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PLAYER TYPE DISTRIBUTIONS AS STATE VARIABLES AND INFORMATION REVELATION IN ZERO SUM REPEATED GAMES WITH DISCOUNTING*

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This paper examines the role of the player type distributions in repeated zero sum games of incomplete information with discounting of payoffs. In particular the strategic "sufficiency" of the posterior distributions for histories and the limiting properties of the posterior sequence are discussed. It is shown that differentiability of the value function is sufficient to allow the posteriors to serve as "state" variables for histories.

The limiting properties of the posterior distributions are considered and a characterization given of the set of possible limit points of the posterior distribution. This characterization is given in terms of the "value" of information in the one-stage game.

1. Introduction. This paper focuses on two aspects of repeated incomplete information zero sum games: the role of the posterior distribution as a state variable and the extent to which information is revealed in equilibrium.

The potential role of the posterior distribution as a state variable can be motivated by the fact that, in this class of game, the value function satisfies a recursive equation. For discounted games the recursion is,

$$v_{\infty}(p, q) = \min_y \max_x \left\{ (1 - \delta) \sum p^k q^r x^k A^{kr} y^r + \delta \sum \bar{x}_i \bar{y}_j v_{\infty}(p(x, i), q(y, j)) \right\}$$

and for finitely repeated games with summation of payoffs,

$$v_n(p, q) = \min_y \max_x \left\{ \sum p^k q^r x^k A^{kr} y^r + \sum \bar{x}_i \bar{y}_j v_{n-1}(p(x, i), q(y, j)) \right\}.$$

(The notation is explained in §2, where these functions are discussed.) One might conjecture from these equations that how players play in successive periods depends only on the past as it affects the posterior distributions—or that one could find equilibria where this is true. For finitely repeated games at least, this is not the case: there are examples of games where a player cannot guarantee the value of the game with a strategy that depends only on histories through the posterior distributions.¹ It is of interest to investigate the circumstances under which the posterior distributions are "sufficient" for histories. This question is developed in §3, where it is shown that,

¹The following example (due to J. F. Mertens) illustrates the point. The game is a game of one-sided information repeated twice. The stage games are:

$$A^1 = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}, \quad A^2 = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$$

and the prior distribution is $p = \frac{1}{2}$. This game has a unique optimal strategy for player I, the informed player and maximizer. Play type independent $(\frac{1}{2}, \frac{1}{2})$ in the first period and play the type dominant strategy

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for the infinitely repeated game, posteriors are "sufficient" for histories when v_∞ is differentiable (see Theorem 2 below for a precise statement of the result).

This result may be used as follows. If one solves the recursive equation for v_∞ the problem of describing the optimal strategies still remains. At this point if v_∞ is differentiable it is sufficient to find the first-period component of the strategies at each (p, q) since these then determine the posterior distributions on which second-period (behaviour) strategies may be defined—and so on. (For some discussion concerning the solution of these recursive equations see Mayberry 1967.)

A second question of interest relates to the revelation of information: in a zero sum game to what extent do players reveal information? This question is discussed in §4. There it is shown (under appropriate smoothness assumptions) that, if in a one-stage game at some type distribution a player can gain by use of information, then such information is used in the repeated game and so the posterior cannot converge to that type distribution. This contrasts sharply with the type of result one obtains if the limit of means payoff criterion is used (see Kohlberg 1975a for a discussion of information revelation in the context of the limit of means payoff criterion).

Information revelation is closely related to strategic information usage and the discussion raises a point not seen in the more commonly discussed model which uses the limiting average criterion. With discounting, information can be used beneficially as long as the order of magnitude of current gain outweighs the order of magnitude of future losses (associated with an opponent being better informed). With the limiting average criterion any gain from information usage must be sustained indefinitely for it to be beneficial. This latter point is illustrated by the following example.

EXAMPLE 1. In this example the informed player and maximizer (player I) has two player types—1 and 2. The corresponding payoff matrices are:

$$A^1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

In this game, with the limit of means payoff criterion, for any prior distribution p , $p_\infty = p$ almost surely in any equilibrium. With discounting of payoffs the posterior p_∞ is in the set $\{0, 1\}$ with probability 1 in any equilibrium. The explanation is as follows. At any prior distribution p , the informed player can guarantee an expected payoff of $u(p) = p(1 - p)$ by playing a type independent, nonrevealing strategy.² However to guarantee a higher payoff at any stage requires that he use a type dependent strategy. But, given any strategy σ for player I which reveals information, player II can find a strategy which exhausts almost all the information in σ within some fixed length of time N (say). Then, given any posterior, p_N (depending on the history to N), II can in each subsequent period play the stage game strategies which are optimal in the zero sum game with payoff matrix $A(p_N) = p_N A^1 + (1 - p_N) A^2$: with the limit of means criterion this achieves over the entire game, a payoff roughly equal to $E_{\sigma\tau p}\{u(p_N)\}$.

in the second and last period—i.e. top if type 1, bottom if type 2. For player II any optimal strategy requires playing $(\frac{1}{2}, \frac{1}{2})$ in the first period. In the second period there are four possible histories $h_1 = (1, 1)$, $h_2 = (2, 1)$, $h_3 = (1, 2)$, $h_4 = (2, 2)$, where (i, j) is a history with $i = 1$ denoting top for I and $j = 1$ denoting left for II. Let $y_j = y(h_j)$ denote the probability that II plays left given history h_j . Then any optimal strategy requires that $y_1 + y_3 = \frac{1}{4}$, and $y_2 + y_4 = \frac{3}{4}$. Thus, although the posterior equals $\frac{1}{2}$ at all four histories, the uninformed player cannot play the same following each history.

²Throughout the paper no attempt is made to distinguish between type independent and nonrevealing strategies. It serves no purpose here. With a nonrevealing strategy, the prior and posterior coincide with probability 1. A type independent strategy requires that each player type plays the same strategy. A nonrevealing strategy may not be type independent at extreme points of the set of player type distributions (see Kohlberg 1975b for a discussion of this matter). The function $u(p)$ is defined below—in §2.

Since u is strictly concave, for n sufficiently large $E_{\sigma\tau p}\{u(p_n)\} < u(p)$. This means that a strategy which reveals no information guarantees a higher payoff to the informed player. The key point is that, with the limit of means criterion, gains from information usage must be sustainable indefinitely. With discounting, since the distant future is unimportant, current gains from information usage may more than offset losses arising from the fact that the opponent is "better informed" in future periods.

2. Description of the game. A matrix is selected from the set $\{A^{kr} | k \in K, r \in R\}$ according to the independent prior distributions $p = (p^1, \dots, p^K) \in \Delta^K$ and $q = (q^1, \dots, q^R) \in \Delta^R$ (Δ^m denotes the simplex of dimension $m - 1$, and without confusion, K will sometimes denote an integer and sometimes the set $\{1, 2, \dots, K\}$; a similar interpretation applies to R): The matrix A^{kr} is chosen with probability $p^k q^r$. Player I is informed of the choice of $k \in K$ and II is informed of the choice of $r \in R$. The game is played repeatedly, each player observes the history $h_t = (i_1, j_1, \dots, i_{t-1}, j_{t-1})$ before moving at time t . Given that the pair (k, r) is chosen and history $h = (i_1, j_1, \dots, i_t, j_t, \dots)$ occurs player I receives $a^{kr}(h)$ from player II, where

$$a^{kr}(h) = (1 - \delta) \sum_{t=1}^{\infty} \delta^{t-1} a_{i_t j_t}^{kr} \left(\text{where } A^{kr} = \{a_{ij}^{kr}\}_{i \in I, j \in J} \right).$$

Histories are denoted $H_t = (I \times J)^{t-1}$, $H_{\infty} = (I \times J)^{\infty}$ and $H^t = \prod_{s=t}^{\infty} (I \times J)$ with the convention, $H_1 = \emptyset$. Thus $h_t \in H_t$, $h \in H$ and $h^t = (i_t, j_t, i_{t+1}, j_{t+1}, \dots) \in H^t$.

