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<td>Levene, Rupert H.; Timoney, Richard M.</td>
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Abstract. It is well-known that if $T$ is a $D_m$–$D_n$ bimodule map on the $m \times n$ complex matrices, then $T$ is a Schur multiplier and $\|T\|_{cb} = \|T\|$. If $n = 2$ and $T$ is merely assumed to be a right $D_2$-module map, then we show that $\|T\|_{cb} = \|T\|$. However, this property fails if $m \geq 2$ and $n \geq 3$. For $m \geq 2$ and $n = 3, 4$ or $n \geq m^2$ we give examples of maps $T$ attaining the supremum $C(m, n) = \sup\{\|T\|_{cb} : T$ a right $D_n$-module map on $M_{m,n}$ with $\|T\| \leq 1\}$, we show that $C(m, m^2) = \sqrt{m}$ and succeed in finding sharp results for $C(m, n)$ in certain other cases. As a consequence, if $H$ is an infinite-dimensional Hilbert space and $D$ is a masa in $B(H)$, then there is a bounded right $D$-module map on $K(H)$ which is not completely bounded.

Keywords: completely bounded, right module map, matrix numerical range, tracial geometric mean, fidelity

MSC (2010): 46L07, 47L25, 15A60, 47A30

1. Introduction

Let $H$ be a Hilbert space, let $B(H)$ be the algebra of bounded linear operators on $H$, let $K(H)$ be the ideal of compact operators and let $D$ be a masa in $B(H)$. If $T: K(H) \to K(H)$ is a bounded $D$-bimodule map, then it is well-known that that $\|T\|_{cb} = \|T\|$ (see [11, 8, 9]). While it would certainly be of use to be able to extend this to larger natural classes than $D$-bimodule maps (generalised Schur multipliers), in the present paper, we consider the effect of relaxing the hypothesis of bimodularity to one-sided modularity over $D$. While we establish a positive result for dimension 2, we give increasing bounds for higher finite dimensions and a negative answer for the following question [4, Remark 7.10]:

**Question 1.1.** If $H$ is infinite-dimensional and $D$ is a masa in $B(H)$, is there a constant $C > 0$ such that $\|T\|_{cb} \leq C\|T\|$ for every bounded, left $D$-module map $T: K(H) \to K(H)$?

By symmetry, this question is unchanged if we replace “left” by “right”, and this makes our notation marginally neater. So we will focus on right $D$-module maps.

Of course, if $H$ is finite dimensional, then the answer to this question is yes even if we discard the modularity condition. It then becomes interesting to estimate the optimal constant $C$. Hence we are led to consider the constants

$$C(m, n) = \sup\{\|T\|_{cb} : T$ a right $D_n$-module map on $M_{m,n}$, $\|T\| \leq 1\}$$

where $M_{m,n}$ is the space of $m \times n$ complex matrices and $D_n$ is the algebra of diagonal $n \times n$ matrices.

The structure of the paper is as follows. We first establish some notation and give some preliminary results in Section 2. In Section 3 we use the second author’s work on elementary operators to show that $C(m, 2) = 1$ for every $m \geq 1$. Section 4 contains some technical results comparing the completely bounded norm to the norm arising from the Hilbert-Schmidt norm, and these are used in Section 5 to find some upper bounds for $C(m, n)$. In the next section we construct examples
which show that $C(m,n)$ grows with $m,n$. This leads naturally to a counterexample (in Corollary 6.12) answering Question 1.1, and we are also able to determine the values of $C(m,n)$ in some cases. Finally, in Section 7 we briefly consider similar problems when we restrict attention to special classes of right module maps.

In the last two sections, we pose several unresolved questions about the behaviour of the constants $C(m,n)$.

2. Preliminaries

If $X$ is a vector space, we write $\mathcal{L}(X)$ for the space of linear maps $X \to X$. If $m,n \in \mathbb{N}$, then $M_{m,n}(X)$ is the vector space of $m \times n$ matrices with entries in $X$.

We will write elements of $M_{m,n}(X)$ as $[x_{ij}]_{1 \leq i \leq m, 1 \leq j \leq n}$ or simply $[x_{ij}]$, where each $x_{ij}$ is in $X$. If $T \in \mathcal{L}(X)$ and $m,n \in \mathbb{N}$, then the $(m,n)$-ampliation of $T$ is the map $T_{m,n} \in \mathcal{L}(M_{m,n}(X))$ given by $T_{m,n}[x_{ij}] = [Tx_{ij}]$. We also write $T_n = T_{n, n}$.

Given a norm $\| \cdot \|$ on $X$, the corresponding operator norm, or simply the norm, of a map $T \in \mathcal{L}(X)$ is

$$\|T\| = \sup \{\|Tx\| : x \in X, \|x\| \leq 1\}.$$ 

If we are given norms on $M_{m,n}(X)$ for all $m,n \in \mathbb{N}$, then the completely bounded norm of $T$ is

$$\|T\|_{cb} = \sup_{m,n \geq 1} \|T_{m,n}\|.$$ 

Provided the inclusions of $M_{m,n}(X)$ into $M_{m+1,n}(X)$ and $M_{m,n+1}(X)$ which pad a matrix with an extra row or column of zeros are isometries, we have

$$\|T\| = \|T_1\| \leq \|T_2\| \leq \|T_3\| \leq \cdots \leq \|T\|_{cb} = \sup_{n \geq 1} \|T_n\|.$$ 

For $n \in \mathbb{N}$ let $\mathbb{C}^n$ denote the Hilbert space of dimension $n$ whose elements are to be thought of as column vectors with $n$ complex entries, with the $\ell^2$ norm, and we will also write $\mathbb{C}^\infty$ for $\ell^2(\mathbb{N})$. For $m,n \in \mathbb{N} \cup \{\infty\}$, we write

$$M_{m,n} = B(\mathbb{C}^n, \mathbb{C}^m) = \{x \in \mathcal{L}(\mathbb{C}^n, \mathbb{C}^m) : \|x\| < \infty\}$$

and $M_m = M_{m,m}$. If $s,t \in \mathbb{N}$, then $M_{s,t}(M_{m,n})$ can be naturally identified with the normed vector space $M_{sm,tn}$, and hence inherits the norm from the latter space. Adding a row or column of zeros is then an isometry.

If $v,w \in \mathbb{C}^n$, then $vw^*$ denotes the rank one operator in $M_n$ given by

$$vw^*(x) = \langle x, w \rangle v \quad \text{for} \quad x \in \mathbb{C}^n.$$ 

For $1 \leq i \leq n$ (or for $i \geq 1$, if $n = \infty$) we write $e_i$ for the $i$th standard basis vector in $\mathbb{C}^n$. Then $D_n$, the diagonal masa of $M_n$, is the von Neumann algebra generated by the diagonal matrix units $e_i e_i^*$.

Let $n \in \mathbb{N} \cup \{\infty\}$, let $b_1, \ldots, b_t \in M_n$ and let $b = \begin{bmatrix} b_1 \\ \vdots \\ b_t \end{bmatrix}$. For $\xi \in \mathbb{C}^n$, let $Q(b, \xi)$ be the positive semi-definite $\ell \times \ell$ matrix

$$Q(b, \xi) = [(b_i \xi, b_j \xi)]_{1 \leq i, j \leq \ell}.$$ 

We recall the definitions from [12] of the matrix numerical range of $b$,

$$W_a(b) = \{Q(b, \xi) : \xi \in \mathbb{C}^n, \|\xi\| = 1\}$$

and the matrix extremal numerical range of $b$,

$$W_{a,e}(b) = \{\beta \in W_a(b) : \text{trace}(\beta) = \|b\|^2\},$$

(where the norm $\|b\|$ is computed with respect to the norm on $M_{\ell,1}(M_n)$ described above). It is easy to see that $W_{a,e}(b)$ is the set of elements of the closure of $W_a(b)$.
of maximal trace. If $n < \infty$ then $W_m(b)$ is a continuous image of the unit sphere of $\mathbb{C}^n$, which is compact. Hence in this case,

$$W_{m,c}(b) = \{Q(b, \xi) : \xi \in \mathbb{C}^n, \|\xi\| = 1, b^*b\xi = \|b\|^2\xi\}.$$ 

Observe that the vectors $\xi$ appearing in this expression are precisely the unit vectors in the eigenspace of $b^*b$ corresponding to its maximal eigenvalue.

If $a = [a_1 \ldots a_\ell]$ and $b = \begin{bmatrix} b_1 \\ \vdots \\ b_\ell \end{bmatrix}$ for some $a_j \in \mathcal{L}(X)$ and $b_j \in \mathcal{L}(Y)$, then we will write $T = a \circ b$ or say that “$a, b$ represent $T$” to mean that $T$ is the elementary operator

$$T : \mathcal{L}(Y, X) \to \mathcal{L}(Y, X), \quad x \mapsto \sum_{j=1}^\ell a_jxb_j.$$ 

Such a representation of $T$ is far from unique due to bilinearity in $(a, b)$; for example, if $T = a \circ b$, then we also have $T = (a\alpha^{-1}) \circ (\alpha b)$ for any invertible matrix $\alpha \in M_\ell$.

