<table>
<thead>
<tr>
<th><strong>Title</strong></th>
<th>A Schwarz lemma and composition operators</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Authors(s)</strong></td>
<td>Mackey, Michael; Mellon, Pauline</td>
</tr>
<tr>
<td><strong>Publication date</strong></td>
<td>2004-04</td>
</tr>
<tr>
<td><strong>Publication information</strong></td>
<td>Integral Equations and Operator Theory, 48 (4): 511-524</td>
</tr>
<tr>
<td><strong>Publisher</strong></td>
<td>Springer</td>
</tr>
<tr>
<td><strong>Item record/more information</strong></td>
<td><a href="http://hdl.handle.net/10197/6284">http://hdl.handle.net/10197/6284</a></td>
</tr>
<tr>
<td><strong>Publisher’s statement</strong></td>
<td>The final publication is available at <a href="http://www.springerlink.com">www.springerlink.com</a></td>
</tr>
<tr>
<td><strong>Publisher's version (DOI)</strong></td>
<td>10.1007/s00020-003-1240-1</td>
</tr>
</tbody>
</table>
A SCHWARZ LEMMA AND COMPOSITION OPERATORS

M. MACKEY AND P. MELLON

Abstract. We give an alternative description of the Carathéodory pseudo-distance on a domain $D$ in an arbitrary complex Banach space. This gives a Schwarz lemma for holomorphic maps of the domain. We specialise to the case of a bounded symmetric domain and obtain some applications. In particular, we give the connected components of the space of composition operators with symbol in a bounded symmetric domain. This generalises results for the space of composition operators on $H^\infty(\Delta)$ in [12] and for $H^\infty(B)$, $B$ the unit ball of a Hilbert space or commutative $C^*$-algebra in [2].

Introduction

Let $D$ be a domain in a complex Banach space $E$ and let $\Delta$ be the open unit disc in $\mathbb{C}$. We define the following pseudo-distance on $D$,

$$d_D(z, w) := \sup\{|f(z) - f(w)| : f : D \to \Delta \text{ holomorphic} \} \quad \text{for } z, w \in D.$$ 

We prove that

$$\log \frac{2 + d_D}{2 - d_D}$$

is in fact the Carathéodory pseudo-distance $C_D$ on $D$. This results in a Schwarz Lemma for holomorphic maps from $D$ to $\Delta$. When we specialise this to $B_E$, the open unit ball of a Banach space $E$, we prove firstly that $d_{B_E}$ can be expressed in terms of holomorphic self-maps of $B_E$, namely,

$$d_{B_E}(z, w) = \sup\{||f(z) - f(w)|| : f : B_E \to B_E \text{ holomorphic} \}.$$ 

Since the Carathéodory distance on $B_E$ satisfies

$$C_{B_E}(z, 0) = \tanh^{-1} ||z||$$

we obtain, among others, the following Schwarz Lemma for all $f : B_E \to B_E$ holomorphic:

$$||f(z) - f(0)|| \leq \frac{2 - 2\sqrt{1 - ||z||^2}}{||z||} \quad \text{for all } z \in B_E.$$ 

If $B$ is a bounded symmetric domain and $f : B \to B$ is holomorphic, we get

$$||f(z) - f(w)|| \leq \frac{2 - 2\sqrt{1 - ||g(z)(w)||^2}}{||g(z)(w)||} \quad \text{for all } z, w \in B.$$ 

The description of $d_{B_E}$ in terms of holomorphic self-maps of $B_E$ makes it suited to the study of composition operators on the space $H^\infty(B_E)$ and, indeed, this is the motivation behind the introduction of $d_{\Delta}$ in the one variable case in [12]. The set-up is as follows:
to every \( \phi : B_E \to B_E \) holomorphic we associate a linear map \( C_\phi \), called a composition operator, on the space \( H^\infty(B_E) \) of all bounded holomorphic functions on \( B_E \) by
\[
C_\phi(f) = f \circ \phi
\]
for \( f \in H^\infty(B_E) \). The idea is to associate the function theoretic properties of \( \phi \) with the properties of \( C_\phi \) as a linear mapping.

For \( B = \Delta \), a survey of the classical theory of composition operators on the Hardy and Bergman spaces is given in [4] and [16]. To extend the classical results where \( \phi \) is taken as a holomorphic function on \( \Delta \) to the case where \( \phi \) is a function of several or even infinitely many variables, one can head in a variety of directions. For example, if \( B_n \) is the open unit ball of \( \mathbb{C}^n \), MacCluer, Shapiro and Luecking, among others have looked at the action of \( C_\phi \) on the Hardy spaces \( H^p(B_n) \), \( 0 < p < \infty \) and the Bergman spaces \( A^p(B_n) \), \( 0 < p < \infty \). Jafari, Li, Russo and others have studied \( C_\phi \) on the Hardy and Bergman spaces of finite dimensional bounded symmetric domains and strongly pseudo-convex domains. We refer to the survey of Russo [15] for references and more information. In the infinite dimensional case, we refer to [1, 2, 7] which study composition operators on the space \( H^\infty(B_E) \), for \( E \) a complex Banach space.