Strategies in the game are sequences of functions $\sigma = (x_1, x_2, \dots, x_t, \dots)$ and $\tau = (y_1, y_2, \dots, y_t, \dots)$ for players I and II respectively, where: $x_t: H_t \times K \rightarrow \Delta^I$ and $y_t: H_t \times R \rightarrow \Delta^J$. The sequence of functions $(x_1^k, x_2^k, \dots, x_t^k, \dots) = \sigma^k$, with $x_t^k: H_t \rightarrow \Delta^I$, denotes a strategy for player I type k . The strategy τ^r for player II type r is defined similarly. Let $\mathcal{F}_t$ be the finite field determined on H_{∞} by H_t and set $\mathcal{F}_{\infty} = \bigvee_{t=1}^{\infty} \mathcal{F}_t$. $\mathcal{F}_t$ represents the information available to players at time t . Let $\mathcal{F}_{\infty} \otimes 2^{K \times R}$ be the sigma field on $H_{\infty} \times K \times R$. A pair of strategies (σ, τ) and a prior distribution (p, q) on $K \times R$ determine a probability measure $\mu_{\sigma\tau pq}$ on $\mathcal{F}_{\infty} \otimes 2^{K \times R}$, with corresponding expectation operator $E_{\sigma\tau pq}$. For fixed $\mu (= \mu_{\sigma\tau pq})$ one may compute the sequence of posterior distributions (p_t, q_t) : $p_t^k = \mu(k | \mathcal{F}_t)$ and $q_t^r = \mu(r | \mathcal{F}_t)$. Sometimes these will be written $p_t^k(h)$ or $p_t^k(h_t)$ (with $h = (h_t, \bar{h})$) to denote $\mu(k | \mathcal{F}_t)(h)$, and similarly with q_t^r . With this notation, the probability of a history $h_t \in H_t$ is given by

$$\mu(\{h_t\} \times H^t) = \sum_{k \in K, r \in R} p^k q^r \left\{ \prod_{s=1}^{t-1} x_{s,i_s}^k(i_1, j_1, \dots, i_{s-1}, j_{s-1}) y_{s,j_s}^r(i_1, j_1, \dots, i_{s-1}, j_{s-1}) \right\}.$$

The posteriors p_t^k, q_t^r , $k \in K, r \in R$ are bounded martingales which converge almost surely. These almost sure limits are denoted $p_{\infty}^k, q_{\infty}^r$.

A strategy pair $(\hat{\sigma}, \hat{\tau})$ is an equilibrium if for all $k \in K$ and $r \in R$,

$$E_{\hat{\sigma}\hat{\tau}pq}\{a^{kr}(h)|k\} \geq \sup_{\sigma} E_{\sigma\hat{\tau}pq}\{a^{kr}(h)|k\} \quad \text{and}$$

$$E_{\hat{\sigma}\hat{\tau}pq}\{a^{kr}(h)|r\} \leq \inf_{\tau} E_{\hat{\sigma}\tau pq}\{a^{kr}(h)|r\}.$$

The term $E_{\hat{\sigma}\hat{\tau}pq}\{a^{kr}(h)|k\}$ will be called the payoff to player I type k and denoted ξ^k .

Let $a_i^{kr}(h) = (1 - \delta) \sum_{s=i}^{\infty} \delta^{s-i} a_{i,j_s}^{kr}$, where $h = (i_1, j_1, \dots, i_t, j_t, \dots)$ and $h_t = (i_1, j_1, \dots, i_{t-1}, j_{t-1})$. Then $E_{\hat{\sigma} \hat{\tau} p q} \{a_i^{kr}(h) | h_t, k\}$ is called the continuation payoff to player I type k given history h_t .

The discussion in the remainder of the paper is based on three functions, v_{∞} , v_n and u , defined $\Delta^K \times \Delta^R$ and denoting respectively the value of infinitely repeated game, the value of the n -stage repeated game and u , the value of the one-stage game where both players are restricted to type independent strategies. Let $x = (x^1, x^2, \dots, x^K)$, $y = (y^1, y^2, \dots, y^R)$ where $x^1, \dots, x^K \in \Delta^I$ and $y^1, \dots, y^R \in \Delta^J$. Denote by $g(x, y)$ the function $\sum_{k,r} p^k q^r x^k A^{kr} y^r$ and let $f(x, y) = (1 - \delta)g(x, y)$. The value of the one-stage game is defined: $v_1(p, q) = \max_x \min_y g(x, y)$. For the game repeated n times with payoff in the t th period weighted by $(1 - \delta)\delta^{t-1}$ the value function v_n , $n \geq 2$, satisfies the recursive formula (using an argument given in Sorin 1980):

$$v_n(p, q) = \max_x \min_y \left\{ (1 - \delta) \sum_{k,r} p^k q^r x^k A^{kr} y^r + \delta \sum_{i,j} \bar{x}_i \bar{y}_j v_{n-1}(p(x, i), q(y, j)) \right\}.$$

The posterior distributions are $p(x, i) = \{p^k(x, i)\}_{k=1}^K$ and $q(y, j) = \{q^r(y, j)\}_{r=1}^R$, where $p^k(x, i) = \{p^k x_i^k / \bar{x}_i\}$ and $q^r(y, j) = \{q^r y_j^r / \bar{y}_j\}$ with $\bar{x}_i = \sum p^k x_i^k$ and $\bar{y}_j = \sum q^r y_j^r$. Thus, for example, $p^k(x, i)$ is the posterior probability that player I is type k , given the strategy x and the outcome i . The function u , defined by

$$u(p, q) = \max_{\tilde{x}} \min_{\tilde{y}} \tilde{x} \left(\sum_{k,r} p^k q^r A^{kr} \right) \tilde{y}$$

where $\tilde{x} \in \Delta^K$ and $\tilde{y} \in \Delta^R$, gives the value of the one-stage game when neither player is allowed to play type dependent. For infinitely repeated games the value function satisfies the recursion (see Theorem 1 below):

$$v_{\infty}(p, q) = \max_x \min_y \left\{ (1 - \delta) \sum_{k,r} p^k q^r x^k A^{kr} y^r + \delta \sum_{i,j} \bar{x}_i \bar{y}_j v_{\infty}(p(x, i), q(y, j)) \right\}.$$

When $R = 1$, these functions become $v_n(p)$, $u(p)$ and $v_{\infty}(p)$ —given by the expressions above with summation only over k and with r superscripts and q variables deleted.

The following result will be used extensively throughout the paper.

THEOREM 1. *The value function, $v_{\infty}(p, q)$, of the infinitely repeated game exists and is continuous in (p, q) . It satisfies $v_{\infty}(p, q) = \lim_{n \rightarrow \infty} v_n(p, q)$ and is the unique function satisfying the recursion:*

$$v_{\infty}(p, q) = \max_x \min_y \left\{ (1 - \delta) \sum_{k,r} p^k q^r x^k A^{kr} y^r + \delta \sum_{i,j} \bar{x}_i \bar{y}_j v_{\infty}(p(x, i), q(y, j)) \right\}.$$

PROOF. Consider the recursion determined by the n -stage game:

$$v_n(p, q) = \max_x \min_y \left\{ f(x, y) + \delta \sum_{i,j} \bar{x}_i \bar{y}_j v_{n-1}(p(x, i), q(y, j)) \right\}.$$

Let $p(x)$ denote the random variable with $\text{prob}(p(x) = p(x, i)) = \bar{x}_i$ (with a similar definition for $q(y)$) and let E_{xy} denote the expectation determined by (x, y) . With

this notation:

$$v_n(p, q) = \max_x \min_y \{f(x, y) + \delta E_{xy}[v_{n-1}(p(x), q(y))]\}.$$

Let $(\tilde{x}, \tilde{y})$ and $(\hat{x}, \hat{y})$ be solutions for the $n+1$ and n period games, respectively. Then

$$\begin{aligned} v_{n+1}(p, q) - v_n(p, q) &= \{f(\tilde{x}, \tilde{y}) + \delta E_{\tilde{x}\tilde{y}}[v_n(p(\tilde{x}), q(\tilde{y}))]\} \\ &\quad - \{f(\hat{x}, \hat{y}) + \delta E_{\hat{x}\hat{y}}[v_{n-1}(p(\hat{x}), q(\hat{y}))]\} \\ &\leq \{f(\tilde{x}, \hat{y}) + \delta E_{\tilde{x}\hat{y}}[v_n(p(\tilde{x}), q(\hat{y}))]\} \\ &\quad - \{f(\tilde{x}, \hat{y}) + \delta E_{\tilde{x}\hat{y}}[v_{n-1}(p(\tilde{x}), q(\hat{y}))]\}. \end{aligned}$$

(The inequality follows since $\hat{y}$ may not be optimal in the $n+1$ period game and $\tilde{x}$ may not be optimal in the n period game.) Thus,

$$v_{n+1}(p, q) - v_n(p, q) \leq \delta E_{\tilde{x}\hat{y}}\{[v_n(p(\tilde{x}), q(\hat{y}))] - [v_{n-1}(p(\tilde{x}), q(\hat{y}))]\}.$$

For each (p, q) , $\exists$ some $(\tilde{x}, \hat{y})$ such that this inequality holds and so:

$$\forall(p, q), \quad v_{n+1}(p, q) - v_n(p, q) \leq \delta \sup |v_n(p, q) - v_{n-1}(p, q)|.$$

Similarly,

$$\forall(p, q), \quad v_{n+1}(p, q) - v_n(p, q) \geq -\delta \sup |v_n(p, q) - v_{n-1}(p, q)|.$$

Using the sup norm—if $f, g: \Delta^K \times \Delta^R \rightarrow \mathbb{R}$, then $\rho(f, g) = \sup |f(p, q) - g(p, q)|$ yields $\rho(v_{n+1}, v_n) \leq \delta \rho(v_n, v_{n-1})$, so the minmax operator is a contraction. Now v_n is continuous in (p, q) since it is a finite zero sum game with parameters varying continuously on $\Delta^K \times \Delta^R$. Noting that $\{v_n\}$ is a cauchy sequence, it converges uniformly to a continuous function—say $v_\infty(p, q)$.

