A SCHWARZ LEMMA AND COMPOSITION OPERATORS

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Abstract. We give an alternative description of the Carathéodory pseudo-distance on a domain $D$ in an arbitrary complex Banach space. This gives a Schwarz lemma for holomorphic maps of the domain. We specialise to the case of a bounded symmetric domain and obtain some applications. In particular, we give the connected components of the space of composition operators with symbol in a bounded symmetric domain. This generalises results for the space of composition operators on $H^\infty(\Delta)$ in [12] and for $H^\infty(B)$, $B$ the unit ball of a Hilbert space or commutative $C^*$-algebra in [2].

Introduction

Let $D$ be a domain in a complex Banach space $E$ and let $\Delta$ be the open unit disc in $\mathbb{C}$. We define the following pseudo-distance on $D$,

$$d_D(z, w) := \sup\{|f(z) - f(w)| : f : D \to \Delta \text{ holomorphic}\} \quad \text{for } z, w \in D.$$ 

We prove that

$$\log \frac{2 + d_D}{2 - d_D}$$

is in fact the Carathéodory pseudo-distance $C_D$ on $D$. This results in a Schwarz Lemma for holomorphic maps from $D$ to $\Delta$. When we specialise this to $B_E$, the open unit ball of a Banach space $E$, we prove firstly that $d_{B_E}$ can be expressed in terms of holomorphic self-maps of $B_E$, namely,

$$d_{B_E}(z, w) = \sup\{\|f(z) - f(w)\| : f : B_E \to B_E \text{ holomorphic}\}.$$ 

Since the Carathéodory distance on $B_E$ satisfies

$$C_{B_E}(z, 0) = \tanh^{-1} \|z\|$$

we obtain, among others, the following Schwarz Lemma for all $f : B_E \to B_E$ holomorphic:

$$\|f(z) - f(0)\| \leq \frac{2 - 2\sqrt{1 - \|z\|^2}}{\|z\|} \quad \text{for all } z \in B_E.$$ 

If $B$ is a bounded symmetric domain and $f : B \to B$ is holomorphic, we get

$$\|f(z) - f(w)\| \leq \frac{2 - 2\sqrt{1 - \|g_{-z}(w)\|^2}}{\|g_{-z}(w)\|} \quad \text{for all } z, w \in B.$$ 

The description of $d_{B_E}$ in terms of holomorphic self-maps of $B_E$ makes it suited to the study of composition operators on the space $H^\infty(B_E)$ and, indeed, this is the motivation behind the introduction of $d_{\Delta}$ in the one variable case in [12]. The set-up is as follows:
to every \( \phi : B_E \to B_E \) holomorphic we associate a linear map \( C_\phi \), called a composition operator, on the space \( H^\infty(B_E) \) of all bounded holomorphic functions on \( B_E \) by

\[
C_\phi(f) = f \circ \phi
\]

for \( f \in H^\infty(B_E) \). The idea is to associate the function theoretic properties of \( \phi \) with the properties of \( C_\phi \) as a linear mapping.

For \( B = \Delta \), a survey of the classical theory of composition operators on the Hardy and Bergman spaces is given in [4] and [16]. To extend the classical results where \( \phi \) is taken as a holomorphic function on \( \Delta \) to the case where \( \phi \) is a function of several or even infinitely many variables, one can head in a variety of directions. For example, if \( B_n \) is the open unit ball of \( \mathbb{C}^n \), MacCluer, Shapiro and Luecking, among others have looked at the action of \( C_\phi \) on the Hardy spaces \( H^p(B_n) \), \( 0 < p < \infty \) and the Bergman spaces \( A^p(B_n) \), \( 0 < p < \infty \). Jafari, Li, Russo and others have studied \( C_\phi \) on the Hardy and Bergman spaces of finite dimensional bounded symmetric domains and strongly pseudo-convex domains. We refer to the survey of Russo [15] for references and more information. In the infinite dimensional case, we refer to [1, 2, 7] which study composition operators on the space \( H^\infty(B_E) \), for \( E \) a complex Banach space.

Our aim is to extend to a bounded symmetric domain \( B \) results of MacCluer, Ohno and Zhao for the one variable case in [12] that determine the connected components of the topological space of composition operators on \( H^\infty(\Delta) \) with the natural uniform norm topology. These results were extended in [2] when \( B \) is the open unit ball of a Hilbert space or commutative \( C^* \)-algebra, and in [17] when \( B \) is the open unit ball of \( \mathbb{C}^n \).

We recall that every bounded symmetric domain \( B \) can be realised as the open unit ball of a Banach space \( Z \), known as a \( JB^* \)-triple [8]. The algebraic properties of \( Z \), in particular the properties of the Bergman operator \( B(z, w) \) and the quasi-inverse map \( z \to z^a \) are then used, together with the distance \( d_B \), to determine the connected components of the space of composition operators on \( H^\infty(B) \). For a general survey and background details on the Poincaré distance, Carathéodory pseudo-distance and \( JB^* \)-triples we refer to [5].