Our aim is to extend to a bounded symmetric domain \( B \) results of MacCluer, Ohno and Zhao for the one variable case in [12] that determine the connected components of the topological space of composition operators on \( H^\infty(\Delta) \) with the natural uniform norm topology. These results were extended in [2] when \( B \) is the open unit ball of a Hilbert space or commutative \( C^* \)-algebra, and in [17] when \( B \) is the open unit ball of \( \mathbb{C}^n \).

We recall that every bounded symmetric domain \( B \) can be realised as the open unit ball of a Banach space \( Z \), known as a \( JB^* \)-triple [8]. The algebraic properties of \( Z \), in particular the properties of the Bergman operator \( B(z, w) \) and the quasi-inverse map \( z \to z^a \) are then used, together with the distance \( d_B \), to determine the connected components of the space of composition operators on \( H^\infty(B) \). For a general survey and background details on the Poincaré distance, Carathéodory pseudo-distance and \( JB^* \)-triples we refer to [5].

1. Notation and Background

We let \( E \) and \( F \) denote complex Banach spaces and \( D \) and \( \tilde{D} \) domains in \( E \) and \( F \) respectively. The set of all holomorphic maps from \( D \) to \( \tilde{D} \) is denoted by \( H(D, \tilde{D}) \). We write \( H^\infty(D) \) for the space of all bounded \( \mathbb{C} \)-valued holomorphic functions on \( D \) and \( \|f\|_\infty := \sup_{z \in D} |f(z)| \) for all \( f \in H^\infty(D) \).

**Definition 1.1.** The Poincaré distance \( \rho \) on \( \Delta \) is
\[
\rho(z, w) := \tanh^{-1} \left| \frac{z - w}{1 - \overline{z}w} \right| \text{ for } z, w \in \Delta.
\]

\(^1\)We thank the referee for drawing our attention to this reference.
The Carathéodory pseudo-distance can be defined on any complex manifold [5], although we restrict our attention here to the case of a domain $D$.

**Definition 1.2.** The Carathéodory pseudo-distance on a domain $D$ is given by

$$C_D(z, w) := \sup \{ \rho(f(z), f(w)) : f \in H(D, \Delta) \} \quad \text{for } z, w \in D.$$

The Carathéodory pseudo-distances form a Schwarz-Pick system (cf. [5]) for which holomorphic functions act as contractions, namely,

$$C_{D_2}(f(z), f(w)) \leq C_{D_1}(z, w) \quad \text{for all } f \in H(D_1, D_2), \ z, w \in D_1.$$

In fact, this is the smallest of all Schwarz-Pick systems. For bounded domains, cf. [5, chapters 4 and 5], it turns out that $C_D$ is continuous and generates the original topology thus ensuring that it is actually a distance on $D$.

We now introduce the class of Banach spaces known as the $JB^*$-triples. We use $H$ and $K$ to denote arbitrary complex Hilbert spaces and $\mathcal{L}(X, Y)$ to denote the space of continuous linear operators from a Banach space $X$ to a Banach space $Y$. We let $\mathcal{L}(X) = \mathcal{L}(X, X)$ and $\text{GL}(X)$ be all invertible elements in $\mathcal{L}(X)$.

**Definition 1.3.** A $JB^*$-triple is a complex Banach space $Z$ with a real trilinear mapping $\{\cdot, \cdot, \cdot\} : Z \times Z \times Z \to Z$ satisfying

(i) $\{x, y, z\}$ is complex linear and symmetric in the outer variables $x$ and $z$, and is complex anti-linear in $y$.

(ii) The map $z \to \{x, x, z\}$, denoted $x \square x$, is Hermitian, $\sigma(x \square x) \geq 0$ and $\|x \square x\| = \|x\|^2$ for all $x \in Z$, where $\sigma$ denotes the spectrum.

(iii) The product satisfies the following “triple identity”

$$\{a, b, \{x, y, z\}\} = \{\{a, b, x\}, y, z\} - \{x, \{b, a, y\}, z\} + \{x, y, \{a, b, z\}\}.$$

Let $Z$ be a $JB^*$-triple. Several types of linear operators on $Z$ arise naturally from the triple product:

- $x \square y \in \mathcal{L}(Z) : z \to \{x, y, z\}$,
- $Q_x \in \mathcal{L}(Z) : z \to \{x, z, z\}$,

and the important Bergman operators

$$B(x, y) = I - 2x \square y + Q_x Q_y \in \mathcal{L}(Z).$$

**Example 1.4.**

(i) $\mathcal{L}(H, K)$ is a $JB^*$-triple for the product $\{x, y, z\} = \frac{1}{2}(xy^*z + zy^*x)$ where $y^*$ denotes the usual adjoint of $y$ and $B(x, y)z = (1 - xy^*)z(1 - y^*x)$.