Finally, observe that given $\epsilon > 0$, $\exists \bar{n}$ such that if $\sigma(n)$ is an optimal strategy for player I in the n -period game ($n \geq \bar{n}$) then in the infinitely repeated game the strategy $\sigma = (\sigma(n), x_{t+1}, x_{t+2}, \dots)$ guarantees $v_n(p, q) - \epsilon$, where $x_{t+\tau}$ is arbitrary for $\tau \geq 1$. (Since payoffs are discounted, when $\bar{n}$ is large the payoff over the remainder of the game—from $\bar{n}$ on—is small (less than ϵ)). Thus player I can guarantee a payoff in the infinitely repeated game of at least $v_n(p, q) - \epsilon$. Similarly, player II can guarantee a payoff no larger than $v_n(p, q) + \epsilon$ for n sufficiently large. Since $\{v_n\}$ converges to v_∞ , in the limit both players can guarantee $v_\infty(p, q)$ and so $v_\infty(p, q)$ is the value of the infinitely repeated game. QED

It should be pointed out that when the limit of means payoff criterion is used, $v_n(p, q)$ still converges—even when the infinitely repeated game has no value. This is discussed in Mertens and Zamir (1971).

3. Posteriors as state variables. This section shows that when the value function is differentiable everywhere in the simplex of player types, then the posterior distribution is a sufficient statistic for histories: there exist equilibria such that if at two different histories the posterior distributions coincide then the strategies of all player types are equal at both histories. In the following discussion $(p_t(h_t), q_t(h_t))$

denote posterior distributions following the history h_t (the determining strategies are not made explicit).

THEOREM 2. *Let $v_\infty(p, q)$ be the value function of an infinitely repeated incomplete information zero sum game with discounted payoffs and player type distributions given by (p, q) . If $v_\infty(p, q)$ is everywhere differentiable on $\Delta^K \times \Delta^R$ (the simplexes of player types), then the posterior distributions (p_t, q_t) are sufficient state variables to determine the strategies of both players. That is, there exists an equilibrium such that if at h_t and h_τ , $(p_t(h_t), q_t(h_t)) = (p_\tau(h_\tau), q_\tau(h_\tau))$, then $x_t^k(h_t) = x_\tau^k(h_\tau)$, $\forall k \in K$ and $y_t^r(h_t) = y_\tau^r(h_\tau)$, $\forall r \in R$.*

PROOF. Denote by $\xi \in \mathbb{R}^K$ and $\zeta \in \mathbb{R}^R$ respectively, vector payoffs to players I and II. Let $\alpha(p, q)$ be the equilibrium correspondence from player type distributions to vector payoffs (i.e. $(\xi, \zeta) \in \alpha(p, q)$ implies that there exists an equilibrium strategy pair (σ, τ) , such that in this equilibrium the expected payoff to player I type k (player II type r) is ξ^k (ζ^r)). The important point in the proof is to show that on $(\Delta^K \times \Delta^R)^\circ$ (the notation X° means the interior of X), $\alpha(p, q)$ is a function—i.e. ξ and ζ are uniquely determined. Thus, if two different histories lead to the same posteriors in $(\Delta^K \times \Delta^R)^\circ$ then the expected payoff to each player type following each of these two histories (the continuation payoff vectors) must be the same. This essentially allows the players to “play the same” at histories h_t, h_τ where the posteriors are the same: $(p_t(h_t), q_t(h_t)) = (p_\tau(h_\tau), q_\tau(h_\tau))$. The only problem arises on the boundary of $(\Delta^K \times \Delta^R)$ where differentiability of v_∞ does not imply uniqueness of the vector payoffs (ξ, ζ) .

To see that α is a function on $(\Delta^K \times \Delta^R)^\circ$, take $\bar{q} \in \Delta^R$, $p \in (\Delta^K)^\circ$ and $(\xi, \zeta) \in \alpha(p, \bar{q})$. The vector ξ satisfies:³ (i) $p \cdot \xi = v_\infty(p, \bar{q})$ and (ii) $\bar{p} \cdot \xi \geq v_\infty(\bar{p}, \bar{q}) \forall \bar{p} \in \Delta^K$. Condition (i) follows directly from $(\xi, \zeta) \in \alpha(p, \bar{q})$. To check that (ii) holds, suppose not. Then there is some $\hat{p}$ with $\hat{p} \cdot \xi < v_\infty(\hat{p}, \bar{q})$ and since $v_\infty(\cdot, \bar{q})$ is continuous there is some $p^* \in (\Delta^K)^\circ$ with $p^* \cdot \xi < v_\infty(p^*, \bar{q})$. In the game with player type distributions $(p, \bar{q})$ pick $\bar{\tau}$, an optimal strategy for II, with

$$(\xi, \zeta) \in \alpha(p, \bar{q}), \quad v_\infty(p, \bar{q}) = \sum_k p^k \sup_{\sigma^k} E_{\sigma^k \bar{q} \bar{\tau}} \{a^{kr}(h) | k\} = \sum_k p^k \xi^k$$

(where $E_{\sigma^k \bar{q} \bar{\tau}}$ is the expectation operator on histories determined by σ^k , $\bar{\tau}$ and $\bar{q}$). Thus II has a strategy bounding above (coordinate-wise) by ξ , the vector payoff to I. Therefore if II uses the strategy $\bar{\tau}$ in the game with priors $(p^*, \bar{q})$, he can hold the expected payoff to I to $p^* \cdot \xi$. This contradicts the assumption that $v_\infty(p^*, \bar{q}) > p^* \cdot \xi$ and so (ii) is verified. Therefore, with $\bar{q}$ fixed, ξ defines a hyperplane tangent to $v_\infty(\cdot, \bar{q})$ at p . Since $v_\infty(\cdot, \bar{q})$ is differentiable, ξ is uniquely determined. A similar argument shows that ζ is also uniquely determined for all $(p, q) \in (\Delta^K \times \Delta^R)^\circ$.

For any (p, q) on the boundary of $(\Delta^K \times \Delta^R)$ there is a unique pair (ξ, ζ) satisfying $\lim \alpha(p_n, q_n) = (\xi, \zeta)$ for all $(p_n, q_n) \in (\Delta^K \times \Delta^R)^\circ$, with $(p_n, q_n) \rightarrow (p, q)$ (since the one-sided derivatives are assumed to exist). Thus, α can be extended to the boundary of $(\Delta^K \times \Delta^R)$ as follows: define the function $\bar{\alpha}$ by $\bar{\alpha}(p, q) = \alpha(p, q)$ on $(\Delta^K \times \Delta^R)^\circ$ and extend $\bar{\alpha}$ to $(\Delta^K \times \Delta^R)$ uniquely by setting $\bar{\alpha}(p, q) = \lim \alpha(p_n, q_n)$ where $(p_n, q_n) \rightarrow (p, q)$, for any (p, q) in $(\Delta^K \times \Delta^R)$. Since the correspondence from priors to equilibrium strategies is upper hemicontinuous, the function $\bar{\alpha}$ associates to any pair of priors (p, q) an equilibrium vector payoff (ξ, ζ) (i.e. $(\xi, \zeta) \in \alpha(p, q)$).

³In one-sided information games with the limit of means payoff criterion, this would follow directly from Blackwell approachability. See Hart (1985). With discounting, approachability theory does not apply. Also, the game may not have a value in two-sided information case with the limit of means criterion.

The function $\bar{\alpha}$ may be used to generate an equilibrium of the infinitely repeated game with priors (p, q) satisfying: (a) $(\xi_1, \zeta_1) \in \bar{\alpha}(p, q)$ where (ξ_1, ζ_1) are first-period vector payoffs and (b) $(\xi_2(i, j), \zeta_2(i, j)) \in \bar{\alpha}(p_2(i, j), q_2(i, j)) \forall i, j$, with $(\xi_2(i, j), \zeta_2(i, j))$ second-period continuation payoffs given history (i, j) and $(p_2(i, j), q_2(i, j))$ second-period posterior distributions given (i, j) . In generating such an equilibrium, a pair of first-period strategies (x, y) will be associated to each (p, q) . Thus the procedure implicitly defines a pair of functions (x, y) on $(\Delta^K \times \Delta^R)$ mapping into the players respective (stage game) strategy spaces. From these functions, the strategies appearing in the statement of the theorem are inductively defined. The remainder of the discussion develops these functions (the discussion is complicated somewhat by the need to take limits to the boundary).