If $D$ is a subring of $M_n$ then $M_{m,n}$ is a right $D$-module. A right $D$-module map on $M_{m,n}$ is a linear map $T \in \mathcal{L}(M_{m,n})$ such that

$$T(xd) = T(x)d \quad \text{for all } x \in M_{m,n} \text{ and all } d \in D.$$ 

We write $\mathcal{L}_D(M_{m,n})$ for the set of all right $D$-module maps on $M_{m,n}$.

**Remark 2.1.** If $n \in \mathbb{N}$ and $T$ is a bounded right $D_n$-module map on $M_{m,n}$, then $T$ is an elementary operator of the form $Tx = \sum_{j=1}^n a_jxb_j$ for some $b_j \in D_n$ and $a_j \in M_m$. Indeed, for each $j$, the map $v \mapsto T(ve_j^*)e_j$ is linear $\mathbb{C}^m \to \mathbb{C}^m$, and it is bounded since $T$ is bounded. Hence there is an operator $a_j \in M_m$ such that $a_jv = T(ve_j^*)e_j$ for $v \in \mathbb{C}^m$. We call the operators $a_j$ the column operators of $T$.

Writing $b_j = e_j^* e_j$, we have

$$\sum_{j=1}^n a_jxb_j = \sum_{j=1}^n a_jxe_j^*e_j^* = \sum_{j=1}^n T(xe_j^*)e_j^*e_j^* = \sum_{j=1}^n T(x)e_j^*e_j^* = T(x).$$

We have found a representation $T = a' \circ b'$ where $a' = [a_1 \ldots a_\ell]$ and $b' = \begin{bmatrix} b_1 \\ \vdots \\ b_\ell \end{bmatrix}$, and each $b_j$ is diagonal. As discussed in [12, §3], there is a representation $T = a \circ b$ where the entries of $a$ and $b$ are linear combinations of the entries of $a'$ and $b'$, respectively, so that

$$\|T\|_{cb} = \|a\| \|b\| = \frac{1}{2}(\|a\|^2 + \|b\|^2).$$ 

Observe that the entries of $b$ are then diagonal, and $\|a\| = \|b\|$ by the arithmetic mean/geometric mean inequality. In [12, Theorem 3.3], the second author shows that a representation $T = a \circ b$ satisfies these equalities if and only if

$$\text{conv } W_{m,c}(a^*) \cap \text{conv } W_{m,c}(b) \neq \emptyset$$

where conv $S$ denotes the convex hull of a subset $S$ of a vector space.

If $n = \infty$, so that $T$ is a bounded right $D_\infty$-module map on $\mathcal{B}(H, \mathbb{C}^n)$ where $H = \ell^2(\mathbb{N})$, then the same argument gives $Tx = \sum_{j=1}^\infty a_jxb_j$, where the operators $a_j \in M_m$ are given by $a_jv = T(ve_j^*)e_j$ and $b_j = e_j^*e_j \in \mathcal{B}(H)$, and the series converges in the strong operator topology.

The relevance of the following lemma to our problem is plain in light of Remark 2.1, and condition (★) in particular.
Lemma 2.2. Let \( n \in \mathbb{N} \), let \( \ell \in \mathbb{N} \) and let \( b_1, \ldots, b_\ell \in D_n \). If \( b = \begin{bmatrix} b_1 \\ \vdots \\ b_\ell \end{bmatrix} \), then

\[
W_{m,e}(b) = \text{conv}\{Q(b, e_p) : 1 \leq p \leq n, \ b^*be_p = \|b\|^2e_p\}.
\]

In particular, \( W_{m,e}(b) \) is convex.

Proof. The matrix \( b^*b = \sum_{j=1}^\ell b_j^*b_j \) is positive semi-definite and diagonal with largest eigenvalue \( \|b\|^2 \). Let \( r \) be the dimension of the corresponding eigenspace. Permuting \( e_1, \ldots, e_n \) if necessary, we have \( b^*b = \|b\|^2(I_r \oplus d) \) for some positive semi-definite \( d \in D_{n-r} \) with \( \|d\| < 1 \). So \( Q(b, \xi) \in W_{m,e}(b) \) if and only if \( \xi = \sum_{p=1}^r \xi_p e_p \) for some \( \xi_p \in \mathbb{C} \) such that \( \sum_{p=1}^r |\xi_p|^2 = 1 \). Each \( b_j \) is diagonal so the vectors \( e_q \) are eigenvectors, hence

\[
Q(b, \xi) = \sum_{p,q=1}^r \langle b_\ell \xi_p e_p, b_j \xi_q e_q \rangle = \sum_{p=1}^r \|b_\ell \xi_p e_p\|^2 = \sum_{p=1}^r |\xi_p|^2Q(b, e_p).
\]

The following argument is essentially contained in any of [8, 9, 11].

Lemma 2.3. Let \( H, K \) be Hilbert spaces, let \( X \) be a subspace of \( B(H, K) \), and let \( A \subseteq B(H) \) be a right norming set for \( X \), meaning that \( xa \in X \) for all \( x \in X \) and \( a \in A \), and for every \( n \geq 1 \) and every \( z \in M_n(X) \), we have

\[
\|z\|_{M_n(X)} = \sup\{\|zb\|_{M_n,h(X)} : b \in M_{n,1}(A), \ |b| \leq 1\}.
\]

If \( T : X \rightarrow X \) is a bounded, linear map such that \( T(xa) = T(x)a \) for all \( x \in X \), \( a \in A \), then \( \|T_n\| = \|T\| \) for all \( n \geq 1 \).

Proof. The inequality \( \|T_{n,1}\| \leq \|T_n\| \) is clear. On the other hand, if \( z \in M_n(X) \) and \( b \in M_{n,1}(A) \), then

\[
T_n(z)b = \left[ \sum_j T(z_{ij})b_j \right]_i = \left[ \sum_j T(z_{ij})b_j \right]_i = T_{n,1}(zb)
\]

and \( \|zb\| \leq \|z\|_{M_n(X)} \|b\|_{M_{n,1}(A)} \). Since \( A \) is a right norming set for \( X \),

\[
\|T_n\| = \sup\{\|T_n(z)\|_{M_n(X)} : z \in M_n(X), \ |z| \leq 1\}
\]

\[
= \sup\{\|T_{n,1}(zb)\|_{M_n,h(X)} : b \in M_{n,1}(A), \ |b| \leq 1, \ z \in M_n(X), \ |z| \leq 1\}
\]

\[
\leq \|T_{n,1}\|.
\]

As shown in [8, 11], the set \( D_n \) of diagonal matrices in \( M_n \) is a right norming set for \( M_{n,n} \). Thus we immediately obtain:

Proposition 2.4. If \( m, n \in \mathbb{N} \cup \{\infty\} \) and \( T \) is a right \( D_n \)-module map on \( M_{m,n} \), then \( \|T\|_{cb} = \sup_{k \geq 1} \|T_k\|_{cb} \).

Remark 2.5. If \( m = 1 \) or \( n = 1 \) (that is, if the matrices on which our maps act have either one row, or one column) then \( \|T\|_{cb} = \|T\| \) for every \( T \in \mathcal{L}(M_{m,n}) \). For if \( n = 1 \), then \( M_{m,n} = \mathbb{C}^m \), and every linear map \( T : \mathbb{C}^m \rightarrow \mathbb{C}^m \) may be written as \( Tx = ax \) for \( a \in \mathbb{C}^m \). Hence \( \|T\| = \|a\| \). Moreover \( T_k : M_k(\mathbb{C}^m) \rightarrow M_k(\mathbb{C}^m) \) is given by left multiplication by a block diagonal matrix \( a^{(k)} \) (with \( k \) copies of \( a \) on the diagonal), and \( \|T_k\| = \|a^{(k)}\| = \|a\| = \|T\| \), so \( \|T\|_{cb} = \|T\| \). If \( m = 1 \), we can apply a similar argument with right multiplication or use the previous case on the map \( T^* : M_{n,m} \rightarrow M_{n,m} \) given by \( T^*(x) = T(x)^* \).
We now show that, surprisingly, the conclusion \( \|T\|_{cb} = \|T\| \) of Remark 2.5 persists for right \( D_2 \)-module maps on \( M_{m,2} \).

**Lemma 3.1.** If \( X \) is a set of positive semi-definite \( 2 \times 2 \) matrices with trace 1 and there is a rank one projection \( p \in \text{conv} \ X \), then \( p \in X \).