(ii) $C_0(X)$, the continuous $\mathbb{C}$-valued functions vanishing at infinity on a locally compact Hausdorff space $X$, is a $JB^*$-triple for the product $\{x, y, z\} = x\bar{y}z$ and $B(x, y)z = (1 - x\bar{y})^2z$. 
As Banach spaces the $JB^*$-triples are characterised by the fact that their open unit balls are homogeneous. In fact, if we let $\text{Aut}(B)$ denote all biholomorphic maps from $B$ to $B$ then for all $a$ in $B$, we have $g_a \in \text{Aut}(B)$ defined by

$$g_a(z) = a + B(a,a)^{\frac{1}{2}}(I + z \Box a)^{-1}z$$

(cf. [9]) which satisfies $g_a(0) = a$, $g_a^{-1} = g_{-a}$ and $g_a'(0) = B(a,a)^{\frac{1}{2}}$ (defined in terms of a functional calculus). We note the fundamental formula [10] for $a \in B$. For $z,a \in B, z^a := (I - z \Box a)^{-1}z$ is called the quasi-inverse of $z$ with respect to $a$ and satisfies

$$\|z^a\| \leq \frac{\|z\|}{1 - \|a\|^2}. \quad (2)$$

The quasi-inverse also satisfies $(z^a)^b = z^{a+b}$ whenever both sides of this equation are well defined. For further details see [11, Chapter 7] or [6].

It is known [8] that every bounded symmetric domain is biholomorphically equivalent to the open unit ball of a $JB^*$-triple and vice versa. From the homogeneity therefore one can easily see that on a bounded symmetric domain $B$ the Carathéodory distance is given by

$$C_B(z,w) = \tanh^{-1}\|g_{-z}(w)\|.$$ For a recent survey of $JB^*$-triples and bounded symmetric domains we refer to [3].

2. The Carathéodory pseudo-distance

To study composition operators on $H^\infty(\Delta)$ MacCluer et al. [12] introduce the distance $d_\infty$ on $\Delta$

$$d_\infty(z,w) := \sup\{||f(z) - f(w)|| : f \in H^\infty(\Delta), ||f||_\infty \leq 1\}, \quad z, w \in \Delta.$$ It is not too difficult to see [13] that

$$d_\infty(z,w) = \frac{2 - 2\sqrt{1 - \beta(z,w)^2}}{\beta(z,w)}$$

for $\beta(z,w) := \left|\frac{z - w}{1 - zw}\right|$ or in terms of the Poincaré metric $\rho$ on $\Delta$

$$\rho(z,w) = \tanh^{-1}\beta(z,w) = \log \frac{2 + d_\infty(z,w)}{2 - d_\infty(z,w)}. \quad (3)$$

Motivated by this we introduce the following pseudo-distance on an arbitrary domain $D$

$$d_D(z,w) := \sup\{||f(z) - f(w)|| : f \in H(D, \Delta)\}.$$ We note that $d_\Delta = d_\infty$ above. Clearly

$$d_D(z,w) = \sup\{||g(f(z)) - g(f(w))|| : g \in H(\Delta, \Delta), f \in H(D, \Delta)\} = \sup_{f \in H(D, \Delta)} d_\infty(f(z), f(w))$$

for $z, w \in D$. 

Since the map \( t \to \log \frac{2 + t}{2 - t} \) is strictly increasing on \([0, 2)\) it follows that
\[
\log \frac{2 + d_D(z, w)}{2 - d_D(z, w)} = \sup_{f \in H(D, \Delta)} \log \frac{2 + d_\infty(f(z), f(w))}{2 - d_\infty(f(z), f(w))} = \sup_{f \in H(D, \Delta)} \rho(f(z), f(w)) \text{ from (3)} = C_D(z, w).
\]

In other words, \( \log \frac{2 + d_D}{2 - d_D} \) is the Carathéodory pseudo-distance on \( D \), or equivalently for any domain \( D \)
\[
d_D(z, w) = \frac{2 - 2\sqrt{1 - (\tanh C_D(z, w))^2}}{\tanh C_D(z, w)} \quad \text{for } z, w \in D. \tag{4}
\]

Throughout, we use \( B_E \) to denote the open unit ball of an arbitrary complex Banach space \( E \) and reserve \( B \) to denote a bounded symmetric domain.

We now present a series of Schwarz Lemmas arising from (4).

