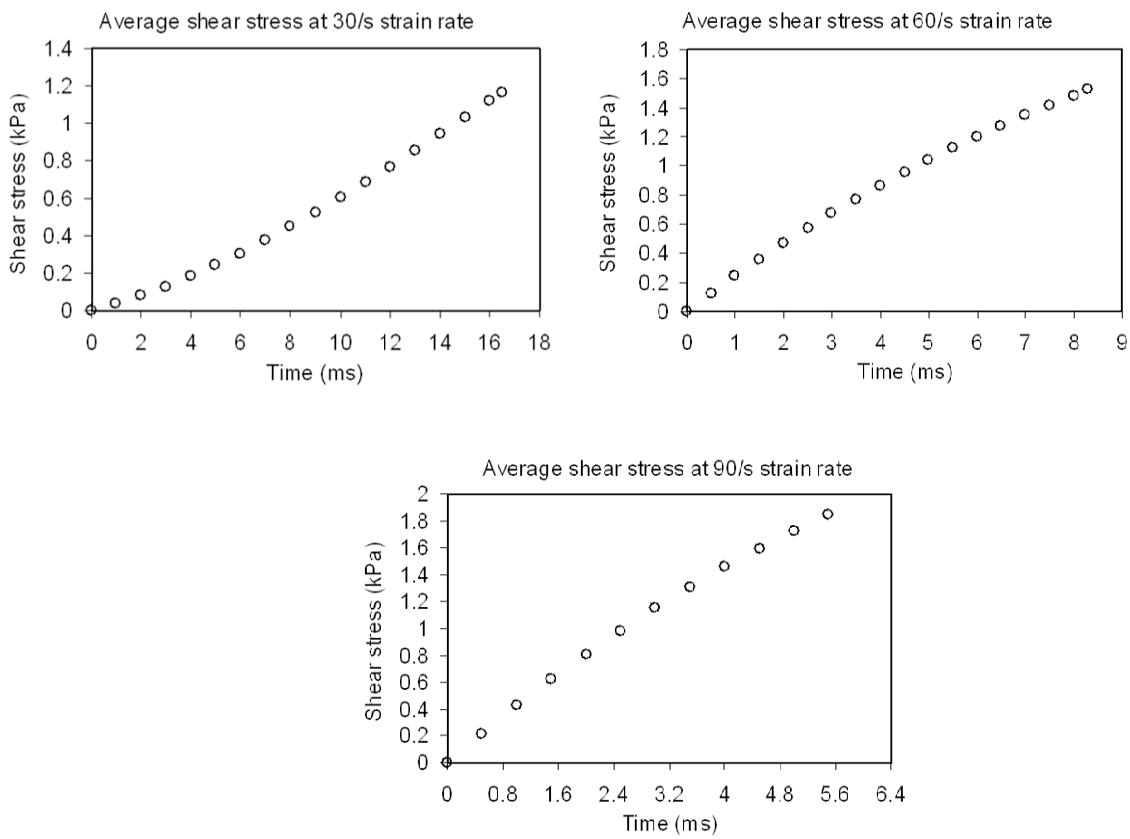

Figure 1: Shear stress response at different strain rates

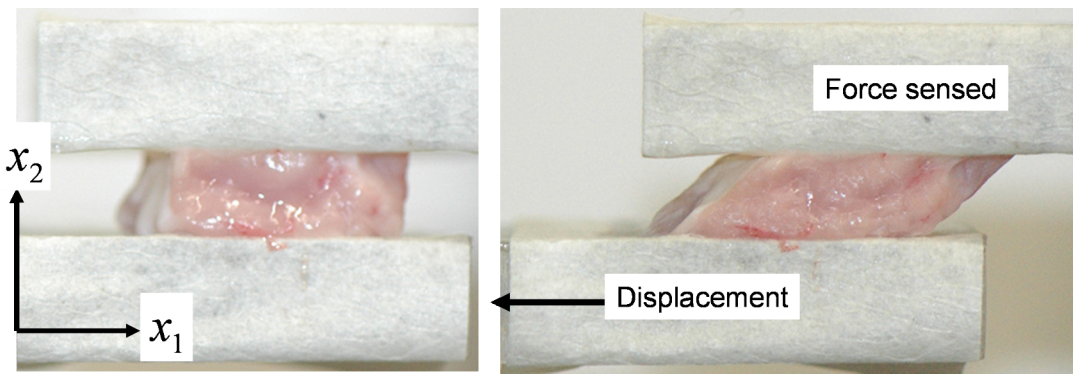

Figure 2: Deformation of a sample of porcine brain tissue before and after a simple shear test

A typical particle, with cartesian coordinates (X, Y, Z) in the undeformed configuration, had coordinates (x, y, z) in the deformed configuration with

$$x = X + \kappa Y, y = Y, z = Z, \tag{37}$$

where $\kappa > 0$ is called the amount of shear. For specimens with dimensions $20 \times 20\,mm$ and a thickness of $4\,mm$, the plots of shear stress versus time are given in Figure 1 for the strain

rates $30/s$, $60/s$ and $90/s$, with $\kappa = 1$ being the maximum amount of shear achieved in each case.