**Proof.** Conjugating by a suitable unitary matrix, we may assume that \( p = e_1 e_1^* \). Now \( p \) is a convex combination of some \( \alpha_1, \ldots, \alpha_k \in X \) and each \( \alpha_j \) is positive semi-definite. Since the \( (2,2) \) entry of \( p \) is zero, the \( (2,2) \) entry of each \( \alpha_j \) is zero, which implies that the off-diagonal entries of each \( \alpha_j \) are also zero. Since trace \( \alpha_j = 1 \), we have \( \alpha_j = p \) for all \( j \). \( \square \)

**Theorem 3.2.** If \( m \in \mathbb{N} \) and \( T : M_{m,2} \to M_{m,2} \) is a right \( D_2 \)-module map, then \( \|T\|_{cb} = \|T\| \).

**Proof.** Suppose \( \|T\|_{cb} = 1 \). By Remark 2.1, \( Tx = a_1 xb_1 + a_2 xb_2 \) for some \( a_1, a_2 \in M_m \) and \( b_1, b_2 \in D_2 \) such that \( \|a\| = \|b\| = 1 \) where \( a = [a_1 \ a_2] \) and \( b = [b_1 \ b_2] \). By Lemma 2.2, \( W_{m,e}(b) \) is convex, so by [12, Theorem 3.3], \( W_{m,e}(b) \) intersects the convex hull of \( W_{m,e}(a^*) \). By [12, Proposition 3.1], it suffices to show that \( W_{m,e}(a^*) \cap W_{m,e}(b) \neq \emptyset \).

Observe that \( b^* b = b_1^* b_1 + b_2^* b_2 \) is a \( 2 \times 2 \) diagonal positive semi-definite matrix of norm 1, so its 1-eigenspace \( E \) dominates its Hilbert-Schmidt norm. Hence \( \|b\| \leq 1 \), so its 1-eigenspace \( E \) has dimension 1 or 2.

If \( \dim E_1(b^* b) = 1 \), then \( b^* b = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \) or \( b^* b = \begin{bmatrix} t & 0 \\ 0 & 1 \end{bmatrix} \) for some \( t \in [0,1] \). If \( b^* b = \begin{bmatrix} t & 0 \\ 0 & 1 \end{bmatrix} \), then
\[
W_{m,e}(b) = \{Q(b, ze_1) : z \in T\} = \{Q(b, e_1)\} = \{e_1 e_1^*\}.
\]
Since \( e_1 e_1^* \) is a rank one projection in \( \text{conv} \ W_{m,e}(a^*) \), we have \( e_1 e_1^* \in W_{m,e}(a^*) \) by Lemma 3.1. Hence \( W_{m,e}(a^*) \cap W_{m,e}(b) \neq \emptyset \). Similarly, if \( b^* b = \begin{bmatrix} t & 0 \\ 0 & 1 \end{bmatrix} \), then \( e_1 e_1^* \in W_{m,e}(a^*) \cap W_{m,e}(b) \neq \emptyset \).

Now suppose that \( \dim E_1(b^* b) = 2 \). Then \( b^* b = I_2 \), so if we write \( \beta_i = Q(b, e_i) \) for \( i = 1, 2 \), then Lemma 2.2 shows that \( W_{m,e}(b) = \text{conv} \{\beta_1, \beta_2\} \). For \( i = 1, 2 \), let us write \( b_i = \begin{bmatrix} b_{i1} & 0 \\ 0 & b_{i2} \end{bmatrix} \) and \( v_i = \begin{bmatrix} b_{i1} \\ b_{i2} \end{bmatrix} \). A simple calculation reveals that \( \|v_i\| = 1 \) and \( \beta_i = v_i \beta_i^* \). If \( \beta_1 = \beta_2 \), then this rank one projection is in \( W_{m,e}(a^*) \) by Lemma 3.1, so \( W_{m,e}(a^*) \cap W_{m,e}(b) \neq \emptyset \). So we may assume that \( \beta_1 \neq \beta_2 \), so that \( W_{m,e}(b) \) is the proper closed line segment joining \( \beta_1 \) and \( \beta_2 \).

For \( t \in \mathbb{R} \), let \( \beta(t) = t \beta_1 + (1-t) \beta_2 \) and consider the closed convex set
\[
S = \{t \in \mathbb{R} : \beta(t) \in \text{conv} \ W_{m,e}(a^*)\}.
\]
Now \( \beta_1 \) and \( \beta_2 \) are distinct and \( \|\beta\|_2 = 1 \), where \( \|\cdot\|_2 \) is the Hilbert-Schmidt norm on \( M_2 \). Moreover, \( (M_2, \|\cdot\|_2) \) is strictly convex, and its closed unit ball contains \( W_{m,e}(a^*) \) since the trace-class norm of every matrix in \( W_{m,e}(a^*) \) is 1, which dominates its Hilbert-Schmidt norm. Hence \( S \subseteq [0,1] \), say \( S = [s_1, s_2] \) where \( 0 \leq s_1 \leq s_2 \leq 1 \), and \( \beta(s_1) \) and \( \beta(s_2) \) are in the boundary of \( \text{conv} \ W_{m,e}(a^*) \), and are the extreme points of \( \text{conv} \ W_{m,e}(a^*) \cap W_{m,e}(b) \).

Given a hermitian \( 2 \times 2 \) matrix \( a \) with trace 1, say \( a = \begin{bmatrix} a & b \\ b & 1-a \end{bmatrix} \), let us write
\[
\theta(a) = (a, \text{Re} b, \text{Im} b) \in \mathbb{R}^3.
\]
Observe that the map \( \theta \) defined on this convex set of matrices is injective and respects convex combinations. Consider
\[
e = \theta(\beta(s_1)), \quad L = \theta(W_{m,e}(b)), \quad W = \theta(W_{m,e}(a^*)), \quad C = \text{conv} W.
\]
By construction, $e$ is an extreme point of $C \cap L$ which lies in the boundary of $C$. Let $\Pi$ be a supporting hyperplane for $e$, so that

$$e \in \Pi = \{x \in \mathbb{R}^3 : \langle x, \eta \rangle = 0\}$$

for some non-zero vector $\eta = (\eta_1, \eta_2, \eta_3) \in \mathbb{R}^3$ and some $r \in \mathbb{R}$, chosen so that

$$C \subseteq \Pi^+ = \{x \in \mathbb{R}^3 : \langle x, \eta \rangle \geq r\}.$$
4. CB norms and Hilbert-Schmidt norms

Given \( n, m \), let \( \mathcal{L}(M_{m,n}) \) be the set of linear maps \( M_{m,n} \to M_{m,n} \). For a map \( T \in \mathcal{L}(M_{m,n}) \), we continue to write

\[
\|T\| = \sup \{ \|Tx\| : x \in M_{m,n}, \|x\| = 1 \}
\]

for the operator norm of \( T \) with respect to the operator norm \( \| \cdot \| \) on \( M_{m,n} \), and we will also consider the quantity

\[
\|||T||| = \sup \{ \|Tx\|_2 : x \in M_{m,n}, \|x\|_2 = 1 \},
\]

that is, the operator norm of \( T \) with respect to the Hilbert-Schmidt norm \( \| \cdot \|_2 \) on \( M_{m,n} \). Note that if \( n = \infty \) or \( m = \infty \), then all of these “norms” may take the value \( \infty \).

For \( T \in \mathcal{L}(M_{m,n}) \), let \( T^* \in \mathcal{L}(M_{n,m}) \) be the map given by

\[
T^*(x) = (T^*)_x, \quad x \in M_{n,m}.
\]

Clearly, \( \|T^*\| = \|T\| \) and \( \|||T^*||| = |||T||| \).

**Remark 4.1.** The norm \( |||\cdot||| \) behaves particularly nicely when we take ampliations: if \( T \in \mathcal{L}(M_{m,n}) \) and \( s, t \in \mathbb{N} \), then viewing \( T_{s,t} \) as a map on \( M_{ms,nt} \), we have \( |||T_{s,t}||| = |||T||| \). Indeed, the inequality \( |||T||| \leq |||T_{s,t}||| \) is trivial, and

\[
|||T_{s,t}||| = \sup \{ \|Tx_{ij}\|_2 : x_{ij} \in M_{ms,nt} \text{ has } \|x_{ij}\|_2 = 1 \}
\]

where the suprema are taken over those \( x_{ij} \in M_{m,n} \) (for \( 1 \leq i \leq s \) and \( 1 \leq j \leq t \)) so that \( [x_{ij}] \in M_{ms,nt} \) has \( \|x_{ij}\|_2 = 1 \).

Below, we show that in many cases, \( |||\cdot||| \) is comparable with the operator norm for the right module maps \( T \) under consideration. This allows us to estimate \( |||T||| \) and \( |||T|||_{cb} \), and these estimates are used to find some upper bounds for \( |||T|||_{cb}/|||T||| \).