**Lemma 2.1.** (i) Let \( D \) be an arbitrary domain and \( f : D \to \Delta \) be holomorphic. Then
\[
|f(z) - f(w)| \leq \frac{2 - 2\sqrt{1 - (\tanh C_D(z, w))^2}}{\tanh C_D(z, w)} \quad \text{for } z, w \in D.
\]

In particular, if \( D = B_E \) is the open unit ball of a Banach space \( E \) then
\[
|f(z) - f(0)| \leq \frac{2 - 2\sqrt{1 - \|z\|^2}}{\|z\|} \quad \text{for } z \in B_E.
\]

(ii) Let \( B \) be a bounded symmetric domain and \( f : B \to \Delta \) be holomorphic. Then
\[
|f(z) - f(w)| \leq \frac{2 - 2\sqrt{1 - \|g_z^{-1}(w)\|^2}}{\|g_z^{-1}(w)\|} = \frac{2\sqrt{\|B_z^{-1}B(w, z)B_z^{-1}\| - 1}}{\sqrt{\|B_z^{-1}B(w, z)B_z^{-1}\| - 1}} \quad \text{for } z, w \in B
\]
where \( g_z \) is an automorphism of \( B \) taking 0 to \( z \) and \( B_z := B(z, z)^{\frac{1}{2}} \).

**Proof.** (i) is immediate from (4). The first part of (ii) follows from (i) since on a bounded symmetric domain \( B \)
\[
C_B(z, w) = \tanh^{-1} \|g_z^{-1}(w)\| \quad \text{for } z, w \in B \tag{5}
\]
where \( g_z \) is an automorphism of \( B \) taking 0 to \( z \).

For the second part of (ii) we rewrite
\[
\frac{2 - 2\sqrt{1 - \|g_z^{-1}(w)\|^2}}{\|g_z^{-1}(w)\|}
\]
in terms of Bergman operators using the fact [14, Proposition 3.1] that
\[ \frac{1}{1 - \|g_{-z}(w)\|^2} = \|B_w^{-1}B(w, z)B_z^{-1}\| \quad \text{for } z, w \in B. \] (6)

For the purpose of studying composition operators on \( H^\infty(B_E) \) the distance we really need on \( B_E \) is written in terms of self-maps of \( B_E \), namely,
\[ \tilde{d}_{B_E}(z, w) := \sup \{ \|f(z) - f(w)\| : f \in H(B_E) \}. \]

**Proposition 2.2.** The distance \( \tilde{d}_{B_E} \) coincides with \( d_{B_E} \).

**Proof.** Fix \( z, w \) in \( B_E \) and \( f \in H(B_E) \). By the Hahn-Banach theorem there exists \( \lambda = \lambda(z, w, f) \in Z^* \), \( \|\lambda\| \leq 1 \) with
\[ \|f(z) - f(w)\| = \lambda(f(z) - f(w)) \]
and hence \( \tilde{d}_{B_E}(z, w) \leq d_{B_E}(z, w) \). On the other hand, if \( g \in H(B_E, \Delta) \) then for any fixed \( u \in \partial B_E \) the map \( z \to g(z)u \) is in \( H(B_E) \) and this implies \( d_{B_E}(z, w) \leq \tilde{d}_{B_E}(z, w) \). \( \Box \)

Proposition 2.2 together with (4), (5) and (6) now easily gives the following.

**Corollary 2.3.** (i) Let \( E \) be a Banach space and \( f : B_E \to B_E \) be holomorphic. Then
\[ \|f(z) - f(w)\| \leq \frac{2 - 2\sqrt{1 - \tanh C_{B_E}(z, w)^2}}{\tanh C_{B_E}(z, w)} \quad \text{for } z, w \in B_E. \]
In particular,
\[ \|f(z) - f(0)\| \leq \frac{2 - 2\sqrt{1 - \|z\|^2}}{\|z\|} \quad \text{for } z \in B_E. \]

(ii) Let \( B \) be a bounded symmetric domain and \( f : B \to B \) be holomorphic. Then
\[ \|f(z) - f(w)\| \leq \frac{2 - 2\sqrt{1 - \|g_{-z}(w)\|^2}}{\|g_{-z}(w)\|} \]
\[ = \frac{2\sqrt{\|B_w^{-1}B(w, z)B_z^{-1}\| - 1}}{\sqrt{\|B_w^{-1}B(w, z)B_z^{-1}\| - 1}} - 1 \quad \text{for } z, w \in B, \]
where \( g_z \) is an automorphism of \( B \) taking 0 to \( z \) and \( B_z := B(z, z)^{\frac{1}{2}} \).

As the Bergman operators play a fundamental role in the holomorphy of \( B \) and \( B(a, a)^{\frac{1}{2}} = g_a'(0) \), \( a \in B \) the inequality [10]
\[ \|B(a, a)^{\frac{1}{2}}\| \leq 1 \]
is crucial to the geometry of \( B \). We are able to obtain a simple direct proof of this result.