1. Notation and Background

We let \( E \) and \( F \) denote complex Banach spaces and \( D \) and \( \tilde{D} \) domains in \( E \) and \( F \) respectively. The set of all holomorphic maps from \( D \) to \( \tilde{D} \) is denoted by \( H(D, \tilde{D}) \). We write \( H^\infty(D) \) for the space of all bounded \( \mathbb{C} \)-valued holomorphic functions on \( D \) and \( \| f \|_\infty := \sup_{z \in D} |f(z)| \) for all \( f \in H^\infty(D) \).

**Definition 1.1.** The Poincaré distance \( \rho \) on \( \Delta \) is

\[
\rho(z, w) := \tanh^{-1} \left\| \frac{z - w}{1 - \overline{z}w} \right\| \quad \text{for } z, w \in \Delta.
\]

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1 We thank the referee for drawing our attention to this reference.
The Carathéodory pseudo-distance can be defined on any complex manifold [5], although we restrict our attention here to the case of a domain \( D \).

**Definition 1.2.** The Carathéodory pseudo-distance on a domain \( D \) is given by

\[
C_D(z, w) := \sup \{ \rho(f(z), f(w)) : f \in H(D, \Delta) \} \quad \text{for } z, w \in D.
\]

The Carathéodory pseudo-distances form a Schwarz-Pick system (cf. [5]) for which holomorphic functions act as contractions, namely,

\[
C_{D_2}(f(z), f(w)) \leq C_{D_1}(z, w) \quad \text{for all } f \in H(D_1, D_2), \ z, w \in D_1.
\]

In fact, this is the smallest of all Schwarz-Pick systems. For bounded domains, cf. [5, chapters 4 and 5], it turns out that \( C_D \) is continuous and generates the original topology thus ensuring that it is actually a distance on \( D \).

We now introduce the class of Banach spaces known as the \( JB^* \)-triples. We use \( H \) and \( K \) to denote arbitrary complex Hilbert spaces and \( L(X, Y) \) to denote the space of continuous linear operators from a Banach space \( X \) to a Banach space \( Y \). We let \( L(X) = L(X, X) \) and \( GL(X) \) be all invertible elements in \( L(X) \).

**Definition 1.3.** A \( JB^* \)-triple is a complex Banach space \( Z \) with a real trilinear mapping \( \{\cdot, \cdot, \cdot\} : Z \times Z \times Z \to Z \) satisfying

(i) \( \{x, y, z\} \) is complex linear and symmetric in the outer variables \( x \) and \( z \), and is complex anti-linear in \( y \).

(ii) The map \( z \to \{x, x, z\} \), denoted \( x \boxtimes x \), is Hermitian, \( \sigma(x \boxtimes x) \geq 0 \) and \( \|x \boxtimes x\| = \|x\|^2 \) for all \( x \in Z \), where \( \sigma \) denotes the spectrum.

(iii) The product satisfies the following “triple identity”

\[
\{a, b, \{x, y, z\}\} = \{\{a, b, x\}, y, z\} - \{x, \{b, a, y\}, z\} + \{x, y, \{a, b, z\}\}.
\]

Let \( Z \) be a \( JB^* \)-triple. Several types of linear operators on \( Z \) arise naturally from the triple product:

\[
x \boxtimes y \in L(Z) : z \to \{x, y, z\},
\]

\[
Q_x \in L_{R}(Z) : z \to \{x, z, z\},
\]

and the important Bergman operators

\[
B(x, y) = I - 2x \boxtimes y + Q_x Q_y \in L(Z).
\]

**Example 1.4.**

(i) \( L(H, K) \) is a \( JB^* \)-triple for the product \( \{x, y, z\} = \frac{1}{2}(xy^*z + zy^*x) \) where \( y^* \) denotes the usual adjoint of \( y \) and \( B(x, y)z = (1 - xy^*)z(1 - y^*x) \).

(ii) \( C_0(X) \), the continuous \( \mathbb{C} \)-valued functions vanishing at infinity on a locally compact Hausdorff space \( X \), is a \( JB^* \)-triple for the product \( \{x, y, z\} = x\overline{y}z \) and \( B(x, y)z = (1 - x\overline{y})^2z \).
As Banach spaces the $JB^*$-triples are characterised by the fact that their open unit balls are homogeneous. In fact, if we let $\text{Aut}(B)$ denote all biholomorphic maps from $B$ to $B$ then for all $a$ in $B$, we have $g_a \in \text{Aut}(B)$ defined by

$$g_a(z) = a + B(a,a)^{1/2}(I + z \Box a)^{-1}z$$

(cf. [9]) which satisfies $g_a(0) = a$, $g_a^{-1} = g_{-a}$ and $g_a'(0) = B(a,a)^{1/2}$ (defined in terms of a functional calculus). We note the fundamental formula [10]

$$\|B(a,a)^{-1/2}\| = \frac{1}{1 - \|a\|^2} \quad (1)$$

for $a \in B$. For $z, a \in B, z^a := (I - z \Box a)^{-1}z$ is called the quasi-inverse of $z$ with respect to $a$ and satisfies

$$\|z^a\| \leq \|z\| \quad (2)$$

The quasi-inverse also satisfies $(z^a)^b = z^{a+b}$ whenever both sides of this equation are well defined. For further details see [11, Chapter 7] or [6].