**Proposition 4.2.** Let \( m, n \in \mathbb{N} \cup \{ \infty \} \). If \( T: M_{m,n} \to M_{m,n} \) is a right \( D_n \)-module map with column operators \( \{a_j : 1 \leq j < n + 1\} \), then

\[
|||T||| = \sup_j \|a_j\| \leq \|T\|.
\]

**Proof.** Recall that the column operators \( a_j \in \mathcal{L}(\mathbb{C}^m) \) of \( T \) were defined in Remark 2.1. Suppose, for convenience of notation, that \( n < \infty \). Let \( a \in \mathcal{L}((\mathbb{C}^m)^n) \) be the diagonal direct sum of \( a_1, \ldots, a_n \), so that \( a(\xi_1, \ldots, \xi_n) = (a_1 \xi_1, \ldots, a_n \xi_n) \) for \( \xi_1, \ldots, \xi_n \in \mathbb{C}^m \). Then \( \|a\| = \max_j \|a_j\| \). Since \( T \) is a right \( D_n \)-module map, we
Proposition 4.4. Let $T$ be observing that if $\parallel M \parallel = \parallel T \parallel$ so $8 RUPERT H. LEVENE AND RICHARD M. TIMONEY$

Suppose $\parallel M \parallel$: $\parallel T \parallel m > n \parallel T x \parallel : x \in M_{m,n}, \parallel x \parallel \leq 1\parallel T \parallel = \sup \{\parallel T x \parallel : x \in M_{m,n}, \parallel x \parallel \leq 1\parallel T \parallel = \sup \{\sum_{j=1}^{n} \parallel T(x) e_j \parallel^2 : x \in M_{m,n}, \parallel x \parallel \leq 1\parallel T \parallel = \sup \{\sum_{j=1}^{n} \parallel T(x e_j) \parallel^2 : x \in M_{m,n}, \sum_{j=1}^{n} \parallel x e_j \parallel^2 \leq 1\parallel T \parallel = \sup \{\sum_{j=1}^{n} \parallel a_j(x e_j) \parallel^2 : x \in M_{m,n}, \sum_{j=1}^{n} \parallel a_j \parallel^2 \leq 1\parallel T \parallel = \sup \{\sum_{j=1}^{n} \parallel a_j \parallel^2 : \parallel a_j \parallel \leq 1\parallel T \parallel = \parallel a \parallel = \max_{j} \parallel a_j \parallel.\}

Moreover, if $\eta \in C^n$ and $1 \leq j \leq n$ then $\parallel \eta e_j \parallel = \parallel \eta \parallel$, so $\parallel T \parallel = \sup \{\parallel T x \parallel : x \in M_{m,n}, \parallel x \parallel \leq 1\parallel T \parallel \geq \sup \{\|T(\eta e_j)\| : \eta \in C^n, \|\eta\| \leq 1\} = \sup \{\|a_j \eta\| : \eta \in C^n, \|\eta\| \leq 1\} = \|a_j\|,$ so $\parallel T \parallel \geq \max_j \|a_j\|.

If $n = \infty$ then the proof is similar. $\square$

The following lemma will be used to obtain a useful inequality in the other direction in Proposition 4.4 below.

Lemma 4.3. Let $m, n \in \mathbb{N} \cup \{\infty\}$ with $k = \min\{m, n\} < \infty$. If $T : M_{m,n} \to M_{m,n}$ is a linear map, then $\|T\| \leq \sqrt{k} \|T\|.$

Proof. Suppose $k = m \leq n$ and $\|T\| = 1$. For $x \in M_{m,n}$, let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$ be the eigenvalues of $xx^*$. We have

$$\|x\|^2 = \lambda_1 \leq \|x\|^2 = \sum_{j=1}^{k} \lambda_j = k \lambda_1 = k \|x\|^2,$$

so $\|x\| \leq \|x\| \leq \sqrt{k} \|x\|$. Hence $\|T(x)\| \leq \|T(x)\| \leq \|x\| \leq \sqrt{k} \|x\|$, so $\|T\| \leq \sqrt{k}.$

If $m > n$, consider the map $T^* \in \mathcal{L}(M_{n,m}). \square$

If $c_1 \in M_{m,n}$ and $k \in \mathbb{N}$, then we write $c_1^{(k)}$ for the block-diagonal operator in $M_k(M_{m,n})$ with $k$ copies of $c_1$ running down the diagonal. Similarly, if $c = \begin{bmatrix} c_1 & \cdots & c_k \end{bmatrix}$

where $c_1, \ldots, c_k \in M_{m,n}$, then $c^{(k)} = \begin{bmatrix} c_1^{(k)} & \cdots & c_k^{(k)} \end{bmatrix}$. The utility of this notation is revealed by observing that if $T = a \oplus b$ and $s, t \in \mathbb{N}$, then $T_{s,t} = a^{(s)} \oplus b^{(t)}$.

Proposition 4.4. Let $\ell, n \in \mathbb{N}$, let $k = \min\{\ell, n\}$ and let $K = \min\{\ell^2, n\}$. If $T : M_{\ell,n} \to M_{\ell,n}$ is a right $D_n$-module map, then

$$\|T\|_{cb} = \|T_{k,1}\| \leq \sqrt{k} \|T\|.$$

In particular, $\|T\|_{cb} = \|T_{n,1}\| \leq \sqrt{n} \|T\|.$
Proof. By Remark 2.1, \( T \) is an elementary operator, and there are matrices \( a_1, \ldots, a_n \in M_k \) and \( b_1, \ldots, b_n \in D_n \) such that \( Tx = \sum_{j=1}^n a_j x b_j \) and

\[
\|T\|_{cb} = \frac{1}{2}(\|a\|^2 + \|b\|^2) \quad \text{where} \quad a = [a_1 \ldots a_n] \text{ and } b = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}.
\]

By [12, Theorem 3.3] we have \( \|T\|_{cb} = \|T\|_t \).

By Lemma 2.2, \( W_{m,e}(b) \) is convex. By [12, Proposition 2.4], the set \( W_{m,e}(a^\ell(t)) \) is convex, so intersects \( W_{m,e}(b(t)) \), so \( \|T\|_{cb} = \|T\|_t \). Hence

\[
\|T_k\| = \min\{\|T\|_t, \|T_n\|\} = \|T\|_{cb}.
\]

By Lemma 2.3, \( \|T\|_{cb} = \|T_{k,1}\| \). By Remark 4.1, the map \( T_{k,1} \in \mathcal{L}_{D_n}(M_{k,n}) \) satisfies \( \|T_{k,1}\| = \|T\| \), hence \( \|T_{k,1}\| \leq \sqrt{k} \|T\| \) by Lemma 4.3.

\[
\square
\]

5. Upper bounds for \( C(m, n) \)

For \( m, n \in \mathbb{N} \cup \{\infty\} \), recall that

\[
C(m, n) = \sup\{\|T\|_{cb}: T \in \mathcal{L}_{D_n}(M_{m,n}), \|T\| \leq 1\}.
\]

We have \( C(m, 1) = C(1, n) = C(m, 2) = 1 \) by Remark 2.5 and Theorem 3.2.

We will now give some upper bounds for \( C(m, n) \).

Proposition 5.1. If \( m \leq m' \) and \( n \leq n' \) then \( C(m, n) \leq C(m', n') \). In other words, \( C \) is an increasing function for the product order.

Proof. Given \( T \in \mathcal{L}_{D_n}(M_{m,n}) \) with \( \|T\| = 1 \), let \( T' \in \mathcal{L}_{D_{n'}}(M_{m',n'}) \) be the map

\[
T'(x) = \begin{bmatrix} T(qxp) & 0_{m \times (n'-n)} \\ 0_{(m'-m) \times n} & 0_{(m'-m) \times (n'-n)} \end{bmatrix}, \quad x \in M_{m',n'},
\]

where

\[
q = [I_m \ 0_{m \times (m'-m)}] \in M_{m,m'} \quad \text{and} \quad p = \begin{bmatrix} I_n \\ 0_{(n'-n) \times n} \end{bmatrix} \in M_{n',n}.
\]

It is easy to see that \( \|T'\| = \|T\| = 1 \) and \( \|T\|_{cb} = \|T'\|_{cb} \). Hence \( C(m, n) \leq C(m', n') \).

\[
\square
\]

Proposition 5.2. If \( m, n, t \geq 1 \) then \( C(m, n)C(s, t) \leq C(ms, nt) \).