**Corollary 2.4.** For \( a \in B \), \( \|B(a, a)^{\frac{1}{2}}\| \leq 1. \)
Proof. Fix $a \in B$. For all $z \in B$

$$\frac{2 - 2\sqrt{1 - \|z\|^2}}{\|z\|} = d_B(z, 0) \geq \|g_a(z) - g_a(0)\| = \|B(a, a)^{\frac{1}{2}} z^{-a}\|.$$ 

Since $z^a \in B$ if $\|z\| < \frac{1}{1 + \|a\|}$ and $(z^a)^{-a} = z$ this implies that

$$\|B(a, a)^{\frac{1}{2}} z\| \leq \frac{2 - 2\sqrt{1 - \|z^a\|^2}}{\|z^a\|}$$

when $\|z\| < \frac{1}{1 + \|a\|}$. Fix $0 < t < \frac{1}{1 + \|a\|}$. For $\|z\| \leq t$, we have from (2) that

$$\|z^a\| \leq \frac{\|z\|}{1 - \|z\| \|a\|} \leq \frac{t}{1 - t \|a\|}$$

and since $h(t) = (2 - 2\sqrt{1 - t^2})/t$ is strictly increasing on $[0, 1)$ this gives

$$\|B(a, a)^{\frac{1}{2}} z\| \leq h(\|z^a\|) \leq h\left(\frac{t}{1 - t \|a\|}\right).$$

Then

$$\|B(a, a)^{\frac{1}{2}}\| = \sup_{\|z\| = 1} \|B(a, a)^{\frac{1}{2}} z\| = \frac{1}{t} \sup_{\|z\| = t} \|B(a, a)^{\frac{1}{2}} z\|$$

$$\leq \frac{1}{t} h\left(\frac{t}{1 - t \|a\|}\right)$$

$$= 2 \left(1 - t \|a\| + \sqrt{(1 - t \|a\|)^2 - t^2}\right)^{-1}.$$ 

As $t \to 0$ this gives $\|B(a, a)^{\frac{1}{2}}\| \leq 1$ as required. 

3. Composition operators on $H^\infty(B)$

In this section we study the connected components of the space of composition operators on $H^\infty(B)$ with the uniform norm topology where $B$ is a bounded symmetric domain. Our motivation was to extend the one variable results in [12] to the case of infinitely many variables. In the case where $B$ is the open unit ball of a hilbert space or of a commutative $C^*$-algebra we refer to [2]. The key to this study is the distance $d_B$ which gives a formula for the hyperbolic distance $C_B$, namely,

$$C_B = \log \frac{2 + d_B}{2 - d_B}.$$ 

Just as the Möbius maps are crucial when studying $\Delta$, so the analogous automorphisms $\{g_a : a \in B\}$ of $B$ are essential here and we establish some simple identities.

Lemma 3.1. For $a, b \in B$,

$$g_{-a}(a + b) = (B(a, a)^{-\frac{1}{2}} b)^a \quad \text{when} \quad a + b \in B, \quad (7)$$

and $$g_{-a}(b) = (B(a, a)^{-\frac{1}{2}} (b - a))^a. \quad (8)$$
Proof. Clearly, the two expressions are equivalent. Recall that \( g_a(z) = a + B(a, a)^{1/2} z^{-a} \). Since the inverse of \( z \to z^a \) is \( z \to z^{-a} \) and the inverse of \( g_a \) is \( g_{-a} \) it follows that \( g_{-a}(b) = g_{a}^{-1}(b) = (B(a, a)^{-1/2}(b - a))^a \).

For \( z, w \in B \), we define \( \beta(z, w) := \|g_z(w)\| \).

Definition 3.2. For \( \phi, \psi \in H(B) \) we let
\[
d_{\beta}(\phi, \psi) := \sup_{z \in B} \beta(\phi(z), \psi(z)).
\]

We note that \( d_{\beta} \) is a metric on \( H(B) \) and, by virtue of the following result, it is the topological structure of \((H(B), d_{\beta})\) that interests us.

Proposition 3.3. Let \( \phi, \psi \in H(B) \). Then
\[
\|C_\phi - C_\psi\| = \frac{2 - 2\sqrt{1 - d_{\beta}(\phi, \psi)^2}}{d_{\beta}(\phi, \psi)}.
\]

In particular, the space of composition operators on \( H^\infty(B) \) with the uniform norm topology is homeomorphic as a topological space to \((H(B), d_{\beta})\).