Consider an infinitely repeated game with prior distribution $(p_n, q_n) \in (\Delta^K \times \Delta^R)^\circ$ and where players are restricted in the first period by the conditions: $x_{1i}^k \geq \epsilon_n$ and $y_{1j}^r \geq \epsilon_n, \forall k, r, i, j$. (There are no restrictions on strategies for the second or later periods.) This game has an equilibrium. Given some equilibrium of this game, denote the first-period vector payoffs (ξ_{1n}, ζ_{1n}) , and second-period continuation vector payoffs following history (i, j) : $(\xi_{2n}(i, j), \zeta_{2n}(i, j))$. To make explicit the dependence of (ξ_{1n}, ζ_{1n}) on (p_n, q_n, ϵ_n) write $(\xi_1(p_n, q_n, \epsilon_n), \zeta_1(p_n, q_n, \epsilon_n))$. Now consider (p_n, q_n) and ϵ_n as sequences converging to $(p, q) \in \Delta^K \times \Delta^R$ and 0 respectively. Then

$$\lim_{\epsilon_n \rightarrow 0} (\xi_1(p_n, q_n, \epsilon_n), \zeta_1(p_n, q_n, \epsilon_n)) = \bar{\alpha}(p_n, q_n)$$

so that

$$\lim_{(p_n, q_n) \rightarrow (p, q)} \lim_{\epsilon_n \rightarrow 0} (\xi_1(p_n, q_n, \epsilon_n), \zeta_1(p_n, q_n, \epsilon_n)) = \bar{\alpha}(p, q).$$

Therefore, take a subsequence $\epsilon_{m(n)}$ of ϵ_n such that

$$\lim_{n \rightarrow \infty} (\xi_1(p_n, q_n, \epsilon_{m(n)}), \zeta_1(p_n, q_n, \epsilon_{m(n)})) = \bar{\alpha}(p, q) = (\xi_1, \zeta_1), \text{ say.}$$

Take convergent (sub)sequences in second-period variables so that

$$\begin{aligned} & (p_{2n}(i, j), q_{2n}(i, j), \xi_{2n}(i, j), \zeta_{2n}(i, j)) \\ & \rightarrow (p_2(i, j), q_2(i, j), \xi_2(i, j), \zeta_2(i, j)) \quad \forall i, j. \end{aligned}$$

With this construction $(\xi_2(i, j), \zeta_2(i, j)) \in \bar{\alpha}(p_2(i, j), q_2(i, j)) \forall i, j$. Finally, define $x_{1i}^k(p, q)$ and $y_{1j}^r(p, q)$ as the limits of the first-period component of the equilibrium strategies: $x_{1i}^k(n)$ and $y_{1j}^r(n)$ (where $x_{1i}^k(n)$ and $y_{1j}^r(n)$ are greater than or equal to $\epsilon_{m(n)}$). Note that with this construction, if for some (i, j) , $(p_2(i, j), q_2(i, j)) = (p, q)$ then $(\xi_2(i, j), \zeta_2(i, j)) = (\xi_1, \zeta_1)$.

Using this procedure functions $x_{1i}^k(p, q)$ and $y_{1j}^r(p, q)$ can be defined on $(\Delta^K \times \Delta^R)$, $\forall i, j, k, r$. These functions may in turn be used to define an equilibrium satisfying the conditions of the theorem. If the priors are (p, q) , let first period strategies and continuation payoffs be determined exactly as above. If history (i_1, j_1) occurs leading to posteriors $(p_2(i_1, j_1), q_2(i_1, j_1))$, the continuation payoffs are uniquely determined as $\bar{\alpha}(p_2(i_1, j_1), q_2(i_1, j_1))$.

Define $x_{2i}^k(i_1, j_1) = x_{1i}^k(p_2(i_1, j_1), q_2(i_1, j_1))$ and $y_{2j}^r(i_1, j_1)$ analogously. Treating $(p_2(i, j), q_2(i, j))$ now as priors and applying the above procedure again gives posteri-

ors and continuation payoffs:

$$(p_3(i_1, j_1, i_2, j_2), q_3(i_1, j_1, i_2, j_2)) \text{ and } \bar{\alpha}(p_3(i_1, j_1, i_2, j_2), q_3(i_1, j_1, i_2, j_2)).$$

Proceeding inductively with this procedure, at each history $h_t = (i_1, j_1, i_2, j_2, \dots, i_{t-1}, j_{t-1})$, the strategies defined there are best responses, given the continuation payoffs. Thus the strategies constructed define an equilibrium of the game. QED

4. Information revelation. Define the set $B = \{(p, q) | v_\infty(p, q) = u(p, q)\}$. This is the set of player type distributions where the value of the infinitely repeated game is the same as the value of the one-stage game where players are not allowed to use their information. Observe that B contains the extreme points of $\Delta^K \times \Delta^R$.

Theorem 3 below asserts that the limits of the posterior distribution must lie in this set almost surely (relative to the equilibrium measure μ). Since v_∞ is concave in p and convex in q this immediately excludes from B (almost surely) any point (p, q) if given q , u is not concave at p and given p , u is not convex at q . However since v_∞ is in general very difficult to compute, an additional set A defined in terms of one-shot games is introduced (in §4.1 and §4.2) and it is shown that if the prior is in this set, then it cannot be in B , and so the posterior must be in the complement of the set A . Thus the set B is used indirectly to identify the set of possible limit points of the posterior distribution.

THEOREM 3. *Let $(\bar{\sigma}, \bar{\tau})$ be equilibrium strategies in the game with prior distributions (p, q) and let $\tilde{\mu} = \mu_{\bar{\sigma}\bar{\tau}pq}$ be the equilibrium measure on $\mathcal{F}_\infty \otimes 2^{K \times R}$. Then $(p_\infty, q_\infty) \in B$ a.s. $\tilde{\mu}$.*

PROOF. The proof is given in four lemmas.

LEMMA 1. *Denote the equilibrium correspondence from priors to strategies by $\varphi(p, q)$. Then $(\sigma, \tau) \in \varphi(p, q)$ and $v_\infty(p, q) \neq u(p, q)$ imply $E_{\sigma\tau pq}\{\|p_2 - p\| + \|q_2 - q\|\} > 0$.⁴*

PROOF. To see this, note that $E_{\sigma\tau pq}\{\|p_2 - p\| + \|q_2 - q\|\} = 0$ implies that $p^k x_{1i}^k = p^k \bar{x}_{1i}$ and $q^r y_{1j}^r = q^r \bar{y}_{1j}$ ($\sigma = (x_1, x_2, \dots, x_t, \dots)$, $x_t = (x_t^1, x_t^2, \dots, x_t^K)$, $\tau = (y_1, y_2, \dots, y_t, \dots)$ and $y_t = (y_t^1, y_t^2, \dots, y_t^R)$). Take $v_\infty(p, q) > u(p, q)$ and note $\sum p^k q^r x_1^k A^{kr} y_1^r = \sum p^k q^r \bar{x}_1 A^{kr} y_1^r$. Thus $\min_y (\sum p^k q^r x_1^k A^{kr}) y \leq u(p, q)$, so let $\hat{y}$ satisfy $(\sum p^k q^r x_1^k A^{kr}) \hat{y} \leq u(p, q)$. If player II plays $\hat{y}$ in period 1, then, on any first-stage history with positive probability the posterior distributions equal the prior distributions and so from then on II can guarantee $v_\infty(p, q)$ following that history. Thus, given σ , an optimal (minimizing) strategy of II guarantees a payoff no larger than $(1 - \delta)u(p, q) + \delta v_\infty(p, q) < v_\infty(p, q)$. This contradicts the assumption $(\sigma, \tau) \in \varphi(p, q)$. Taking $v_\infty(p, q) < u(p, q)$ yields a similar contradiction. QED

LEMMA 2. *Let $m(p, q) = \inf_{\sigma, \tau} \{E_{\sigma\tau pq}\{\|p_2 - p\| + \|q_2 - q\|\} | (\sigma, \tau) \in \varphi(p, q)\}$. If $v_\infty(p, q) \neq u(p, q)$, then for any closed neighbourhood $\mathcal{N}(p, q)$ of (p, q) on which $v_\infty(\bar{p}, \bar{q}) \neq u(\bar{p}, \bar{q})$, for all $(\bar{p}, \bar{q})$ in $\mathcal{N}(p, q)$, $\inf\{m(\bar{p}, \bar{q}) | (\bar{p}, \bar{q}) \in \mathcal{N}(p, q)\} = \bar{m} > 0$.*