Proof. Suppose that \( C(m, n) > \alpha \) and \( C(s, t) > \beta \). There are \( T \in \mathcal{L}_{D_n}(M_{m,n}) \) and \( S \in \mathcal{L}_{D_s}(M_{s,t}) \) with \( \|T\|, \|S\| < 1 \) and \( \|T\|_{cb} > \alpha \) and \( \|S\|_{cb} > \beta \). Consider the tensor product map \( T \otimes S \in \mathcal{L}(M_{m,n} \otimes M_{s,t}) \), which is defined on elementary tensors by \( T \otimes S(x \otimes y) = T(x) \otimes S(y) \). This is a right \( D_n \otimes D_t \) module map, and identifying \( M_{m,n} \otimes M_{s,t} \) isometrically with \( M_{ms,nt} \) in the usual way, we have \( D_n \otimes D_t = D_{nt} \) and it follows that \( T \otimes S \in \mathcal{L}_{D_{nt}}(M_{ms,nt}) \) with \( \|T \otimes S\| = \|T\| \|S\| < 1 \) and \( \|T \otimes S\|_{cb} = \|T\|_{cb} \|S\|_{cb} > \alpha \beta \). So \( C(ms, nt) > \alpha \beta \) whenever \( C(m, n) > \alpha \) and \( C(s, t) > \beta \), hence \( C(ms, nt) \geq C(m, n)C(s, t) \).

\[
\square
\]

Lemma 5.3. Let \( n \in \mathbb{N} \) with \( n \geq 2 \). If \( y \in M_{m,n} \) and \( \|y(e_i e_j^* + e_j e_i^*)\| \leq 1 \) for \( 1 \leq i < j \leq n \), then \( \|y\| \leq \sqrt{n/2} \).

Proof. Let \( p_{ij} = e_i e_j^* + e_j e_i^* \) for \( 1 \leq i < j \leq n \). Since each \( p_{ij} \) is a projection, we have \( \|yp_{ij} y^*\| = \|(yp_{ij})(yp_{ij})^*\| = \|yp_{ij}\|^2 \leq 1 \). Moreover,

\[
\sum_{1 \leq i < j \leq n} p_{ij} = (n - 1)I_n.
\]
Hence
\[ \|y\| = \sqrt{\|yy^*\|} = \sqrt{\frac{1}{n-1} \left\| \sum_{1 \leq i < j \leq n} y_{ij}y_{ij}^* \right\|} \leq \sqrt{\frac{1}{n-1} \sum_{1 \leq i < j \leq n} \|y_{ij}y_{ij}^*\|} \]
\[ \leq \sqrt{\frac{1}{n-1} \left( \frac{n}{2} \right)} = \sqrt{\frac{n}{2}}. \]
\[ \square \]

The following simple estimate applies to arbitrary linear maps between operator spaces, and is analogous to the well-known bound \( \|T_n\| \leq n \|T\| \) ([8, Exercise 3.11], due to Smith).

**Lemma 5.4.** Let \( k, m, n \in \mathbb{N} \). For any \( T \in \mathcal{L}(M_{m,n}) \), we have
\[ \|Tk,1\| \leq \sqrt{k} \|T\|. \]

**Proof.** There is \( x \in M_{k,1}(M_{m,n}) \), say \( x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \end{bmatrix} \) (where \( x_j \in M_{m,n} \) for \( 1 \leq j \leq k \)), with \( \|x\| = 1 \) and \( \|Tk,1(x)\| = \|Tk,1\| \). Clearly we can write \( Tk,1(x) = \begin{bmatrix} Tx_1 \\ Tx_2 \\ \vdots \\ Tx_k \end{bmatrix} \), and since \( \|x_j\| \leq \|x\| \leq 1 \) for \( 1 \leq j \leq n \), we have
\[ \|Tk,1\| = \|Tk,1(x)\| \]
\[ = \sqrt{\|Tk,1(x)^*Tk,1(x)\|} \]
\[ = \sqrt{\sum_{j=1}^{k} \|T(x_j)^*T(x_j)\|} \]
\[ \leq \sqrt{\sum_{j=1}^{k} \|T(x_j)^*T(x_j)\|} \]
\[ = \sqrt{\sum_{j=1}^{k} \|T(x_j)\|^2} \]
\[ \leq \|T\| \sqrt{\sum_{j=1}^{k} \|x_j\|^2} \]
\[ \leq \sqrt{k} \|T\|. \]
\[ \square \]

**Theorem 5.5.** If \( m, n \in \mathbb{N} \) and \( n \geq 2 \) then \( C(m, n) \leq \sqrt{\min\{m, n/2\}} \).

**Proof.** Let \( T \in \mathcal{L}_{D_n}(M_{m,n}) \) with \( \|T\| = 1 \) and \( \|T\|_{cb} = C(m, n) \). By Proposition 4.4 and Lemma 5.4, we have
\[ C(m, n) = \|T\|_{cb} = \|Tk,1\| \leq \sqrt{m} \|T\| = \sqrt{m} \]
so it only remains to show that \( C(m, n) \leq \sqrt{n/2} \).
Since \( \|T\|_{cb} = \|T_{a,1}\| \) by Proposition 4.4, there is \( x \in M_{m,1}(M_{m,n}) \) with \( \|x\| = 1 \) such that \( y = T_{m,1}(x) \) has \( \|y\| = C(m,n) \). Let \( a_1, \ldots, a_n \) be the column operators of \( T \), so that

\[
T = [a_1 \ldots a_n] \odot \begin{bmatrix} e_1 e^*_1 \\ \vdots \\ e_n e^*_n \end{bmatrix}.
\]

Given \( i, j \) with \( 1 \leq i < j \leq n \), consider

\[
S = [a_i \ a_j] \odot \begin{bmatrix} e_i e^*_i \\ e_j e^*_j \end{bmatrix} \in \mathcal{L}_{D_2}(M_{m,2}).
\]

Clearly, \( \|S\| \leq \|T\| \). If \( w = [xe_i \ x e_j] \in M_{n^2,2} \), then \( \|w\| \leq \|x\| = 1 \). Moreover, \( y(e_i e^*_i + e_j e^*_j) \in M_{m^2,n} \) can be recovered from \( S_{m,1}(w) \in M_{m^2,2} \) by padding with \( n - 2 \) columns of zeros, so \( \|S_{m,1}(w)\| = \|y(e_i e^*_i + e_j e^*_j)\| \). Since \( \|S\| = \|S\|_{cb} \) by Theorem 3.2, we have

\[
1 = \|T\| \geq \|S\| = \|S\|_{cb} \geq \|S_{m,1}(w)\| = \|y(e_i e^*_i + e_j e^*_j)\|.
\]

By Lemma 5.3,

\[
C(m,n) = \|y\| \leq \sqrt{n/2}.
\]

Fix \( n \in \mathbb{N} \). The sequence \( C(2,n), C(3,n), C(4,n), \ldots \) is increasing by Proposition 5.1. We will now show that it is eventually constant.

In [13, Theorem 1.3], the second author establishes an exact formula for the norm of an elementary operator \( T \), which we now recall. If \( \ell \in \mathbb{N} \) and \( X, Y \) are positive semi-definite elements of \( M_{\ell} \), then the tracial geometric mean of \( X \) and \( Y \) is

\[
\text{tgm}(X,Y) = \sqrt{\sqrt{X} \sqrt{Y}}_1 = \text{trace} \sqrt{\sqrt{X} \sqrt{Y}} X
\]

where \( \| \cdot \|_1 \) denotes the trace-class norm on \( M_{\ell} \). If \( T \) is an elementary operator on \( M_m \) which is represented by \( a \in M_{1,\ell}(M_m) \) and \( b \in M_{\ell,1}(M_m) \), then the formula is:

\[
\|T\| = \sup \{ \text{tgm}(X,Y) : X \in W_m(a^*), Y \in W_m(b) \}.
\]

In fact, a generalisation of this formula is shown to hold for elementary operators on any C*-algebra \( A \).

We need to show that (\( \dagger \)) holds in the rectangular case, too. If \( T \) is an elementary operator on \( M_{m,n} \) with \( n > m \) which is represented by \( a \in M_{1,\ell}(M_m) \) and \( b \in M_{\ell,1}(M_n) \), consider the map

\[
\tilde{T} : M_m \to M_n, \quad x \mapsto \begin{bmatrix} T(px) \\ 0_{(n-m)\times n} \end{bmatrix},
\]

where \( p \in \mathcal{B}(\mathbb{C}^m, \mathbb{C}^m) \) is the orthogonal projection onto the linear span of \( \{e_1, \ldots, e_m\} \), which is viewed simultaneously as \( \mathbb{C}^m \) and as a subspace of \( \mathbb{C}^n \). That is, \( \tilde{T} \) is “\( T \) applied to the first \( m \) rows, and zero on the remaining rows”. Clearly, \( \|T\| = \|\tilde{T}\| \). If \( a = [a_1 \ldots a_2] \) and \( \tilde{a} = [\tilde{a}_1 \ldots \tilde{a}_2] \) where \( \tilde{a}_j = \begin{bmatrix} a_j \ 0 \end{bmatrix} \in M_n \) is “\( a_j \) padded with \( n - m \) zero rows and columns”, then \( \tilde{T} \) is represented by \( \tilde{a}, b, \) and \( W_m(\tilde{a}^*) = \{ rX : r \in [0,1], X \in W_m(a^*) \} \). So, since \( \text{tgm}(rX,Y) = \sqrt{r} \text{tgm}(X,Y) \) for \( r \in [0,1] \),

\[
\|T\| = \|\tilde{T}\| = \sup \{ \text{tgm}(X,Y) : X \in W_m(a^*), Y \in W_m(b) \}.
\]

If \( T \) is an elementary operator on \( M_{m,n} \) with \( n < m \), then (\( \dagger \)) still holds, as may be seen by considering \( T^* \).
Remarks 5.6. The tracial geometric mean (or, sometimes, its square) is called fidelity in quantum information theory [3, 6, 7, 14], where it is interpreted as a measure of the closeness of two quantum states (positive semi-definite trace-class operators with trace 1).