Since these experiments are strain-controlled, the appropriate quasi-static criterion is that given in (35). Since brain tissue is primarily water, it will be assumed that $\rho = 1000\, kg\, m^{-3}$. S is taken to be the maximum shear stress. The quasi-static parameter $\mathcal{Q}_h$, together with the corresponding S value, is given below for each of the three strain rates:

strain rate (s^{-1})	$S\,(Pa)$	$\mathcal{Q}_h$
30	1200	0.01
60	1500	0.04
90	1800	0.07

Table 1: $\mathcal{Q}_h$ values for brain tissue at different strain rates

The $\mathcal{Q}_h$ values are in the intermediate range where they could be acceptable if the corresponding $\mathcal{Q}_{2v}$ values, calculated in the next section, are infinitesimal. These $\mathcal{Q}_h$ values could be made smaller by repeating the experiments for smaller specimen thicknesses. For example, to ensure that $\mathcal{Q}_h < 1/100$ for the $90\, s^{-1}$ strain rate, assuming that S remains the same, requires that $L < 1.5\, mm$, which, for most laboratories, would not be feasible.

As previously discussed, there will always be an initial time interval, bounded above by t_i, say, and given in (16) for the viscoelastic material (21), for which inertial effects must be accounted for, even if the quasi-static criteria for the experiment are satisfied. For the three strain rates considered here, these upper bounds are given by

strain rate (s^{-1})	$t_i\,(ms)$
30	4
60	3
90	3

Table 2: Time upper bounds for which inertial effects are present

Somewhat counterintuitively, it is seen that as the strain rate increases, the time interval for which inertial effects are important decreases. This is because, as shown in Section 3, the time interval is in fact *independent* of the strain rate and, instead, is inversely proportional to $S^{1/2}$. Data from Table 1 suggest that as the strain rate increases, S increases and t_i therefore decreases. Of course, higher strain rates result in higher $\mathcal{Q}_h$ parameters and therefore quasi-static effects become increasingly important outside the boundary layer bounded above by t_i, even as the size of this boundary layer decreases. For high enough strain-rates, the boundary layer where inertial effects are important essentially disappears, but for $t > t_i$ inertial effects must be accounted for because of the large $\mathcal{Q}_h$ value.

4.3 Determining the viscosity from the brain tissue experiments

Accepting for the moment that $\mathcal{Q}_h << 1$ for the brain tissue in simple shear, one must still ensure that $\mathcal{Q}_{2v} << 1$, where $\mathcal{Q}_{2v}$ is defined by $(25)_2$, before one can conclude that quasi-static conditions hold. Determination of this second parameter requires knowledge of the viscosity of brain tissue. A search of the literature reveals a wide range of possible values. For example, Fallenstein et al. [6] quote values in the range $15.5 - 96\, P$. We prefer to obtain an estimate based on the data in Figure 1.

It will first be shown that the constitutive assumption (21) for an NVE material yields a good fit with the albeit limited experimental data of Figure 1. Note that for the simple shear experiments of interest here, it follows from (21) that

$$S = \boldsymbol{T}_e|_{\kappa=1} + \nu\dot{\gamma}, \tag{38}$$

where S is the maximum shear stress. Thus the assumption of an NVE constitutive model implies that there is a linear relation between maximum shear stress and strain rate. The available experimental data suggest that this is indeed the case. A plot of S versus strain rate is given in Figure 3 below for the data in Table 1:

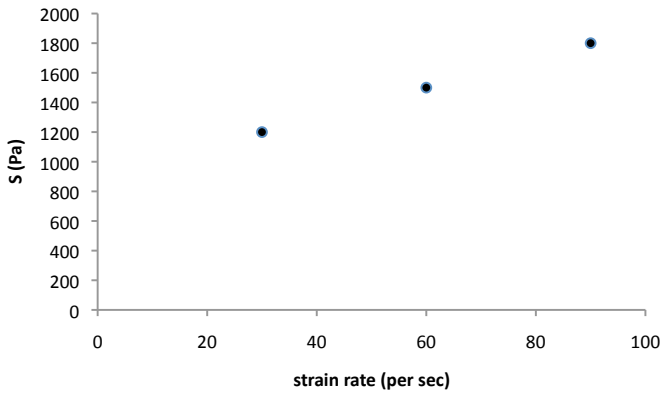

Figure 3: Shear stress response at different strain rates

The perfectly linear plot is partially a result of the fact that only three strain rates are available. Nonetheless the available data suggest that (38) is a valid approximation in the given range of strain rates. Accordingly, ν is calculated to be

$$\nu = \frac{1800 - 1200}{90 - 30} = 10\,P, \tag{39}$$

which is at the lower end of the range of previous estimates. The corresponding $\mathcal{Q}_{2v}$ values are therefore given by

strain rate (s^{-1})	$S\,(Pa)$	$\mathcal{Q}_{2v}$
30	1200	0.19
60	1500	0.24
90	1800	0.29

Table 3: $\mathcal{Q}_{2v}$ values for brain tissue at different strain rates

These $\mathcal{Q}_{2v}$ values are *not* infinitesimal and suggest that the experimental data contain a significant inertial component. This might be an explanation as to why our measured value for the viscosity of brain tissue is lower than other estimates in the literature. One conclusion is that the NVE assumption, (21), is not a convenient assumption for the modelling of brain tissue for this range of strain rates and that a more complex rate dependence is appropriate. The search for a better model will be conducted elsewhere. It is sufficient to note that the analysis of quasi-static behaviour introduced here can be a powerful tool in the mathematical modelling of soft biological tissue.

5 Concluding remarks

Quasi-static behaviour is assumed for the vast majority of material characterisation tests and in many mathematical analyses of the mechanical response of continuous bodies. Despite its importance, there has been little or no attempt to rigorously define what precisely quasi-static conditions are. A first attempt at this is presented here, focusing on elastic and simple viscoelastic soft tissue. The analysis shows that an accurate mathematical formulation of this concept is non-trivial. One reassuring aspect of the approach suggested here is that the Reynolds number criterion for quasi-static flow of Newtonian fluids is recovered as a special case.

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