Proof. Proposition 2.2 together with (4) and (5) gives
\[
d_B(z, w) = \sup\{\|f(z) - f(w)\| : f \in H(B)\} = \frac{2 - 2\sqrt{1 - \beta(z, w)^2}}{\beta(z, w)}.
\]

Since \( h(t) = 2(1 - \sqrt{1 - t^2})/t \) is an increasing function on \([0, 1] \) we have
\[
\|C_\phi - C_\psi\| = \sup\{\|C_\phi(f) - C_\psi(f)\|_\infty : f \in H^\infty(B), \|f\|_\infty \leq 1\}
\]
\[
= \sup\{\|f \circ \phi - f \circ \psi\|_\infty : f \in H(B)\}
\]
\[
= \sup\{\|f(\phi(z)) - f(\psi(z))\| : f \in H(B), z \in B\}
\]
\[
= \sup_{z \in B} d_B(\phi(z), \psi(z))
\]
\[
= \sup_{z \in B} \frac{2 - 2\sqrt{1 - \beta(\phi(z), \psi(z))^2}}{\beta(\phi(z), \psi(z))}
\]
\[
= \frac{2 - 2\sqrt{1 - d_{\beta}(\phi, \psi)^2}}{d_{\beta}(\phi, \psi)}.
\]

Our aim is to determine the connected components of the space of composition operators on \( H^\infty(B) \). The above result means that we can now do this by examining the space \((H(B), d_{\beta})\). In order to achieve this, we use \( JB^*\)-triple tools such as Bergman operators and the quasi-inverse map as a substitute for the algebra structure used when \( B = \Delta [12] \) or \( B \) is the unit ball of \( C_0(X) [2] \) and as a substitute for the inner product used when \( B \) is a Hilbert ball [2].
To begin with we note that the $d_\beta$-topology on $H(B)$ is stronger than the $\| \cdot \|_\infty$ topology. Indeed from (8) we have that
\[
(g_{-w}(z))^{-w} = B(w, w)^{-\frac{1}{2}}(z - w)
\]
for $z, w \in B$ and hence we may write
\[
z - w = B(w, w)^\frac{1}{2}(g_{-w}(z))^{-w}.
\]
Since $\|B(w, w)^{\frac{1}{2}}\| \leq 1$ and
\[
\| (g_{-w}(z))^{-w} \| \leq \frac{\|g_{-w}(z)\|}{1 - \|w\|\|g_{-w}(z)\|} \leq \frac{\|g_{-w}(z)\|}{1 - \|g_{-w}(z)\|}
\]
this gives that
\[
\| z - w \| \leq \frac{\beta(z, w)}{1 - \beta(z, w)}
\]
for all $z, w \in B$. Therefore for $\phi, \psi \in H(B)$ we have
\[
\sup_{z \in B} \| \phi(z) - \psi(z) \| \leq \sup_{z \in B} \frac{\beta(\phi(z), \psi(z))}{1 - \beta(\phi(z), \psi(z))}
\]
and hence $\| \phi - \psi \|_\infty \leq \frac{d_{\beta}(\phi, \psi)}{1 - d_{\beta}(\phi, \psi)}$. In particular, if $d_{\beta}(\phi, \psi, t) \rightarrow 0$ then $\| \phi - \psi \|_\infty \rightarrow 0$. The converse however is not true. For example, in $\Delta$, $\beta(a, e^{it}a) = |g_{-a}(e^{it}a)| = \frac{|(e^{it} - 1)a|}{1 - e^{it}|a|^2}$ which implies $d_{\beta}(\text{id}, e^{it}\text{id}) = 1$ for all $t \in (0, 2\pi)$, even though $\|\text{id} - e^{it}\text{id}\|_\infty \rightarrow 0$ as $t \rightarrow 0$.

However, the two topologies do agree on the set of holomorphic functions which map $B$ strictly inside $B$. In other words, if $\|\phi\|_\infty < 1$ then $\|\phi - \psi \|_\infty \rightarrow 0$ if and only if $d_{\beta}(\phi, \psi) \rightarrow 0$. To see this, we note from (8) that
\[
\|g_{-a}(b)\| \leq \|(B(a, a)^{-\frac{1}{2}}(b - a))\| \leq \frac{\|b - a\|}{1 - \|a\|^2 - \|a\|\|b - a\|}
\]
from repeated use of (1) and (2) when $\|b - a\|$ is sufficiently small. Therefore if $\|\phi\|_\infty < 1$ we have
\[
d_{\beta}(\phi, \psi, t) = \sup_{z \in B} \|g_{-\phi(z)}(\psi_t(z))\| \leq \sup_{z \in B} \frac{\|\phi(z) - \psi_t(z)\|}{1 - \|\phi(z)\|^2 - \|\phi(z)\|\|\phi(z) - \psi_t(z)\|} \leq \frac{\|\phi - \psi_t\|_\infty}{1 - \|\phi\|_\infty^2 - \|\phi\|_\infty\|\phi - \psi_t\|_\infty}
\]
and hence $\|\phi - \psi_t\|_\infty \rightarrow 0$ implies that $d_{\beta}(\phi, \psi, t) \rightarrow 0$ as well.