It is known [8] that every bounded symmetric domain is biholomorphically equivalent to the open unit ball of a $JB^*$-triple and vice versa. From the homogeneity therefore one can easily see that on a bounded symmetric domain $B$ the Carathéodory distance is given by $C_B(z,w) = \tanh^{-1}\|g_{-z}(w)\|$. For a recent survey of $JB^*$-triples and bounded symmetric domains we refer to [3].

2. The Carathéodory pseudo-distance

To study composition operators on $H^\infty(\Delta)$ MacCluer et al. [12] introduce the distance $d_\infty$ on $\Delta$

$$d_\infty(z, w) := \sup\{|f(z) - f(w)| : f \in H^\infty(\Delta), \|f\|_\infty \leq 1\}, \quad z, w \in \Delta.$$

It is not too difficult to see [13] that

$$d_\infty(z, w) = \frac{2 - 2\sqrt{1 - \beta(z,w)^2}}{\beta(z,w)} \quad \text{for} \quad \beta(z,w) := \left|\frac{z - w}{1 - \overline{z}w}\right|$$

or in terms of the Poincaré metric $\rho$ on $\Delta$

$$\rho(z, w) = \tanh^{-1}\beta(z,w) = \log \frac{2 + d_\infty(z,w)}{2 - d_\infty(z,w)}. \quad (3)$$

Motivated by this we introduce the following pseudo-distance on an arbitrary domain $D$

$$d_D(z, w) := \sup\{|f(z) - f(w)| : f \in H(D, \Delta)\}.$$  

We note that $d_\Delta = d_\infty$ above. Clearly

$$d_D(z, w) = \sup\{|g(f(z)) - g(f(w))| : g \in H(\Delta, \Delta), f \in H(D, \Delta)\} = \sup_{f \in H(D, \Delta)} d_\infty(f(z), f(w)) \quad \text{for} \quad z, w \in D.$$
Since the map \( t \to \log \frac{2 + t}{2 - t} \) is strictly increasing on \([0, 2)\) it follows that
\[
\log \frac{2 + d_D(z, w)}{2 - d_D(z, w)} = \sup_{f \in H(D, \Delta)} \log \frac{2 + d_\infty(f(z), f(w))}{2 - d_\infty(f(z), f(w))} = \sup_{f \in H(D, \Delta)} \rho(f(z), f(w)) \text{ from (3)}
\]
\[= \sup_{f \in H(D, \Delta)} \rho(f(z), f(w)) = C_D(z, w).
\]
In other words, \( \log \frac{2 + d_D}{2 - d_D} \) is the Carathéodory pseudo-distance on \( D \), or equivalently for any domain \( D \)
\[
d_D(z, w) = \frac{2 - 2\sqrt{1 - (\tanh C_D(z, w))^2}}{\tanh C_D(z, w)} \quad \text{for } z, w \in D. \tag{4}
\]
Throughout, we use \( B_E \) to denote the open unit ball of an arbitrary complex Banach space \( E \) and reserve \( B \) to denote a bounded symmetric domain.

We now present a series of Schwarz Lemmas arising from (4).

**Lemma 2.1.** (i) Let \( D \) be an arbitrary domain and \( f : D \to \Delta \) be holomorphic. Then
\[
|f(z) - f(w)| \leq \frac{2 - 2\sqrt{1 - (\tanh C_D(z, w))^2}}{\tanh C_D(z, w)} \quad \text{for } z, w \in D.
\]
In particular, if \( D = B_E \) is the open unit ball of a Banach space \( E \) then
\[
|f(z) - f(0)| \leq \frac{2 - 2\sqrt{1 - \|z\|^2}}{\|z\|} \quad \text{for } z \in B_E.
\]
(ii) Let \( B \) be a bounded symmetric domain and \( f : B \to \Delta \) be holomorphic. Then
\[
|f(z) - f(w)| \leq \frac{2 - 2\sqrt{1 - \|g_{z}(w)\|^2}}{\|g_{z}(w)\|} = \frac{2\sqrt{\|B_w^{-1}B(w, z)B_z^{-1}\| - 1}}{\sqrt{\|B_w^{-1}B(w, z)B_z^{-1}\| - 1}} \quad \text{for } z, w \in B
\]
where \( g_z \) is an automorphism of \( B \) taking 0 to \( z \) and \( B_z := B(z, z)^\frac{1}{2} \).