PROOF. Since $v_\infty(\bar{p}, \bar{q}) \neq u(\bar{p}, \bar{q})$ for all $(p, q) \in \mathcal{N}(p, q)$, we can partition $\mathcal{N}(p, q)$ into two disjoint sets, $\mathcal{N}_1(p, q) = \mathcal{N}(p, q) \cap \{(\bar{p}, \bar{q}) | v_\infty(\bar{p}, \bar{q}) > u(\bar{p}, \bar{q})\}$ and $\mathcal{N}_2(p, q) = \mathcal{N}(p, q) \cap \{(\bar{p}, \bar{q}) | v_\infty(\bar{p}, \bar{q}) < u(\bar{p}, \bar{q})\}$. Both of these sets are closed. For example, if $(p_n, q_n) \rightarrow (\bar{p}, \bar{q})$ and $(p_n, q_n) \in \mathcal{N}(p, q) \cap \{(\bar{p}, \bar{q}) | v_\infty(\bar{p}, \bar{q}) > u(\bar{p}, \bar{q})\}$ then $(\bar{p}, \bar{q})$ is in $\{(\bar{p}, \bar{q}) | v_\infty(\bar{p}, \bar{q}) \geq u(\bar{p}, \bar{q})\}$, using continuity of v_∞ and u . Since $\mathcal{N}(p, q)$ is closed, $(\bar{p}, \bar{q})$ is also in $\mathcal{N}(p, q)$. This then implies that $v_\infty(\bar{p}, \bar{q}) > u(\bar{p}, \bar{q})$ and so $(\bar{p}, \bar{q}) \in \mathcal{N}_1(p, q)$. Suppose $\mathcal{N}_1(p, q) \neq \emptyset$. Since $E_{\sigma\tau pq}\{\|p_2 - p\| +$

⁴ $\|p_2 - p\| = \sum_{k \in K} |p_2^k - p^k|$, where $|x|$ denotes the absolute value of x .

$\|q_2 - q\|$ is a continuous function of (σ, τ) and φ is a closed valued correspondence, the function $m(p, q)$ is lower-semicontinuous. Therefore $m(p, q)$ has a minimum on $\mathcal{N}_1(p, q)$ —say at $(\hat{p}, \hat{q})$ with $m(\hat{p}, \hat{q}) = \bar{m}_1$. If $\bar{m}_1 = 0$ then there exist $(\hat{\sigma}, \hat{\tau}) \in \varphi(\hat{p}, \hat{q})$ and $E_{\hat{\sigma}\hat{\tau}pq}(\|p_2 - p\| + \|q_2 - q\|) = 0$, contradicting Lemma 1, so $\bar{m}_1 > 0$. Similarly, if $\mathcal{N}_2(p, q) \neq \emptyset$, then $m(p, q)$ has a minimum $\bar{m}_2 > 0$, on $\mathcal{N}_2(p, q)$. If both $\mathcal{N}_1(p, q)$ and $\mathcal{N}_2(p, q)$ are nonempty $\bar{m} = \min(\bar{m}_1, \bar{m}_2) > 0$ otherwise one set, $\mathcal{N}_i(p, q)$ is nonempty and then $\bar{m} = m_i > 0$. QED

LEMMA 3. Let μ be a measure on $\mathcal{F}_\infty$. There exists a set $H(\infty) \subset H_\infty$, $H(\infty) \in \mathcal{F}_\infty$ such that $\mu(H(\infty)) = 1$ and for any $h \in H(\infty)$, where $h = (h_t, h') \in H_t \times H'$, $\mu(\{h_t\} \times H') > 0$ for all t .

PROOF. Let $H(t) = \{(h_t, h') \in H_\infty | \mu(\{h_t\} \times H') > 0\}$. Clearly, $\mu(H(t)) = 1$. Note that $H(t+1) \subset H(t)$. Define $H(\infty) = \bigcap_{t=1}^\infty H(t)$. Since the measure μ is continuous from above, $\lim_{t \rightarrow \infty} \mu(H(t)) = \mu(H(\infty)) = 1$. Pick any t and $h = (h_t, h') \in H(\infty)$ and note that since $H(\infty) \subset H(t)$, $h \in H(t)$ and so $\mu(\{h\} \times H') > 0$. QED

LEMMA 4. Let $(\sigma, \tau) \in \varphi(p, q)$ and let μ be the measure on $H_\infty \otimes 2^{K \times R}$ determined by (σ, τ) and (p, q) . Then $(p_\infty, q_\infty) \in B$ a.s. μ .

PROOF. The posteriors converge almost surely to (p_∞, q_∞) so pick a set H with measure 1 on which the posteriors converge pointwise—i.e., $H \subseteq H_\infty$, $\mu(H) = 1$ and $(p_t(h), q_t(h)) \rightarrow (p_\infty(h), q_\infty(h))$, $\forall h \in H$. Let $H(\infty)$ be determined by μ , as in Lemma 3, set $H^* = H \cap H(\infty)$ and observe that $\mu(H^*) = 1$. If for some $h \in H^*$, $v_\infty(p_\infty(h), q_\infty(h)) > u(p_\infty(h), q_\infty(h))$ (the following argument can also be applied to the case $v_\infty(p_\infty(h), q_\infty(h)) < u(p_\infty(h), q_\infty(h))$), then pick any compact neighbourhood of $(p_\infty(h), q_\infty(h))$, $\mathcal{N}(p_\infty(h), q_\infty(h))$ with $v_\infty(p, q) > u(p, q)$, $\forall (p, q) \in \mathcal{N}(p_\infty(h), q_\infty(h))$ and such that $\exists \eta > 0$ with $\{(p, q) | \|p - p_\infty(h)\| + \|q - q_\infty(h)\| < \eta\} \subseteq \mathcal{N}(p_\infty(h), q_\infty(h))$. (This can be done using the continuity of v_∞ and u .) From Lemma 2, this implies that $\exists \bar{m} > 0$ such that $\forall (p, q) \in \mathcal{N}(p_\infty(h), q_\infty(h))$, $E_{\sigma\tau pq}(\|p_2 - p\| + \|q_2 - q\|) \geq \bar{m}$, $\forall (\sigma, \tau) \in \varphi(p, q)$. Since $(p_t(h), q_t(h)) \rightarrow (p_\infty(h), q_\infty(h))$, there is some t^* and $p_t(h), q_t(h) \in \mathcal{N}(p_\infty(h), q_\infty(h))$, $\forall t \geq t^*$. Pick $t \geq t^*$ and note that since $\mu(\{h_t\} \times H') > 0$, the equilibrium strategy pair which determines μ , induces equilibrium strategies on the subform⁵ determined by h_t and the posterior distributions following history h_t . Consequently, using Lemma 2,⁶ $E_{\sigma\tau pq}(\|p_{t+1} - p_t\| + \|q_{t+1} - q_t\| | \mathcal{F}_t)(h) \geq \bar{m}$, $t \geq t^*$, contradicting $E_{\sigma\tau pq}(\|p_{t+1} - p_t\| + \|q_{t+1} - q_t\| | \mathcal{F}_t)(h) \rightarrow 0$. Therefore $v_\infty(p_\infty(h), q_\infty(h)) = u(p_\infty(h), q_\infty(h))$ a.s. μ so $(p(h), q(h)) \in B$ a.s. μ . QED

Surprisingly, Theorem 3 does not hold when the limit of means is used (and $v_\infty(p, q)$ exists). The following example illustrates this fact in the one-sided information game.

EXAMPLE 2. Player I, the row player, has two types 1 and 2 and is type 1 with probability p .

$$\begin{array}{cc} & \begin{array}{cc} A^1 & A^2 \end{array} \\ \begin{array}{cc} 1 & 2 \end{array} & \begin{array}{cc} 1 & 2 \end{array} \\ 1 & \begin{bmatrix} -1 & 0 \\ 2 & 0 \end{bmatrix} \\ 2 & \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \end{array}$$

⁵Essentially, the continuation game determined by the posterior distributions at h and the branches of the extensive form, from there on. Note that, by restricting attention to H^* , we focus only on equilibrium paths.

⁶This argument makes use of the special structure of the game: there is full monitoring, so at time t the history up to this time is fully observed and the player type distributions are independent.

Thus $u(p) = -p(1-p)$ and $v_1(p) = v_x(p) = 0, \forall p \in [0, 1]$. Define strategies σ and τ for players I and II respectively as follows. Given a history $h_t = (i_1, j_1, i_2, j_2, \dots, i_{t-1}, j_{t-1})$, let $\sigma = (x_1, x_2, \dots, x_t, \dots)$ be given by

- $x_t(h_t, k) = (1, 0)$ for $k = 1, 2$ if $j_\tau = 2$ for all $\tau \leq t-1$. (Both types play 1.)
- $x_t(h_t, 1) = (0, 1)$ if $j_\tau = 1$ for some $\tau \leq t-1$. (Type 1 plays 2.)
- $x_t(h_t, 2) = (1, 0)$ if $j_\tau = 1$ for some $\tau \leq t-1$. (Type 2 plays 1.)

For $\tau = (y_1, y_2, \dots, y_t, \dots)$, $y(h_t) = (0, 1)$ if $i_\tau = 1$ for all $\tau \leq t-1$. Otherwise $y(h_t) = (\frac{1}{2}, \frac{1}{2})$. Both strategies guarantee 0—the value of the game. However, for any $p \in (0, 1)$, $E_{\sigma\tau p}\{p_\infty - p\} = 0$ —the posterior limit coincides almost surely with the prior.