Theorem 5.7. If $1 \leq n < m \leq \infty$, then $C(m, n) = C(n, n)$.

Proof. Suppose first that $m < \infty$. The supremum $C(m, n)$ is then attained, so there is $T \in \mathcal{L}_{D_n}(M_{m,n})$ with $\|T\| = 1$ and $\|T\|_{cb} = C(m, n)$. By Proposition 4.4, $\|T\|_{cb} = \|T_{n,1}\|$. By (i), there are unit vectors $\xi_1, \ldots, \xi_n$ in $\mathbb{C}^m$ and $\eta \in \mathbb{C}^n$, and $r_1, \ldots, r_n \in [0,1]$ with $\sum_{j=1}^n r_j^2 = 1$ such that the vector $\xi = \begin{bmatrix} r_1 \xi_1 \\ \vdots \\ r_n \xi_n \end{bmatrix}$ satisfies

$$\|T\|_{cb} = \text{tgm} \left( Q((a^*)^{(n)}), Q(b, \eta) \right).$$

Let $K$ be an $n$-dimensional subspace of $\mathbb{C}^m$ containing $\xi_1, \ldots, \xi_n$, and let us identify $K$ with $\mathbb{C}^n$. Then writing $p$ for the orthogonal projection of $\mathbb{C}^m$ onto $K$, let $\tilde{a}_j = p a_j |_K$, let $\tilde{a} = [\tilde{a}_1 \ldots \tilde{a}_n]$ and let $\tilde{T}$ be the elementary operator on $M_n$ represented by $\tilde{a}, b$. By our choice of $K$, we have $\|\tilde{T}\|_{cb} = \|T\|_{cb}$ and $Q(\tilde{a}^*, \xi) = Q(a, \xi)$ for $\xi \in K$, so

$$\|\tilde{T}\| = \sup \{ \text{tgm} \left( Q(\tilde{a}^*, \xi), Q(b, \eta) \right) : \xi \in K, \eta \in \mathbb{C}^n, \|\xi\|, \|\eta\| = 1 \}$$

$$\leq \sup \{ \text{tgm} \left( Q(a^*, \xi), Q(b, \eta) \right) : \xi \in \mathbb{C}^m, \eta \in \mathbb{C}^n, \|\xi\|, \|\eta\| = 1 \}$$

$$= \|T\|.$$

Since $\tilde{T}$ is a right $D_n$-module map, we have

$$C(n, n) \geq \frac{\|\tilde{T}\|_{cb}}{\|T\|} \geq \frac{\|T\|_{cb}}{\|T\|} = C(m, n) \geq C(n, n)$$

and hence $C(n, n) = C(m, n)$.

The case $m = \infty$ now follows by the argument of Corollary 3.3.

Remark 5.8. This reduces Theorem 3.2 to the $2 \times 2$ case, but does not appear to greatly simplify the proof.

6. More than two columns

We now give some examples which establish non-trivial lower bounds for $C(m, n)$ when $n \geq 3$. The matrix extremal numerical range of an $\ell$-tuple $[a_1 \ldots a_\ell]^*$ is closely connected to the joint numerical range of the operators $a_i a_i^*$ for $1 \leq i < j \leq \ell$. Moreover, the joint numerical range of three matrices (even three hermitian matrices) need not be convex, and an explicit example of this phenomenon is given in [5, Example 1.1].

Let

$$a_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad a_2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad \text{and} \quad a_3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$  

It is easy to see that the joint numerical range of the operators $a_i a_i^*$ for $1 \leq i < j \leq 3$ is affinely equivalent to a 2-sphere, so is not convex. Our first example is the map whose column operators are $a_1, a_2, a_3$.

Example 6.1. The map $T: M_{2,3} \to M_{2,3}$,

$$T: \begin{bmatrix} a & c & e \\ b & d & f \end{bmatrix} \mapsto \begin{bmatrix} a & c & f \\ b & -d & e \end{bmatrix}$$

is a right $D_3$-module map with

$$\sqrt{2} = \|T\| < \|T_{2,1}\| = \|T\|_{cb} = \sqrt{3}.$$
So $C(2,3) = \sqrt{3/2}$.

**Proof.** $T$ is a right $D_3$-module map and a Hilbert-Schmidt isometry. By Lemma 4.3, $\|T\| \leq \sqrt{2}$, and we have equality since $\|T_1 \circ 0_0 1\| = \sqrt{2}$.

By Proposition 4.4, $\|T\|_{cb} \leq \sqrt{3}$, and if

\[
x = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\]

then

\[
T_{2,1}(x) = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\]

and $\|x\| = 1$ while $\|T_{2,1}(x)\| = \sqrt{3}$. So $\|T\|_{cb} = \sqrt{3} = \|T_{2,1}\|$. Hence $C(2,3) \geq \sqrt{3/2}$, and we have equality by Theorem 5.5. $\square$

Extending the previous example by one column yields:

**Example 6.2.** The map $T: M_{2,4} \to M_{2,4}$,

\[
T: \begin{bmatrix} a & c & e & g \\ b & d & f & h \end{bmatrix} \mapsto \begin{bmatrix} a & c & f & h \\ b & -d & e & -g \end{bmatrix}
\]

is a right $D_4$-module map with

\[
\sqrt{2} = \|T\| < \|T_{2,1}\| = \|T\|_{cb} = 2.
\]

So $C(2,4) = \sqrt{2}$.

**Proof.** $T$ is a right $D_4$-module map and a Hilbert-Schmidt isometry. By Lemma 4.3, $\|T\| \leq \sqrt{2}$, and we have equality since $\|T_1 \circ 0_0 0\| = \sqrt{2}$. By Proposition 4.4, $\|T\|_{cb} \leq 2$, and if

\[
x = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \\ 1 & -1 & 0 & 0 \end{bmatrix}
\]

then

\[
T_{2,1}(x) = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}
\]

and $\|x\| = 1$ while $\|T_{2,1}(x)\| = 2$. So $\|T\|_{cb} = \|T_{2,1}\|$. Hence $C(2,4) \geq \sqrt{2}$, and Theorem 5.5 gives the reverse inequality. $\square$

Example 6.2 may be generalised as follows:

**Theorem 6.3.** For each $m \in \mathbb{N}$ with $m > 1$, there is a right $D_{m^2}$-module map $T \in \mathcal{L}(M_{m^2})$ with $\sqrt{m} = \|T\| < \|T_{m,1}\| = \|T\|_{cb} = m$. Hence $C(m,m^2) = \sqrt{m}$.

**Proof.** We have $C(m,m^2) \leq \sqrt{m}$ by Theorem 5.5.

Let $\rho = e^{2\pi i/m}$ and consider the $m \times m$ matrices

\[
g = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & \rho & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \rho^{m-1} \end{bmatrix}, \quad h = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}
\]

so that $h$ is the matrix for the $m$-cycle permutation $\alpha = (1\ 2\ \ldots\ m)$.

For $1 \leq j \leq m^2$, let $0 \leq r < m$ and $1 \leq s \leq m$ with $j = mr + s$, and define

\[
a_j = g^{-(s-1)}h^{-r}.
\]

Take $T$ to be the right $D_{m^2}$-module map with column operators $a_j \ (1 \leq j \leq m^2)$.

Since each $a_j$ is unitary, $T$ is a Hilbert-Schmidt isometry and so $\|T\| \leq \sqrt{m}$ by Lemma 4.3. (By Proposition 4.4, $\|T\|_{cb} \leq m$, but we will not actually need that.)
For $1 \leq i \leq m$ and $1 \leq j \leq m^2$, let $v_j^i \in \mathbb{C}^m$ be the vector
\[ v_j^i = \rho^{(i-1)(s-1)}e_{r(i)} \] where $j = mr + s$ with $0 \leq r < m$, $1 \leq s \leq m$ and define $x^i \in M_{m,m^2}$ by
\[ x^i = \sum_{j=mr+s}^{j<s=m} v_j^i e_j^* \]
Observe that $a_jv_j^i = e_i$ for every $j$. Hence, $Tx^i = e_iw^*$ where $w = \sum_{j=1}^{m^2} e_j^* e_j \in \mathbb{C}^{m^2}$, and so $\|Tx^i\| = \|e_i\| \|w\| = m$.