**Proof.** (i) is immediate from (4). The first part of (ii) follows from (i) since on a bounded symmetric domain \( B \)
\[
C_B(z, w) = \tanh^{-1} \|g_{z}(w)\| \quad \text{for } z, w \in B \tag{5}
\]
where \( g_z \) is an automorphism of \( B \) taking 0 to \( z \).

For the second part of (ii) we rewrite
\[
\frac{2 - 2\sqrt{1 - \|g_{z}(w)\|^2}}{\|g_{z}(w)\|}
\]
in terms of Bergman operators using the fact [14, Proposition 3.1] that
\[
\frac{1}{1 - \|g_{-z}(w)\|^2} = \|B_w^{-1}B(w, z)B_z^{-1}\| \quad \text{for } z, w \in B.
\] (6)

For the purpose of studying composition operators on \(H^\infty(B_E)\) the distance we really
need on \(B_E\) is written in terms of self-maps of \(B_E\), namely,
\[
\tilde{d}_{B_E}(z, w) := \sup\{\|f(z) - f(w)\| : f \in H(B_E)\}.
\]

**Proposition 2.2.** The distance \(\tilde{d}_{B_E}\) coincides with \(d_{B_E}\).

**Proof.** Fix \(z, w\) in \(B_E\) and \(f \in H(B_E)\). By the Hahn-Banach theorem there exists \(\lambda = \lambda(z, w, f) \in Z^*, \|\lambda\| \leq 1\) with
\[
\|f(z) - f(w)\| = \lambda(f(z) - f(w))
\]
and hence \(\tilde{d}_{B_E}(z, w) \leq d_{B_E}(z, w)\). On the other hand, if \(g \in H(B_E, \Delta)\) then for any fixed \(u \in \partial B_E\) the map \(z \to g(z)u\) is in \(H(B_E)\) and this implies \(d_{B_E}(z, w) \leq \tilde{d}_{B_E}(z, w)\). \(\square\)

Proposition 2.2 together with (4), (5) and (6) now easily gives the following.

**Corollary 2.3.** (i) Let \(E\) be a Banach space and \(f : B_E \to B_E\) be holomorphic. Then
\[
\|f(z) - f(w)\| \leq \frac{2 - 2\sqrt{1 - \tanh C_{B_E}(z, w)^2}}{\tanh C_{B_E}(z, w)} \quad \text{for } z, w \in B_E.
\]

In particular,
\[
\|f(z) - f(0)\| \leq \frac{2 - 2\sqrt{1 - \|z\|^2}}{\|z\|} \quad \text{for } z \in B_E.
\]

(ii) Let \(B\) be a bounded symmetric domain and \(f : B \to B\) be holomorphic. Then
\[
\|f(z) - f(w)\| \leq \frac{2 - 2\sqrt{1 - \|g_{-z}(w)\|^2}}{\|g_{-z}(w)\|} \leq \frac{2\sqrt{\|B_w^{-1}B(w, z)B_z^{-1}\| - 1}}{\sqrt{\|B_w^{-1}B(w, z)B_z^{-1}\| - 1}} \quad \text{for } z, w \in B,
\]
where \(g_z\) is an automorphism of \(B\) taking 0 to \(z\) and \(B_z := B(z, z)^{\frac{1}{2}}\).

As the Bergman operators play a fundamental role in the holomorphy of \(B\) and \(B(a, a)^{\frac{1}{2}} = g'_a(0), \ a \in B\) the inequality [10]
\[
\|B(a, a)^{\frac{1}{2}}\| \leq 1
\]
is crucial to the geometry of \(B\). We are able to obtain a simple direct proof of this result.

**Corollary 2.4.** For \(a \in B\), \(\|B(a, a)^{\frac{1}{2}}\| \leq 1\).
Proof. Fix $a \in B$. For all $z \in B$

$$
\frac{2 - 2\sqrt{1 - \|z\|^2}}{\|z\|} = d_B(z, 0) \geq \|g_a(z) - g_a(0)\| = \|B(a, a)\frac{1}{2}z^{-a}\|.
$$

Since $z^a \in B$ if $\|z\| < \frac{1}{1 + \|a\|}$ and $(z^a)^{-a} = z$ this implies that

$$
\|B(a, a)\frac{1}{2}z\| \leq \frac{2 - 2\sqrt{1 - \|z^a\|^2}}{\|z^a\|}
$$

when $\|z\| < \frac{1}{1 + \|a\|}$. Fix $0 < t < \frac{1}{1 + \|a\|}$. For $\|z\| \leq t$, we have from (2) that

$$
\|z^a\| \leq \frac{\|z\|}{1 - \|z\|\|a\|} \leq \frac{t}{1 - t\|a\|}
$$

and since $h(t) = (2 - 2\sqrt{1 - t^2})/t$ is strictly increasing on $[0, 1)$ this gives

$$
\|B(a, a)\frac{1}{2}z\| \leq h(\|z^a\|) \leq h\left(\frac{t}{1 - t\|a\|}\right).
$$