The following two sections deal with the one- and two-sided information games separately.

4.1. *One-sided information.* In this case take $R = 1$ and define the set A as:

$A = \{p \mid$ (i) For every $\epsilon > 0$ and sufficiently small, $\exists x = (x^1, x^2, \dots, x^K)$ such that $\min_y \sum p^k x^k A^k y = u(p) + O^+(\epsilon)$ and $\sum_{i \in I} \|p(x, i) - p\| \leq O^+(\epsilon)$,
(ii) u is differentiable relative to the face in which p lies}.

(Here $O^+(\epsilon)$ means a positive term of the same order of magnitude as ϵ , $O(\epsilon)$ and $o(\epsilon)$ mean term of the same and smaller orders, respectively. If p is not an interior point of Δ^K then u is differentiable when viewed as a function only of those elements of p that are strictly positive. Note that condition (i) implies that $v_1(p) > u(p)$. In the context of one-sided information games, the set B becomes: $B = \{p \mid v_x(p) = u(p)\}$. Denote by $\bar{X}$, the complement of a set X .

THEOREM 4. $B \subseteq \bar{A}$.

PROOF. It is enough to show that: $p \in A \Rightarrow v_x(p) > u(p)$ (so that $B \subseteq \bar{A}$). Therefore, take $p \in A$ and consider the following strategy for I (the informed player). In period 1 play a strategy x which satisfies (i) in the definition of A . If in the second period, the posterior is $p(x, i)$ (i.e. I chooses i in the first period) play a type independent strategy guaranteeing $u(p(x, i))$ in that and subsequent periods. This strategy guarantees an expected payoff to player I of $(1 - \delta)[u(p) + O^+(\epsilon)] + \delta \sum_{i \in I} \bar{x}_i u(p(x, i))$. Since u is differentiable relative to the face in which p lies and all posteriors derived from p must lie in this face also, expanding u around p for $p(x, i)$ close to p gives

$$u(p(x, i)) = u(p) + \nabla u(p) \cdot [p(x, i) - p] + o[\|p(x, i) - p\|].$$

Thus, since $\sum_{i \in I} \bar{x}_i \cdot [p(x, i) - p] = 0$,

$$\sum_{i \in I} \bar{x}_i u(p(x, i)) = u(p) + o\left[\sum_{i \in I} \bar{x}_i \|p(x, i) - p\|\right] = u(p) + o(\epsilon).$$

Therefore, this strategy guarantees:

$$u(p) + \{(1 - \delta) + \delta[o(\epsilon)/O^+(\epsilon)]\}O^+(\epsilon).$$

Since $[o(\epsilon)/O^+(\epsilon)] \rightarrow 0$ as $\epsilon \rightarrow 0$ and $O^+(\epsilon) > 0$ for all $\epsilon > 0$, for ϵ sufficiently small the last expression strictly exceeds $u(p)$. Thus I has a strategy guaranteeing a higher payoff than $u(p)$, so that $v_x(p) > u(p)$ and therefore $B \subseteq \bar{A}$. QED

In view of Theorem 3, Theorem 4 has the following corollary.

COROLLARY. Let $(\bar{\sigma}, \bar{\tau})$ be equilibrium strategies in the game with prior distribution p and let $\bar{\mu} = \mu_{\bar{\sigma}\bar{\tau}p}$ be the equilibrium measure on $\mathcal{F}_\infty \otimes 2^K$. Then $p_\infty \in \bar{A}$ a.s. $\bar{\mu}$.

Referring back to Example 1, it may be confirmed that $A = (0, 1)$ and so $p_\infty \in \bar{A} = \{0, 1\}$ almost surely. In that example, fix a discount rate δ and prior $p \in (0, 1)$. Then, if in stage 1 the informed player plays $\left[\frac{(1-p)+\epsilon}{p+\epsilon}\right]$ if type 1, $\left[\frac{(1-p)-\epsilon}{p-\epsilon}\right]$ if type 2 (ϵ sufficiently small), and then plays an optimal nonrevealing strategy thereafter, this strategy guarantees a payoff of

$$p(1-p) + \left\{ (1-\delta) - \delta \left(\frac{(2p-1)^2 - 1}{[(1-p) + \epsilon(2p-1)][p - \epsilon(2p-1)]} \right) \epsilon \right\}.$$

For ϵ sufficiently small this is greater than $p(1-p)$.

One might conjecture that the set A could more simply be defined as $A = \{p | v_1(p) > u(p)\}$, however this is not the case—as Example 3 below illustrates. The key property of this example is that if the informed player is to achieve a higher payoff than $u(p)$ (the payoff achievable with no information usage by a type independent strategy) he must reveal a “lot” of information: any strategy guaranteeing the informed player a stage game payoff higher than $u(p)$ leads to a posterior distribution close to either 0 or 1. However, in either case the posterior distribution is strategically very unfavourable to the informed player in the sense that $v_\infty(p)$ is relatively small for p close to 0 or 1. This leads to the informed player having an optimal nonrevealing strategy in the infinitely repeated game. In this example at some calculations show that $v_1(\frac{1}{2}) = 4$ and $u(\frac{1}{2}) = 0$. However, in the repeated game any strategy for the informed player guaranteeing more than 0 in the first period leads to posteriors (in the second period) which are either greater than $51/55$ or less than $4/55$. On the set $E = [0, 4/55] \cup [51/55, 1]$, $v_1(p) \leq -1$ and so $v_\infty(p) \leq v_1(p) \leq -1$ on this set also. To push the payoff above 0 in the first period, the informed player must “spread” the posteriors into the set E and so obtains a payoff no greater than -1 in the remainder of the game. When $\delta > \frac{4}{5}$ (so that the remainder of the game is important) the unique optimal strategy in the first period is a type independent strategy. This implies that $v_\infty(\frac{1}{2}) = 0$. Thus, condition (1) in the definition of A is a necessary condition. The details are as follows:

EXAMPLE 3. In this example there are two player types for player I, with prior probability p that player I is type 1. For the infinitely repeated game, take $\delta \in (\frac{4}{5}, 1)$.

A^1	A^2	$A(p) = pA^1 + (1-p)A^2$
$\begin{bmatrix} 10 & -2 \\ 2 & -2 \\ -100 & -2 \end{bmatrix}$	$\begin{bmatrix} -2 & -100 \\ -2 & 2 \\ -2 & 10 \end{bmatrix}$	$\begin{bmatrix} 12p - 2 & -100 + 98p \\ 2(2p - 1) & 2(1 - 2p) \\ -2 - 98p & 10 - 12p \end{bmatrix}$

Thus,

$$\begin{aligned} v_1(p) &= -2 + 12p, & p &\leq \frac{1}{2}, \\ v_1(p) &= 10 - 12p, & p &> \frac{1}{2}. \end{aligned}$$

In particular, $v_1(0) = v_1(1) = -2$, $v_1(\frac{1}{2}) = 4$ and $v_1(\frac{1}{12}) = v_1(\frac{11}{12}) = -1$. Some calculation yields:

$$\begin{aligned} u(p) &= 2(1-2p)\{[8p - 102(1-p)]/[102(1-p) + 8p]\}, & p &\leq \frac{1}{2}, \\ u(p) &= 2(1-2p)\{[102p - 8(1-p)]/[8(1-p) + 102p]\} & p &> \frac{1}{2}. \end{aligned}$$

Let $p = \frac{1}{2}$ and recall that $v_\infty(\frac{1}{2})$ satisfies:

$$v_\infty(\frac{1}{2}) = \max_x \left\{ \min_y (1 - \delta) \left(\frac{1}{2} x^1 A^1 y + \frac{1}{2} x^2 A^2 y \right) + \delta \sum_i \bar{x}_i v_\infty(p(i)) \right\},$$

where $\bar{x}_i = \frac{1}{2}(x_i^1 + x_i^2)$. Therefore,

$$v_\infty(\frac{1}{2}) \leq \max_x \left\{ (1 - \delta) \frac{1}{2} \left(x^1 A^1 \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} + x^2 A^2 \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \right) + \delta \sum_i \bar{x}_i v_\infty(p(i)) \right\}.$$

Since the centre row of $A(\frac{1}{2})$ is $(0, 0)$, $v_\infty(\frac{1}{2}) \geq 0$. We show that $v_\infty(\frac{1}{2}) = 0$, and this is achievable only by a type independent strategy: only a type independent strategy ensures that the last expression is nonnegative. Let

$$x^1 = \begin{bmatrix} x_1^1 \\ x_2^1 \\ x_3^1 \end{bmatrix}, \quad x^2 = \begin{bmatrix} x_1^2 \\ x_2^2 \\ x_3^2 \end{bmatrix}$$

and note that

$$A^1 \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ -51 \end{bmatrix} \quad \text{and} \quad A^2 \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} -51 \\ 0 \\ 4 \end{bmatrix}$$

in the present example.