If $x = \begin{bmatrix} x^1 \\ x^2 \\ \vdots \\ x^{m^2} \end{bmatrix} \in M_{m,1}(M_{m,m^2})$ then $T_{m,1}(x) = \begin{bmatrix} e_1 w^* \\ e_2 w^* \\ \vdots \\ e_{m^2} w^* \end{bmatrix} = \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_m \end{bmatrix} w^*$,
so $\|T_{m,1}(x)\| = \|\begin{bmatrix} e_1^* \ldots e_{m^2}^* \end{bmatrix}w\| = m^{3/2}$. On the other hand, a calculation shows that the rows of $x$ are mutually orthogonal and have norm $\sqrt{m}$, and so $\|x\| = \sqrt{m}$. Hence $m \leq \|T_{m,1}\| = \|T\|_{cb}$ and $C(m,m^2) \geq \|T\|_{cb}/\|T\| \geq m/\sqrt{m} = \sqrt{m}$.

**Remark 6.4.** By Lemma 5.4,
\[ \|T_{m-1,1}\| \leq \sqrt{m-T\|T\|} < \|T_{m,1}\| = \|T\|_{cb} = \sqrt{m}\|T\| = m \|T\| \]
for the $D_{m^2}$-module maps $T$ on $M_{m,m^2}$ constructed in this proof. Thus the estimates of Proposition 4.4 and Theorem 5.5 are sharp, at least for $m = m^2$.

**Corollary 6.5.** If $m,n \in \mathbb{N}$ with $n \geq m^2$ then $C(m,n) = \sqrt{m}$.

**Proof.** Since $C(m,n)$ is increasing in $n$, this is an immediate consequence of Theorems 5.5 and 6.3.

**Theorem 6.6.** If $m,n \in \mathbb{N}$ with $2 \leq n \leq m^2$, then $C(m,n) \geq \sqrt{n/m}$.

**Proof.** Let $T \in L_{D_{m^2}}(M_{m,m^2})$ and $x \in M_{m^2}$ be as in the proof of Theorem 6.3. Let $a_1, \ldots, a_{m^2}$ be the column operators of $T$ and consider the map $S \in L_{D_{m}}(M_{m,n})$ whose column operators are $a_1, \ldots, a_n$. Also, let $x_j = xe_j \in \mathbb{C}^{mn}$ be the $j$th column of $x$ and let $y = [x_1 \ldots x_n] \in M_{m^2,n}$. By following the earlier argument, it is not hard to see that $S_{m,1}(y)$ is the matrix in $M_{m^2,n}$ whose columns are the first $n$ columns of $T_{m,1}(x)$, and hence that $\|S_{m,1}(y)\| = \sqrt{n}$. Since $\|y\| \leq \|x\| = 1$, we have
\[ \|S\|_{cb} \geq \|S_{m,1}\| \geq \|S_{m,1}(y)\| = \sqrt{n}. \]
Clearly $\|S\| \leq \|T\| \leq \sqrt{m}$. Hence
\[ C(m,n) \geq \frac{\|S\|_{cb}}{\|S\|} \geq \frac{\sqrt{n}}{m}. \]

**Remark 6.7.** For $(m,n) = (2,3)$, the operator $S$ in this proof was considered in Example 6.1, and we have equality in the bounds $\|S\|_{cb} \geq \sqrt{n}$ and $\|S\| \leq \sqrt{m}$ in this case.

We now summarise the best bounds we have obtained for $C(m,n)$.

**Corollary 6.8.** Let $m,n \in \mathbb{N} \cup \{\infty\}$.
\begin{enumerate}[(i)]  
  
  (i) If $m = 1$ or $n \in \{1,2\}$ then $C(m,n) = 1$.

  (ii) If $m \geq n$ then $C(m,n) = C(n,n)$.

  (iii) If $n \geq m^2$ then $C(m,n) = \sqrt{m}n$.

  (iv) If $2 \leq n \leq m^2$ then $\sqrt{\max\{\sqrt{m},n/\sqrt{n}\}} \leq C(m,n) \leq \min\{m,n/2\}$.
\end{enumerate}
Proof. Statements (i)–(iii) and the upper bound in (iv) have been discussed already, in Remark 2.5, Theorem 3.2, Corollary 6.5 and Theorem 5.5.

Suppose that $2 \leq n \leq m^2$. Let $k = \lceil \sqrt{n} \rceil$. Then $m \geq \sqrt{n} \geq k$ and $n \geq k^2$, and $C$ is increasing by Proposition 5.1, so $C(m, n) \geq C(k, k^2) = \sqrt{k}$ by Theorem 6.3. Similarly, if we write $\ell = \lceil \sqrt{n} \rceil$ then $m \geq \ell$ so $C(m, n) \geq C(\ell, n) \geq \sqrt{n}/\ell$ by Theorem 6.6.

**Question 6.9.** If $m \leq 2$ or $n \leq 4$ then these bounds yield exact values of $C(m, n)$, but we have been unable to find the exact values of $C(m, n)$ in many other cases. In particular, what is $C(3, 5)$?

**Remark 6.10.** It seems improbable that the lower bounds we have obtained could be sharp in general. In particular, it would seem surprising if $C(6, 6)$ turned out to be no larger than $C(2, 4) = \sqrt{2}$.

**Question 6.11.** Is $C(n, n)$ strictly increasing in $n$ for $n \geq 2$?

We now answer Question 1.1 in the negative. Recall that a masa in $B(H)$ is said to be discrete if it is generated by its minimal projections. Moreover [10], if $H$ is separable and a masa $D$ in $B(H)$ is not discrete, then $D$ is unitarily equivalent to the direct sum of a discrete masa and a diffuse masa, namely the masa $L^\infty[0, 1]$ in $B(L^2[0, 1])$.

**Corollary 6.12.** If $H$ is an infinite-dimensional Hilbert space and $D$ is a masa in $B(H)$, then there is a bounded right $D$-module map $T : K(H) \to K(H)$ which is not completely bounded.

**Proof.** By restricting attention to a separable subspace if necessary, we may assume that $H$ is separable.

First suppose that $D$ is discrete. By considering the minimal projections in $D$, we may identify $H$ with $\bigoplus_{m \geq 1} H_m$, where $H_m = \mathbb{C}^{m^2}$ for $m \geq 1$, in such a way that the minimal projections of $D$ are identified with the coordinate projections of $H_m$.

Let $p_m$ be the orthogonal projection in $D = \bigoplus_{m \geq 1} H_m$ with range $H_m$. By Theorem 6.3, there is a right $D_m$-module map $T_{(m)} : B(H_m) \to B(H_m)$ with $\|T_{(m)}\| = 1$ and $\|T_{(m)}\|_{cb} \geq \sqrt{m}$. The map $T : K(H) \to K(H)$, $x \mapsto \sum_{m \geq 1} T_{(m)}(p_mxp_m)$ has $\|T\| = 1$ and $\|T\|_{cb} \geq \sup_{m \geq 1} \|T_{(m)}\|_{cb} = \infty$.

If $D$ is not discrete, then $D$ is unitarily equivalent to the direct sum of a discrete masa in $B(H_0)$ for some Hilbert space $H_0$ (possibly zero) and a diffuse masa. If $D_1 \subseteq B(H_1)$ and $D_2 \subseteq B(H_2)$ are two diffuse masas where $H_1$ and $H_2$ are separable Hilbert spaces, then $D_1$ and $D_2$ are unitarily equivalent [10, Lemma 2.3.6]. Restricting attention to the diffuse part, we may assume that $H = L^2[0, 1] \otimes \ell^\infty(N)$ and $D = L^\infty[0, 1] \otimes \ell^\infty(N)$.

Let $\iota$ be the identity map on $K(L^2[0, 1])$ and let $\theta$ be the contraction on $K(\ell^2(N))$ given by applying the construction of first part of the proof to the masa $\ell^\infty(N)$ in $B(\ell^2(N))$. Under the natural identification $K(H) = K(L^2[0, 1]) \otimes K(\ell^2(N))$, the mapping $\Theta = \iota \otimes \theta$ is then a contractive right $D$-module map on $K(H)$ which is not completely bounded.

**Remark 6.13.** Under the same hypotheses, using weakly convergent sums in place of the norm convergent sums in this construction provides a bounded right $D$-module map $B(H) \to B(H)$ which is not completely bounded.