Then

$$
\|B(a, a)\frac{1}{2}z\| = \sup_{\|z\| = 1} \|B(a, a)\frac{1}{2}z\| = \frac{1}{t} \sup_{\|z\| = t} \|B(a, a)\frac{1}{2}z\|
$$

$$
\leq \frac{1}{t} h\left(\frac{t}{1 - t\|a\|}\right)
$$

$$
= 2 \left(1 - t\|a\| + \sqrt{(1 - t\|a\|)^2 - t^2}\right)^{-1}.
$$

As $t \to 0$ this gives $\|B(a, a)\frac{1}{2}\| \leq 1$ as required. \qed

3. Composition operators on $H^\infty(B)$

In this section we study the connected components of the space of composition operators on $H^\infty(B)$ with the uniform norm topology where $B$ is a bounded symmetric domain. Our motivation was to extend the one variable results in [12] to the case of infinitely many variables. In the case where $B$ is the open unit ball of a Hilbert space or of a commutative $C^*$-algebra we refer to [2]. The key to this study is the distance $d_B$ which gives a formula for the hyperbolic distance $C_B$, namely,

$$
C_B = \log \frac{2 + d_B}{2 - d_B}.
$$

Just as the Möbius maps are crucial when studying $\Delta$, so the analogous automorphisms $\{g_a : a \in B\}$ of $B$ are essential here and we establish some simple identities.

Lemma 3.1. For $a, b \in B$,

$$
g_{-a}(a + b) = (B(a, a)^{-\frac{1}{2}}b)^a \quad \text{when } a + b \in B, \quad (7)
$$

and

$$
g_{-a}(b) = (B(a, a)^{-\frac{1}{2}}(b - a))^a. \quad (8)
$$
Proof. Clearly, the two expressions are equivalent. Recall that \( g_a(z) = a + B(a, a)^{\frac{1}{2}} z^{-a} \). Since the inverse of \( z \rightarrow z^a \) is \( z \rightarrow z^{-a} \) and the inverse of \( g_a \) is \( g_{-a} \) it follows that \( g_{-a}(b) = g^{-1}_a(b) = (B(a, a)^{-\frac{1}{2}} (b - a))^a \).

For \( z, w \in B \), we define \( \beta(z, w) := \|g_z(w)\| \).

Definition 3.2. For \( \phi, \psi \in H(B) \) we let

\[
d_\beta(\phi, \psi) := \sup_{z \in B} \beta(\phi(z), \psi(z)).
\]

We note that \( d_\beta \) is a metric on \( H(B) \) and, by virtue of the following result, it is the topological structure of \((H(B), d_\beta)\) that interests us.

Proposition 3.3. Let \( \phi, \psi \in H(B) \). Then

\[
\|C_\phi - C_\psi\| = \frac{2 - 2\sqrt{1 - d_\beta(\phi, \psi)^2}}{d_\beta(\phi, \psi)}.
\]

In particular, the space of composition operators on \( H^\infty(B) \) with the uniform norm topology is homeomorphic as a topological space to \((H(B), d_\beta)\).

Proof. Proposition 2.2 together with (4) and (5) gives

\[
d_B(z, w) = \sup \{\|f(z) - f(w)\| : f \in H(B)\} = \frac{2 - 2\sqrt{1 - \beta(z, w)^2}}{\beta(z, w)}.
\]

Since \( h(t) = 2(1 - \sqrt{1 - t^2})/t \) is an increasing function on \([0, 1)\) we have

\[
\|C_\phi - C_\psi\| = \sup\{\|C_\phi(f) - C_\psi(f)\|_\infty : f \in H^\infty(B), \|f\|_\infty \leq 1\}
= \sup\{\|f \circ \phi - f \circ \psi\|_\infty : f \in H(B)\}
= \sup\{\|f(\phi(z)) - f(\psi(z))\| : f \in H(B), z \in B\}
= \sup_{z \in B} d_B(\phi(z), \psi(z))
= \sup_{z \in B} \frac{2 - 2\sqrt{1 - \beta(\phi(z), \psi(z))^2}}{\beta(\phi(z), \psi(z))}
= \frac{2 - 2\sqrt{1 - d_\beta(\phi, \psi)^2}}{d_\beta(\phi, \psi)}.
\]