Given $\phi, \psi \in H(B)$, it is obvious from the definition that $d_{\beta}(\phi, \psi) \leq 1$. Later results will show the importance of determining whether $d_{\beta}(\phi, \psi) < 1$. We remark therefore that if $\phi$
maps $B$ strictly inside $B$, then the condition $d_\beta(\phi, \psi) < 1$ is satisfied for every $\psi \in H(B)$ which also maps $B$ strictly inside $B$. To see this, we use (6) to write
\[
\frac{1}{1 - \|g_{-\phi}(\psi(z))\|^2} = \|B_{\phi(z)}^{-1}B(\phi(z), \psi(z))B_{\psi(z)}^{-1}\|
\]
and hence $d_\beta(\phi, \psi) = \sup_{z \in B} \|g_{-\phi}(\psi(z))\| < 1$ if and only if
\[
\sup_{z \in B} \|B_{\phi(z)}^{-1}B(\phi(z), \psi(z))B_{\psi(z)}^{-1}\| < \infty.
\]
Since from (1)
\[
\|B_{\phi(z)}^{-1}B(\phi(z), \psi(z))B_{\psi(z)}^{-1}\| \leq \frac{\|B(\phi(z), \psi(z))\|}{(1 - \|\psi(z)\|^2)(1 - \|\phi(z)\|^2)}
\]
and $\|B(\phi(z), \psi(z))\| \leq (1 + \|\phi(z)\|\|\psi(z)\|)\|\phi(z)\|^2$ for all $z \in B$, it follows that if $\|\phi\|_\infty < 1$ and $\|\psi\|_\infty < 1$ then
\[
\sup_{z \in B} \|B_{\phi(z)}^{-1}B(\phi(z), \psi(z))B_{\psi(z)}^{-1}\| \leq \frac{4}{1 - \|\phi\|_\infty^2} \frac{1}{1 - \|\psi\|_\infty^2} < \infty
\]
and hence $d_\beta(\phi, \psi) < 1$.
Since $d_\beta(\phi, 0) = \|\phi\|$, the converse is also true. In other words, for $\phi \in H(B)$
\[
d_\beta(\phi, \psi) < 1 \text{ for all } \psi \text{ with } \|\psi\|_\infty < 1 \text{ if and only if } \|\phi\|_\infty < 1. \tag{9}
\]

**Theorem 3.4.** Let $\phi, \psi \in H(B)$ with $d_\beta(\phi, \psi) < 1$. Then the map
\[
t \mapsto \phi_t := t\phi + (1 - t)\psi
\]
is a $d_\beta$-continuous path joining $\phi$ to $\psi$.

The proof breaks into two parts proved below. The first part (Lemma 3.6) shows that any convex combination $\phi_t$ of $\phi$ and $\psi$ satisfies $d_\beta(\phi_t, \psi) \leq d_\beta(\phi, \psi)$. The second part (Lemma 3.7) then uses this to show that the map $t \mapsto \phi_t$ is $d_\beta$-continuous which proves the theorem. As a result of the homeomorphism $\phi \mapsto C_\phi$ guaranteed by Proposition 3.3, this theorem immediately implies the following.

**Corollary 3.5.** Let $\phi, \psi \in H(B)$ with $d_\beta(\phi, \psi) < 1$. Then $C_\phi$ and $C_\psi$ are in the same path connected component in the space of composition operators on $H^\infty(B)$.

In particular, from (9), we have that the set of composition operators $C_\phi$ with $\|\phi\|_\infty < 1$ is path connected. This is proved in a more general setting in [2]

**Lemma 3.6.** Let $\phi, \psi \in H(B)$ satisfy $d_\beta(\phi, \psi) = \lambda < 1$. Then for any $t \in [0, 1]$ we have $d_\beta(\phi_t, \psi) \leq \lambda$, where $\phi_t = t\phi + (1 - t)\psi$. 
From Lemma 3.6 we have that $b = t(\phi(z) - \psi(z))$ to obtain

$$\beta(\psi(z), \phi_t(z)) = \|g - \psi(\psi + t(\phi - \psi))\|$$

$$= \|B(\psi, \psi)^{-\frac{1}{2}}(t(\phi - \psi))\|^\psi$$

$$= t\|B(\psi, \psi)^{-\frac{1}{2}}(\phi - \psi)\|^\psi$$

(since $(tx)^y = tx^y$)

$$= t\left(\left|B(\psi, \psi)^{-\frac{1}{2}}(\psi - \phi)\right|^\psi\right)^{(t-1)^\psi}$$

(since $(x^y)^z = x^{y+z}$)

$$= t\|g - \psi(\phi)^{(t-1)^\psi}(z)\|$$

from (8).