Expanding the right-hand side of the expression gives:

$$\begin{aligned} & \frac{1}{2} \max \{ (1 - \delta) [4x_1^1 - 51x_3^1 + 4x_3^2 - 51x_1^2] \\ & + \delta [(x_1^1 + x_1^2) v_\infty(p(1)) + (x_3^1 + x_3^2) v_\infty(p(3))] \\ & + ((1 - x_1^1 - x_1^2) + (1 - x_3^1 - x_3^2)) v_\infty(p(2)) \}. \end{aligned}$$

The coefficient on $(1 - \delta)$ may also be written: $[(4x_1^1 - 51x_3^1) + (4x_3^2 - 51x_1^2)]$. Consider the following two cases separately: $x_2^1 + x_2^2 = 2$ and $x_2^1 + x_2^2 < 2$.

In the first case ($x_2^1 + x_2^2 = 2$), the coefficient on $(1 - \delta)$ is 0 and $p(i) = \frac{1}{2}$, for all i . In this case then, the strategy of player I with $x_2^1 + x_2^2 = 2$ ensures a payoff of $\delta v_\infty(\frac{1}{2})$. (Note the " $\frac{1}{2}$ " outside the "max" expression.) Since $v_\infty(\frac{1}{2}) \geq 0$, if $x_2^1 + x_2^2 = 2$ is optimal then $v_\infty(\frac{1}{2}) = 0$.

In the second case ($x_2^1 + x_2^2 < 2$), one of the terms x_1^1 , x_3^1 , x_1^2 or x_3^2 must be strictly positive. The coefficient on δ is no greater than $v_\infty(\frac{1}{2})$ since $v_\infty(\frac{1}{2}) \geq v_\infty(p)$, $\forall p$. (This follows from two important facts— v_∞ is always concave and for a certain class of games, such as Example 3, $v_\infty(p) = v_\infty(1 - p)$. See the appendix for some discussion.) Thus in this case, the payoff cannot be higher than in the first case unless the coefficient on $(1 - \delta)$ is strictly positive. This implies that at least one of the terms $(4x_1^1 - 51x_3^1)$, $(4x_3^2 - 51x_1^2)$ is strictly positive.

First consider the case where one term is positive and one is negative, taking $(4x_1^1 - 51x_3^1) > 0$ and $(4x_3^2 - 51x_1^2) < 0$. Since $(4x_1^1 - 51x_3^1) > 0$ we may write

$$x_1^1 = \frac{51}{4} x_3^2 + \eta_1 \quad \text{with } \eta_1 > 0.$$

Note that

$$\begin{aligned} p(1) &= [x_1^1/(x_1^1 + x_1^2)] = [1/(1 + (x_1^2/x_1^1))] \\ &> [1/(1 + (4/51))] = (51/55), \end{aligned}$$

using the fact that $(x_1^2/x_1^1) < (4/51)$. The associated payoff is:

$$\begin{aligned} &\frac{1}{2} \left\{ (1 - \delta) [4\eta_1 + (4x_3^2 - 51x_3^1)] + \delta \left[\left(1 + \frac{51}{4} \right) x_1^2 + \eta_1 \right] v_\infty(p(1)) \right. \\ &\quad + \delta (x_3^1 + x_3^2) v_\infty(p(3)) \\ &\quad \left. + \delta \left[1 - \left(1 + \frac{51}{4} \right) x_1^2 - \eta_1 + 1 - x_3^1 - x_3^2 \right] v_\infty(p(2)) \right\}. \end{aligned}$$

This may be rewritten as:

$$\begin{aligned} &\frac{1}{2} [(1 - \delta)4 + \delta v_\infty(p(1))] \eta_1 + \frac{1}{2} (4x_3^2 - 51x_3^1)(1 - \delta) \\ &\quad + \delta \left\{ \frac{1}{2} \left[\left(1 + \frac{51}{4} \right) x_1^2 \right] v_\infty(p(1)) + \frac{1}{2} (x_3^1 + x_3^2) v_\infty(p(3)) \right. \\ &\quad \left. + \frac{1}{2} \left[1 - \left(1 + \frac{51}{4} \right) x_1^2 - \eta_1 + 1 - x_3^1 - x_3^2 \right] v_\infty(p(2)) \right\}. \end{aligned}$$

Consider the third term $(\delta \{ \cdot \})$ in this expression: the coefficient on each $v_\infty(p(i))$ (the probability of i) is nonnegative and since $v_\infty(\frac{1}{2}) \geq v_\infty(p)$ and $v_\infty(\frac{1}{2}) \geq 0$, the third term is no greater than $\delta v_\infty(\frac{1}{2})$. Thus the whole expression is no greater than

$$\frac{1}{2} [(1 - \delta)4 + \delta v_\infty(p(1))] \eta_1 + \frac{1}{2} (4x_3^2 - 51x_3^1)(1 - \delta) + \delta v_\infty(\frac{1}{2}).$$

Recall that $p(1) > 51/55$ and observe that $-1 = v_1(\frac{11}{12}) \geq v_\infty(\frac{51}{55}) \geq v_\infty(p(1))$. Consequently, since $(1 - \delta)4 < \delta$ this expression is negative, as we have assumed that $(4x_3^2 - 51x_3^1) \leq 0$.

In the case where both of the terms $(4x_1^1 - 51x_1^2)$ and $(4x_3^2 - 51x_3^1)$ are positive then with

$$x_1^1 = \frac{51}{4} x_1^2 + \eta_1 \quad \text{and} \quad x_3^2 = \frac{51}{4} x_3^1 + \eta_3$$

so that $p(1) > \frac{51}{55}$ and $p(3) < \frac{4}{55}$. A calculation similar to that above gives the payoff:

$$\begin{aligned} &\frac{1}{2} \left\{ (1 - \delta) [4\eta_1 + 4\eta_3] + \delta \left[\left(1 + \frac{51}{4} \right) x_1^2 + \eta_1 \right] v_\infty(p(1)) \right. \\ &\quad + \delta \left[\left(1 + \frac{51}{4} \right) x_3^1 + \eta_3 \right] v_\infty(p(3)) \\ &\quad \left. + \delta \left[1 - \left(1 + \frac{51}{4} \right) x_1^2 - \eta_1 + 1 - \left(1 + \frac{51}{4} \right) x_3^1 - \eta_3 \right] v_\infty(p(2)) \right\}. \end{aligned}$$

This may be rewritten as:

$$\begin{aligned} & \frac{1}{2}[(1-\delta)4 + \delta v_{\infty}(p(1))]\eta_1 + \frac{1}{2}[(1-\delta)4 + \delta v_{\infty}(p(3))]\eta_3 \\ & + \delta \left\{ \frac{1}{2} \left[\left(1 + \frac{51}{4}\right) x_1^2 \right] v_{\infty}(p(1)) + \frac{1}{2} \left[\left(1 + \frac{51}{4}\right) x_3^2 \right] v_{\infty}(p(3)) \right. \\ & \quad \left. + \frac{1}{2} \left[2 - \left(1 + \frac{51}{4}\right) (x_1^2 + x_3^2) - \eta_1 - \eta_3 \right] v_{\infty}(p(2)) \right\}. \end{aligned}$$

Again the third expression is no greater than $\delta v_{\infty}(\frac{1}{2})$ and with $(1-\delta)4 < \delta$, each of the first two expressions is negative (since $p(1)$ and $p(3)$ are both less than -1). Thus, when $\frac{4}{5} < \delta$, $v_{\infty}(\frac{1}{2}) \leq \delta v_{\infty}(\frac{1}{2})$. Since $v_{\infty}(\frac{1}{2}) \geq 0$, this implies that $v_{\infty}(\frac{1}{2}) = 0$ and this value is achieved only by a nonrevealing strategy. This completes the example.

4.2. Two-sided information. The previous discussion relied on the fact that an informed player could control the variation of the posterior distribution on player types. In the two-sided information case, actions of one player affect the information revealed by an opponent indirectly through the history. Difficulties of this sort are avoided by perturbing the game. Let: $X_{\epsilon} = \{x = (x^1, x^2, \dots, x^K) | x^k \in \Delta^I, x_i^k \geq \epsilon, \forall i \in I\}$ and $Y_{\epsilon} = \{y = (y^1, y^2, \dots, y^R) | y^r \in \Delta^J, y_j^r \geq \epsilon, \forall j \in J\}$, where ϵ is a small positive number. With this restriction, define functions $u^{\epsilon}(p, q)$, $v_n^{\epsilon}(p, q)$ and $v_{\infty}^{\epsilon}(p, q)$ analogous to the functions defined in §2: thus, for example, $v_1^{\epsilon}(p, q) = \max_x \min_y \sum p^k q^r x^k A^{kr} y^r$, where $x = (x^1, x^2, \dots, x^K) \in X_{\epsilon}$ and similarly for y . Call these games ϵ -restricted games and define the sets, $A_{\epsilon} = \{(p, q) | v_1^{\epsilon}(p, q) \neq u^{\epsilon}(p, q)\}$ and $u^{\epsilon}(p, q)$ is differentiable at (p, q) and $B_{\epsilon} = \{(p, q) | v_{\infty}^{\epsilon}(p, q) = u^{\epsilon}(p, q)\}$.