Our aim is to determine the connected components of the space of composition operators on \( H^\infty(B) \). The above result means that we can now do this by examining the space \((H(B), d_\beta)\). In order to achieve this, we use \( JB^*\)-triple tools such as Bergman operators and the quasi-inverse map as a substitute for the algebra structure used when \( B = \Delta \) \([12]\) or \( B \) is the unit ball of \( C_0(X) \) \([2]\) and as a substitute for the inner product used when \( B \) is a Hilbert ball \([2]\).
To begin with we note that the $d_\beta$-topology on $H(B)$ is stronger than the $\| \cdot \|_\infty$ topology. Indeed from (8) we have that
\[(g_{-w}(z))^{-w} = B(w, w)^{-\frac{1}{2}}(z - w)\]
for $z, w \in B$ and hence we may write
\[z - w = B(w, w)^{\frac{1}{2}}(g_{-w}(z))^{-w}.\]
Since $\|B(w, w)^{\frac{1}{2}}\| \leq 1$ and
\[\|(g_{-w}(z))^{-w}\| \leq \frac{\|g_{-w}(z)\|}{1 - \|w\|\|g_{-w}(z)\|} \leq \frac{\|g_{-w}(z)\|}{1 - \|g_{-w}(z)\|}\]
this gives that
\[\|z - w\| \leq \frac{\beta(z, w)}{1 - \beta(z, w)}\]
for all $z, w \in B$. Therefore for $\phi, \psi \in H(B)$ we have
\[\sup_{z \in B} \|\phi(z) - \psi(z)\| \leq \sup_{z \in B} \frac{\beta(\phi(z), \psi(z))}{1 - \beta(\phi(z), \psi(z))}\]
and hence $\|\phi - \psi\|_\infty \leq \frac{d_\beta(\phi, \psi)}{1 - d_\beta(\phi, \psi)}$. In particular, if $d_\beta(\phi, \psi) \rightarrow 0$ then $\|\phi - \psi\|_\infty \rightarrow 0$.

The converse however is not true. For example, in $\Delta$, $\beta(a, e^{it}a) = |g_{-a}(e^{it}a)| = \left| \frac{e^{it} - 1}{e^{it}a - 1} \right|$ which implies $d_\beta(id, e^{it}id) = 1$ for all $t \in (0, 2\pi)$, even though $\|id - e^{it}id\|_\infty \rightarrow 0$ as $t \rightarrow 0$.

However, the two topologies do agree on the set of holomorphic functions which map $B$ strictly inside $B$. In other words, if $\|\phi\|_\infty < 1$ then $\|\phi - \psi_t\|_\infty \rightarrow 0$ if and only if $d_\beta(\phi, \psi_t) \rightarrow 0$. To see this, we note from (8) that
\[\|g_{-a}(b)\| \leq \|(B(a, a)^{-\frac{1}{2}}(b - a))^a\| \leq \frac{\|b - a\|}{1 - \|a\|^2 - \|a\|\|b - a\|}\]
from repeated use of (1) and (2) when $\|b - a\|$ is sufficiently small. Therefore if $\|\phi\|_\infty < 1$ we have
\[d_\beta(\phi, \psi_t) = \sup_{z \in B} \|g_{-\phi(z)}(\psi_t(z))\| \leq \sup_{z \in B} \frac{\|\phi(z) - \psi_t(z)\|}{1 - \|\phi(z)\|^2 - \|\phi(z)\|\|\phi(z) - \psi_t(z)\|} \leq \frac{\|\phi - \psi_t\|_\infty}{1 - \|\phi\|_\infty^2 - \|\phi\|_\infty\|\phi - \psi_t\|_\infty}\]
and hence $\|\phi - \psi_t\|_\infty \rightarrow 0$ implies that $d_\beta(\phi, \psi_t) \rightarrow 0$ as well.

Given $\phi, \psi \in H(B)$, it is obvious from the definition that $d_\beta(\phi, \psi) \leq 1$. Later results will show the importance of determining whether $d_\beta(\phi, \psi) < 1$. We remark therefore that if $\phi$
maps $B$ strictly inside $B$, then the condition $d_\beta(\phi, \psi) < 1$ is satisfied for every $\psi \in H(B)$ which also maps $B$ strictly inside $B$. To see this, we use (6) to write

$$
\frac{1}{1 - \|g_{-\phi(z)}(\psi(z))\|^2} = \|B_{\phi(z)}^{-1} B(\phi(z), \psi(z))B_{\psi(z)}^{-1}\|
$$

and hence $d_\beta(\phi, \psi) = \sup_{z \in B} \|g_{-\phi(z)}(\psi(z))\| < 1$ if and only if

$$
\sup_{z \in B} \|B_{\phi(z)}^{-1} B(\phi(z), \psi(z))B_{\psi(z)}^{-1}\| < \infty.
$$

Since from (1)

$$
\|B_{\phi(z)}^{-1} B(\phi(z), \psi(z))B_{\psi(z)}^{-1}\| \leq \frac{\|B(\phi(z), \psi(z))\|}{(1 - \|\psi(z)\|^2)(1 - \|\phi(z)\|^2)}
$$

and $\|B(\phi(z), \psi(z))\| \leq (1 + \|\phi(z)\|\|\psi(z)\|)^2$ for all $z \in B$, it follows that if $\|\phi\|_\infty < 1$ and $\|\psi\|_\infty < 1$ then

$$
\sup_{z \in B} \|B_{\phi(z)}^{-1} B(\phi(z), \psi(z))B_{\psi(z)}^{-1}\| \leq \frac{4}{1 - \|\phi\|_\infty^2} \frac{1}{1 - \|\psi\|_\infty^2} < \infty
$$

and hence $d_\beta(\phi, \psi) < 1$.