Since $\sup_{z \in B} \|g - \psi(z)\| = d_\beta(\phi, \psi) = \lambda < 1$ and $\|(t - 1)^\psi\| \leq 1$ we use (2) to get

$$d_\beta(\psi, \phi_t) \leq \frac{t\lambda}{1 - (1 - t)\lambda} \leq \lambda.$$  \hfill (10)

**Lemma 3.7.** Let $\phi, \psi \in H(B)$ satisfy $d_\beta(\phi, \psi) = \lambda < 1$. Then for $t, t + \delta \in [0, 1]$ we have

$$\lim_{|\delta| \to 0} d_\beta(\phi_t, \phi_{t+\delta}) = 0.$$

**Proof.** Assume firstly that $t > 0$. (Again, we write $f$ for $f(z)$ where convenient.) Notice that $\phi_{t+\delta} = (t + \delta)\phi + (1 - t - \delta)\psi = \phi_t + \delta(\phi - \psi)$. We apply (7) with $a = \phi_t(z)$ and $b = \delta(\phi(z) - \psi(z)) = \frac{\delta}{t}(\phi_t - \psi)$. We then have

$$g_{-\phi_t}(\phi_{t+\delta}) = \varepsilon \left[B(\phi_t, \phi_t)^{-\frac{1}{2}}(\psi - \phi_t)\right]^{(\varepsilon - 1)\phi_t}$$

where $\varepsilon := -\frac{\delta}{t}$

which from (8)

$$= \varepsilon(g_{-\phi_t}(\psi)(\varepsilon - 1)\phi_t).$$  \hfill (11)

From Lemma 3.6 we have that $d_\beta(\phi_t, \psi) \leq d_\beta(\phi, \psi)$ and hence $\|g_{-\phi_t}(\psi(z))\| \leq \lambda < 1$ for all $z \in B$ and we can choose $\delta$, and hence $\varepsilon$, sufficiently small so that $|\lambda(\varepsilon - 1)| \leq \lambda' < 1$.

In particular,

$$\|g_{-\phi_t}(\psi(z))\| |(\varepsilon - 1)\phi_t(z)\| \leq \lambda' < 1$$

for all $z \in B$. We then have

$$d_\beta(\phi_t, \phi_{t+\delta}) = \sup_{z \in B} \|g_{-\phi_t}(\phi_{t+\delta}(z))\|$$

$$= \sup_{z \in B} |\varepsilon(g_{-\phi_t}(\psi(z))(\varepsilon - 1)\phi_t(z))|$$

(from (11))
In particular, $\lim_{|\delta| \to 0} d_\beta(\phi_t, \phi_{t+\delta}) = 0$.

If $t = 0$ then $\phi_t = \phi_0 = \psi$ and $\phi_{t+\delta} = \phi_\delta$. Then (10) shows that $d_\beta(\phi_t, \phi_{t+\delta}) \to 0$ as $|\delta| \to 0$. \hfill \Box

To get the following result we can now adapt the one dimensional proof in [12] to the infinite dimensional setting of a bounded symmetric domain.

**Theorem 3.8.** Let $B$ be a bounded symmetric domain and let $\phi, \psi \in H(B)$. The following are equivalent.

(i) $C_\phi$ and $C_\psi$ are in the same path connected component of the space of composition operators on $H^\infty(B)$,

(ii) $d_\beta(\phi, \psi) < 1$,

(iii) $\|C_\phi - C_\psi\| < 2$.

**Proof.** (ii) $\iff$ (iii) is immediate from Proposition 3.3. Corollary 3.5 is precisely the statement that (ii) implies (i).

To show (i) implies (ii), let $\phi \in H(B)$ and let $[\phi] = \{\psi \in H(B), d_\beta(\phi, \psi) < 1\}$. We recall that the map $\phi \mapsto C_\phi$ is a homeomorphism. Then, since (ii) implies (i), $[\phi]$ is contained in the path connected component of $\phi$ in $H(B)$. In fact $[\phi]$ is the path connected component of $\phi$. To see this, choose $\omega \in H(B)$ which is not in $[\phi]$. Then $d_\beta(\phi, \omega) = \sup_{z \in B} \beta(\phi(z), \omega(z)) = 1$ which implies that

$$\sup_{z \in B} C_B(\phi(z), \omega(z)) = \sup_{z \in B} \tanh^{-1} \beta(\phi(z), \omega(z)) = \infty.$$ 

For $\psi \in [\phi]$, $\sup_{z \in B} C_B(\phi(z), \psi(z)) < \infty$ and so the triangle inequality for the Carathéodory distance $C_B$ implies that

$$\sup_{z \in B} C_B(\psi(z), \omega(z)) \geq \sup_{z \in B} (C_B(\phi(z), \omega(z)) - C_B(\phi(z), \psi(z))) = \infty.$$ 

Thus $d_\beta(\psi, \omega) = 1$. In other words, the $d_\beta$-distance between any element of $[\phi]$ and any element not in $[\phi]$ is equal to 1. In particular, there can be no $d_\beta$-continuous path from an element of $[\phi]$ to an element not in $[\phi]$. We conclude $[\phi]$ is the path-connected component of $\phi$ in $H(B)$. Again by the homeomorphism of Proposition 3.3 the path component of $C_\phi$ in the space of composition operators is $\{C_\psi : \psi \in [\phi]\}$ and thus (i) implies (ii). \hfill \Box

**References**