THEOREM 5. Let $(\bar{\sigma}_{\epsilon}, \bar{\tau}_{\epsilon})$ be equilibrium strategies of the infinitely repeated ϵ -restricted game with prior distributions (p, q) and let $\bar{\mu}_{\epsilon} = \mu_{\bar{\sigma}_{\epsilon}, \bar{\tau}_{\epsilon}, p, q}$. Then

$$(p_{\infty}, q_{\infty}) \in \bar{A}_{\epsilon} \cap B_{\epsilon} \quad a.s. \bar{\mu}_{\epsilon}.$$

PROOF. The proof uses essentially the same ideas as the proof of Theorem 2 and is omitted.

Appendix. Let a be a $(I \times 1)$ vector, b a $(1 \times J)$ vector and A an $I \times J$ matrix. Thus

$$a = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_I \end{bmatrix}, \quad b = [b_1 \dots b_J],$$

$$A = \begin{bmatrix} a_{11}, a_{12}, \dots, a_{1J} \\ a_{21}, a_{22}, \dots, a_{2J} \\ \vdots \\ a_{I1}, a_{I2}, \dots, a_{IJ} \end{bmatrix} = \begin{bmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \vdots \\ \mathbf{a}_I \end{bmatrix}$$

where $\mathbf{a}_i$ is the i th row of A . Let

$$\beta(a) = \begin{bmatrix} a_I \\ a_{I-1} \\ \vdots \\ a_1 \end{bmatrix}, \quad \beta(b) = [b_J, b_{J-1}, \dots, b_1] \quad \text{and}$$

$$\beta(A) = \begin{bmatrix} \beta(\mathbf{a}_I) \\ \beta(\mathbf{a}_{I-1}) \\ \vdots \\ \beta(\mathbf{a}_1) \end{bmatrix}.$$

Note that β is symmetric in the sense that $\beta(\beta(X)) = X$, X a vector or matrix. This is essential for the following result—the proposition is false when β is an arbitrary permutation.

PROPOSITION 1. *Let A^1, A^2 be two $I \times J$ matrices and $(p, 1-p)$ a probability distribution on the indices 1 and 2. Let $v_\infty(p)$ be the value of the infinitely repeated game defined by A^1, A^2 and p . Then $\beta(A^2) = A^1$ implies that $v_\infty(p) = v_\infty(1-p)$.*

PROOF. Denote by $v_n(p)$, the n -period repeated game with weight $(1-\delta)\delta^{t-1}$ on the t th period payoff. Then $v_n(p) = v_n(1-p)$ implies that $v_{n+1}(p) = v_{n+1}(1-p)$. To see this, let ξ^1, ξ^2 be optimal first-period strategies for 1 in the $n+1$ period game with prior p . Thus, with $\bar{\xi}_i = p\xi_i^1 + (1-p)\xi_i^2$,

$$v_{n+1}(p) = \min_y \left\{ (1-\delta) [p\xi^1 A^1 + (1-p)\xi^2 A^2] y \right. \\ \left. + \delta \sum \bar{\xi}_i v_n \left(\frac{p\xi_i^1}{p\xi_i^1 + (1-p)\xi_i^2} \right) \right\}.$$

Now, we show that $v_{n+1}(1-p) \geq v_{n+1}(p)$. Put $\eta^1 = \beta(\xi^2)$, $\eta^2 = \beta(\xi^1)$. Then

$$v_{n+1}(1-p) \geq \min_y \left\{ (1-\delta) [(1-p)\eta^1 A^1 + p\eta^2 A^2] y \right. \\ \left. + \delta \sum_i [(1-p)\eta_i^1 + p\eta_i^2] v_n \left(\frac{(1-p)\eta_i^1}{(1-p)\eta_i^1 + p\eta_i^2} \right) \right\}.$$

Observe that $\eta^1 A^1 = (\eta^1 a_1^1, \dots, \eta^1 a_J^1)$ with $A^1 = (a_1^1, \dots, a_J^1)$. Also, $\eta^1 = \beta(\xi^2)$, $a_1^1 = \beta(a_J^2)$, $a_2^1 = \beta(a_{J-1}^2), \dots, a_J^1 = \beta(a_1^2)$.

Thus,

$$\eta^1 A^1 = (\beta(\xi^2)\beta(a_J^2), \beta(\xi^2)\beta(a_{J+1}^2), \dots, \beta(\xi^2)\beta(a_1^2)) \\ = (\xi^2 a_J^2, \dots, \xi^2 a_1^2) = \beta(\xi^2 A^2),$$

so that $\eta^1 A^1 = \beta(\xi^2 A^2)$. Similarly, $\eta^2 A^2 = \beta(\xi^1 A^1)$ and therefore

$$(1-\delta) [(1-p)\eta^1 A^1 + p\eta^2 A^2] = (1-\delta) \beta [p\xi^1 A^1 + (1-p)\xi^2 A^2].$$

Since $\min_{y \in \Delta} \alpha \cdot y = \min_{y \in \Delta} \beta(\alpha) \cdot y$,

$$\min_y (1 - \delta) [(1 - p)\eta^1 A^1 + p\eta^2 A^2] y = \min_y (1 - \delta) [p\xi^1 A^1 + (1 - p)\xi^2 A^2] y.$$

Also, given any $i, \exists j$ such that $(1 - p)\eta_i^1 + p\eta_i^2 = (1 - p)\xi_j^2 + p\xi_j^1$ and

$$\frac{(1 - p)\eta_i^1}{(1 - p)\eta_i^1 + p\eta_i^2} = \frac{(1 - p)\xi_j^2}{(1 - p)\xi_j^2 + p\xi_j^1}.$$

This last fact implies that:

$$1 - \frac{(1 - p)\eta_i^1}{(1 - p)\eta_i^1 + p\eta_i^2} = \frac{p\xi_j^1}{p\xi_j^1 + (1 - p)\xi_j^2}.$$

Symmetry of v_n implies then

$$v_n \left(\frac{(1 - p)\eta_i^1}{(1 - p)\eta_i^1 + p\eta_i^2} \right) = v_n \left(\frac{p\xi_j^1}{p\xi_j^1 + (1 - p)\xi_j^2} \right).$$

Thus

$$\begin{aligned} \sum_i [(1 - p)\eta_i^1 + p\eta_i^2] v_n \left(\frac{(1 - p)\eta_i^1}{(1 - p)\eta_i^1 + p\eta_i^2} \right) \\ = \sum_j (p\xi_j^1 + (1 - p)\xi_j^2) v_n \left(\frac{p\xi_j^1}{p\xi_j^1 + (1 - p)\xi_j^2} \right). \end{aligned}$$

Therefore $v_{n+1}(1 - p) \geq v_{n+1}(p)$. Reversing the argument given above yields $v_{n+1}(p) \geq v_{n+1}(1 - p)$ and so $v_{n+1}(1 - p) = v_{n+1}(p)$. The discussion above implies directly that $v_1(p) = v_1(1 - p)$. Consequently, for any n , $v_{n+1}(1 - p) = v_{n+1}(p)$. By Theorem 1, $\lim_{n \rightarrow \infty} v_{n+1}(p) = \lim_{n \rightarrow \infty} v_{n+1}(1 - p) = v_\infty(p) = v_\infty(1 - p)$. QED

PROPOSITION 2. v_∞ is concave in p .

PROOF. This result is given in Aumann and Maschler (1966) (see Zamir 1974 for a short proof) in the case where the limiting average payoff criterion is used. The same proof can be applied with discounting—a short sketch is as follows. Consider a finitely repeated game of length n . Take two prior distributions p_1 and p_2 and some $\alpha \in [0, 1]$ and let $p = \alpha p_1 + (1 - \alpha)p_2$. Consider two games. In the first a prior, either p_1 or p_2 , is chosen with probability α and $(1 - \alpha)$ respectively. Player II is informed which prior is chosen, I is fully informed and they play the n stage game. The second game differs from the first in that player 2 is not informed as to which prior (p_1 or p_2) is chosen—so player II's information is represented by the distribution $p = \alpha p_1 + (1 - \alpha)p_2$. The first game has value $\alpha v_n(p_1) + (1 - \alpha)v_n(p_2)$ and the second has value $v_n(p)$. Since player II, the minimizer is better informed in the first case $\alpha v_n(p_1) + (1 - \alpha)v_n(p_2) \leq v_n(p)$. Hence, $\forall n$, $v_n(\cdot)$ is a concave function and so the uniform limit $v_\infty(\cdot)$ is also concave. QED

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