Since $d_\beta(\phi, 0) = \|\phi\|$, the converse is also true. In other words, for $\phi \in H(B)$

$$
d_\beta(\phi, \psi) < 1 \text{ for all } \psi \text{ with } \|\psi\|_\infty < 1 \text{ if and only if } \|\phi\|_\infty < 1.
$$

(9)

**Theorem 3.4.** Let $\phi, \psi \in H(B)$ with $d_\beta(\phi, \psi) < 1$. Then the map

$$
t \mapsto \phi_t := t\phi + (1 - t)\psi
$$

is a $d_\beta$-continuous path joining $\phi$ to $\psi$.

The proof breaks into two parts proved below. The first part (Lemma 3.6) shows that any convex combination $\phi_t$ of $\phi$ and $\psi$ satisfies $d_\beta(\phi_t, \psi) \leq d_\beta(\phi, \psi)$. The second part (Lemma 3.7) then uses this to show that the map $t \mapsto \phi_t$ is $d_\beta$-continuous which proves the theorem. As a result of the homeomorphism $\phi \mapsto C_\phi$ guaranteed by Proposition 3.3, this theorem immediately implies the following.

**Corollary 3.5.** Let $\phi, \psi \in H(B)$ with $d_\beta(\phi, \psi) < 1$. Then $C_\phi$ and $C_\psi$ are in the same path connected component in the space of composition operators on $H^\infty(B)$.

In particular, from (9), we have that the set of composition operators $C_\phi$ with $\|\phi\|_\infty < 1$ is path connected. This is proved in a more general setting in [2]

**Lemma 3.6.** Let $\phi, \psi \in H(B)$ satisfy $d_\beta(\phi, \psi) = \lambda < 1$. Then for any $t \in [0, 1]$ we have $d_\beta(\phi_t, \psi) \leq \lambda$, where $\phi_t = t\phi + (1 - t)\psi$. 

Proof. Note first that \( \phi_t = \psi + t(\phi - \psi) \). Fix \( z \in B \). For ease of notation we write \( f(\phi) \) rather than \( f(\phi(z)) \) whenever \( f \) is a function on \( B \). Now we use (7) with \( a = \psi(z) \) and \( b = t(\phi(z) - \psi(z)) \) to obtain

\[
\beta(\psi(z), \phi_t(z)) = \|g_{-\psi}(\psi + t(\phi - \psi))\| \\
= \| \left[ B(\psi, \psi) - \frac{t}{t}(t(\phi - \psi)) \right]^{\psi} \| \\
= t \| \left[ B(\psi, \psi) - \frac{1}{t}(\phi - \psi) \right]^{\psi} \| \\
= t \| \left[ B(\psi, \psi) - \frac{1}{t}(\phi - \psi) \right]^{\psi} \|^{(t-1)\psi} \\
= t \| (g_{-\psi}(\phi))^{(t-1)\psi}(z) \| \quad \text{from (8)}.
\]

Since \( \sup_{z \in B} \| g_{-\psi(z)}(\phi(z)) \| = d_\beta(\phi, \psi) = \lambda < 1 \) and \( \| (t-1)\psi \|_\infty \leq 1 \) we use (2) to get

\[
d_\beta(\psi, \phi_t) \leq \frac{t\lambda}{1 - (1 - t)\lambda} \leq \lambda.
\]

(10)

Lemma 3.7. Let \( \phi, \psi \in H(B) \) satisfy \( d_\beta(\phi, \psi) = \lambda < 1 \). Then for \( t, t + \delta \in [0, 1] \) we have

\[
\lim_{|\delta| \to 0} d_\beta(\phi_t, \phi_{t+\delta}) = 0.
\]