**Remark 6.14.** If $T : M_{m,n} \to M_{m,n}$ is a right $D_n$-module map with $1 = \|T\| < \|T\|_{cb}$, then just as in Proposition 5.2, the $k$th tensor power of $T$, that is, the map $T^\otimes k : M_{m^k,n^k} \to M_{m^k,n^k}$ is a right $D_n$-module map with $\|T^\otimes k\| = 1$ and $\|T^\otimes k\|_{cb} = \|T\|^k_{cb} \to \infty$ as $k \to \infty$. Thus Example 6.1 may be used in place of Theorem 6.3 to establish Corollary 6.12.
7. Subsets of the right module maps

For \( m, n \in \mathbb{N} \), let \( \mathcal{S}(m, n) \) be a subset of \( \mathcal{L}(M_{m,n}) \) containing a nonzero mapping and let

\[
C_S(m, n) = \sup \left\{ \frac{\|T\|_{cb}}{\|T\|} : T \in \mathcal{S}(m, n), \ T \neq 0 \right\}.
\]

Above, we have considered the case \( \mathcal{S}(m, n) = \mathcal{L}_{D_n}(M_{m,n}) \) and have shown that the corresponding function \( C = C_S \) can take values larger than 1. On the other hand, if \( \mathcal{S} \) is the class of Schur multipliers, then \( C_S \) is identically 1. It seems natural to ask for which classes of operators \( \mathcal{S} \) we still have \( C_S(m, n) > 1 \) for some \( m, n \). Of course, if \( \mathcal{S}(m, n) \subseteq \mathcal{L}_{D_n}(M_{m,n}) \) then \( 1 \leq C_S(m, n) \leq C(m, n) \).

Let \( m, n \in \mathbb{N} \). Given \( \alpha \in S_n \), let \( u_{\alpha} \) be the corresponding permutation unitary satisfying \( u_{\alpha}(e_i) = e_{\alpha(i)} \) for \( 1 \leq i \leq m \). Let

\[
\mathcal{P}(m, n) = \left\{ u_{\alpha_1} \cdots u_{\alpha_n} \circ \begin{bmatrix} e_1 e_1^* \\ \vdots \\ e_n e_n^* \end{bmatrix} : \alpha_j \in S_m \right\} \subseteq \mathcal{L}_{D_n}(M_{m,n}).
\]

This is a natural class of right \( D_n \)-module maps in which to seek maps with larger cb norm than norm. Indeed, if we drop the right modularity requirement, then the classic example of such a map is the transpose of a square matrix, which is a carefully chosen permutation of the matrix entries; \( \mathcal{P} \) is precisely the set of right \( D_n \)-module maps which are permutations of the matrix entries. We initially looked for examples in this class, and having had no luck, were eventually led to Examples 6.1 and 6.2, and so to Theorem 6.3. Since we concentrated on the \( 2 \times n \) and the \( 3 \times 3 \) cases, it is nice to be able to offer the following explanation for this initial failure.

Proposition 7.1. \( C_{\mathcal{P}}(2, 2) = C_{\mathcal{P}}(3, 3) = 1 \).

Proof. By Theorem 3.2, \( C_{\mathcal{P}}(2, 2) \leq C(2, 2) = 1 \), so \( C_{\mathcal{P}}(2, 2) = 1 \). Alternatively, since \( S_2 \) is abelian, this is an immediate consequence of [12, Remark 2.5].

Now consider \( u \circ e \in \mathcal{P}(3, 3) \) where \( u = [u_1 \ u_2 \ u_3] \) and \( u_j = u_{\alpha_j} \) for some \( \alpha_j \in S_3 \). Observe that if \( u_0 \) is a unitary matrix in \( M_m \) then the norms and completely bounded norms of \( u \circ e \) and \( u_0 u \circ e \) coincide. So, taking \( u_0 = u_1^{-1} \), we may assume that \( \alpha_1 \) is the identity permutation. Similarly, conjugating each \( \alpha_j \) by some \( \alpha_0 \in S_3 \) will not change the norm or completely bounded norm of the corresponding elementary operator. Hence up to symmetry there are three cases to consider:

1. \( \alpha_2 = (1 \ 2 \ 3) \) and \( \alpha_3 = (1 \ 3 \ 2) = \alpha_2^{-1}; \)
2. \( \alpha_2 = (1 \ 2) \) and \( \alpha_3 = (1 \ 2 \ 3); \) and
3. \( \alpha_2 = (1 \ 2) \) and \( \alpha_3 = (1 \ 3). \)

In the first case, the unitaries all commute and hence \( \|T\| = \|T\|_{cb} \) by [12, Remark 2.5]. In both of the latter two cases,

\[
U = \{ u_i u_j^* : 1 \leq i < j \leq 3 \} = \{ u_{(1 \ 2)}, u_{(1 \ 2 \ 3)}, u_{(1 \ 3)} \}
\]

and the joint numerical range of these three unitaries contains zero, since for every \( u \in U \) we have \( \langle u e_1, e_1 \rangle = 0 \). Hence \( W_{m,e}(\frac{1}{\sqrt{3}}u^*) \) contains a positive semidefinite diagonal \( 3 \times 3 \) matrix of trace 1, and Lemma 2.2 shows that \( W_{m,e}(e) \) consists of all such matrices. Hence

\[
W_{m,e}(\frac{1}{\sqrt{3}}u^*) \cap W_{m,e}(e) \neq \emptyset
\]

and so \( T = u \circ e \) has \( \|T\|_{cb} = \|T\| \).
However, a more persistent search reveals that $C_P$ is not constant.

**Example 7.2.** If

\[
T = [u(1) \ u(2) \ u(1 \ 3) \ u(2 \ 3)] \odot \begin{bmatrix} e_1 e_1^* \\ e_2 e_2^* \\ e_3 e_3^* \\ e_4 e_4^* \end{bmatrix} \in \mathcal{L}_{D_4}(M_{3,4}),
\]

then $\|T\| = \sqrt{3}$ and $\|T_{2,1}\| > 1.0775\sqrt{3}$. Hence

\[
C_P(3,4) \geq \frac{\|T_{2,1}\|}{\|T\|} > 1.0775.
\]

**Proof.** We have $\|T\| \leq \sqrt{3} |||T|||$ by Lemma 4.3, and the lower bound is given by considering the norm one matrix $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. Let

\[
x = \begin{bmatrix} -2/3 & 0 & 0 & 2/3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1/\sqrt{2} & 0 \\ 1/3 & 0 & 0 & 2/3 \\ -2/3 & 0 & 0 & -1/3 \end{bmatrix},
\]

Observe that $\|x\| = 1$, since we can reorder the rows and columns to recognise it as the direct sum of two $3 \times 2$ matrices with orthonormal columns. Now

\[
T_{2,1}(x) = \begin{bmatrix} -2/3 & 1 & 1/\sqrt{2} & 2/3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1/3 & 0 & 0 & -1/3 \\ -2/3 & 0 & 1/\sqrt{2} & 2/3 \end{bmatrix},
\]

and a computation with Mathematica reveals that $\|T_{2,1}(x)\|^2$ is the largest root of $18x^3 - 72x^2 + 33x - 2 = 0$, and hence that $\|T_{2,1}(x)\| > 1.0775\sqrt{3}$.

**Remark 7.3.** Numerical estimates obtained from a GNU Octave program using the tracial geometric mean formula (†) give an improved lower bound for $\|T_{2,1}\|$ for the operator $T$ in the preceding example of $1.13\|T\|$.

**Corollary 7.4.** $C_P(\infty, \infty) = \infty$.

**Proof.** Let $T$ be the map of Example 7.2. Considering the tensor powers $T^{\otimes k}$, we see that $T^{\otimes k} \in \mathcal{P}(3^k, 4^k)$ and

\[
C_P(\infty, \infty) \geq \sup_{k \geq 1} \frac{\|T^{\otimes k}\|_{cb}}{\|T\|^{\otimes k}} = \sup_{k \geq 1} \left( \frac{\|T\|_{cb}}{\|T\|} \right)^k = \infty.
\]

**Question 7.5.** If $\min\{m, n\} < \infty$, is it ever true that $1 < C_P(m, n) = C(m, n)$?

Finally, we pose a question about the class of module maps whose column operators are unitary:

\[
\mathcal{U}(m,n) = \left\{ [u_1 \ldots u_n] \odot \begin{bmatrix} e_1 e_1^* \\ \vdots \\ e_n e_n^* \end{bmatrix} : u_j \in \mathcal{U}(M_m), \ 1 \leq j \leq n \right\} \subseteq \mathcal{L}_{D_n}(M_{m,n})
\]

where $\mathcal{U}(M_m)$ is the set of unitary operators in $M_m$. The examples constructed in Theorem 6.3 are in $\mathcal{U}(m, m^2)$, so $C(m, m^2) = C_{\mathcal{U}}(m, m^2)$ for all $m \geq 1$.

**Question 7.6.** Is $C(m, n) = C_{\mathcal{U}}(m, n)$ for all $m, n \geq 1$?
References


E-mail address: levene@maths.tcd.ie, richardt@maths.tcd.ie

School of Mathematics, Trinity College Dublin, Dublin 2, Ireland