Proof. Assume firstly that \( t > 0 \). (Again, we write \( f \) for \( f(z) \) where convenient.) Notice that \( \phi_{t+\delta} = (t + \delta)\phi + (1 - t - \delta)\psi = \phi_t + \delta(\phi - \psi) \). We apply (7) with \( a = \phi_t(z) \) and \( b = \delta(\phi(z) - \psi(z)) = \frac{\delta}{t}(\phi_t - \psi) \) to get

\[
g_{-\phi_t}(\phi_{t+\delta}) = (B(\phi_t, \phi_t) - \frac{\delta}{t}(\phi_t - \psi))^{\phi_t} = \left( \varepsilon B(\phi_t, \phi_t) - \frac{1}{t}(\phi_t - \psi) \right)^{\phi_t} \quad \text{where} \quad \varepsilon := -\frac{\delta}{t}
\]

\[
= \varepsilon \left( [B(\phi_t, \phi_t) - \frac{1}{t}(\phi_t - \psi)]^{\phi_t} \right)^{(t-1)\phi_t} \quad \text{which from (8)}
\]

\[
= \varepsilon (g_{-\phi_t}(\psi))^{(t-1)\phi_t}.
\]

(11)

From Lemma 3.6 we have that \( d_\beta(\phi_t, \psi) \leq d_\beta(\phi, \psi) \) and hence \( \|g_{-\phi_t(z)}(\psi(z))\| \leq \lambda < 1 \) for all \( z \in B \) and we can choose \( \delta \), and hence \( \varepsilon \), sufficiently small so that \( |\lambda(\varepsilon - 1)| \leq \lambda' < 1 \). In particular,

\[
\|g_{-\phi_t(z)}(\psi(z))\| \|(\varepsilon - 1)\phi_t(z)\| \leq \lambda' < 1
\]

for all \( z \in B \). We then have

\[
d_\beta(\phi_t, \phi_{t+\delta}) = \sup_{z \in B} \|g_{-\phi_t(z)}(\phi_{t+\delta}(z))\| \\
= \sup_{z \in B} \|\varepsilon (g_{-\phi_t(z)}(\psi(z)))^{(\varepsilon - 1)\phi_t(z)}\| \quad \text{(from (11))}
\]
\[
\leq |\varepsilon| \frac{\lambda}{1 - \lambda'} = \frac{|\delta\lambda|}{t(1 - \lambda')}
\] (from (2)).

In particular, \(\lim_{|\delta| \to 0} d_\beta(\phi_t, \phi_{t+\delta}) = 0\).

If \(t = 0\) then \(\phi_t = \phi_0 = \psi\) and \(\phi_{t+\delta} = \phi_\delta\). Then (10) shows that \(d_\beta(\phi_t, \phi_{t+\delta}) \to 0\) as \(|\delta| \to 0\).

To get the following result we can now adapt the one dimensional proof in [12] to the infinite dimensional setting of a bounded symmetric domain.

**Theorem 3.8.** Let \(B\) be a bounded symmetric domain and let \(\phi, \psi \in H(B)\). The following are equivalent.

1. \(C_\phi\) and \(C_\psi\) are in the same path connected component of the space of composition operators on \(H^\infty(B)\).
2. \(d_\beta(\phi, \psi) < 1\).
3. \(\|C_\phi - C_\psi\| < 2\).

**Proof.** (ii) \(\iff\) (iii) is immediate from Proposition 3.3. Corollary 3.5 is precisely the statement that (ii) implies (i).

To show (i) implies (ii), let \(\phi \in H(B)\) and let \([\phi] = \{\psi \in H(B), d_\beta(\phi, \psi) < 1\}\). We recall that the map \(\phi \mapsto C_\phi\) is a homeomorphism. Then, since (ii) implies (i), \([\phi]\) is contained in the path connected component of \(\phi\) in \(H(B)\). In fact \([\phi]\) is the path connected component of \(\phi\). To see this, choose \(\omega \in H(B)\) which is not in \([\phi]\). Then \(d_\beta(\phi, \omega) = \sup_{z \in B} \beta(\phi(z), \omega(z)) = 1\) which implies that

\[
\sup_{z \in B} C_B(\phi(z), \omega(z)) = \sup_{z \in B} \tanh^{-1} \beta(\phi(z), \omega(z)) = \infty.
\]

For \(\psi \in [\phi], \sup_{z \in B} C_B(\phi(z), \psi(z)) < \infty\) and so the triangle inequality for the Carathéodory distance \(C_B\) implies that

\[
\sup_{z \in B} C_B(\psi(z), \omega(z)) \geq \sup_{z \in B} (C_B(\phi(z), \omega(z)) - C_B(\phi(z), \psi(z))) = \infty.
\]

Thus \(d_\beta(\psi, \omega) = 1\). In other words, the \(d_\beta\)-distance between any element of \([\phi]\) and any element not in \([\phi]\) is equal to 1. In particular, there can be no \(d_\beta\)-continuous path from an element of \([\phi]\) to an element not in \([\phi]\). We conclude \([\phi]\) is the path-connected component of \(\phi\) in \(H(B)\). Again by the homeomorphism of Proposition 3.3 the path component of \(C_\phi\) in the space of composition operators is \(\{C_\psi : \psi \in [\phi]\}\) and thus (i) implies (ii).

**References**


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