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On the low-dimensional homology of $\text{SL}_2(k[t, t^{-1}])$

Kevin Hutchinson

Abstract. We prove analogues of the fundamental theorem of $K$-theory for the second and third homology of $\text{SL}_2$ over an infinite field. The statements of the theorems involve Milnor-Witt $K$-theory and refined scissors congruence groups. We use these results to calculate the low-dimensional homology of $\text{SL}_2$ of Laurent polynomials over certain fields.

1. Introduction

Our goal in this paper is to prove unstable analogues of the fundamental theorem of $K$-theory for the second and third homology of $\text{SL}_2$ over an infinite field $k$, and to use these results to calculate the low-dimensional homology of $\text{SL}_2$ of Laurent polynomials over certain fields.

We follow the approach of K. Knudson ([8]) who studied the homology of $\text{SL}_2(k[t, t^{-1}])$ via the Mayer-Vietoris sequence associated to its natural decomposition as an amalgamated product. What allows us to advance on the calculations of Knudson in the present article is the understanding that has been gained in the intervening years of the connections between the homology of $\text{SL}_2$, Milnor-Witt $K$-theory and scissors congruence groups.

Let $k$ be an infinite field. When $n \geq 3$ then $H_2(\text{SL}_n(k), \mathbb{Z}) \cong K_2(k)$. Furthermore, by [9 Corollary 6.11]

$$H_2(\text{SL}_n(k[t, t^{-1}]), \mathbb{Z}) \cong K_2(k[t, t^{-1}]) \text{ for } n \geq 3.$$  

The fundamental theorem of $K$-theory, combined with the fact that $K_n(A) \cong K_n(A[t])$ for regular rings $A$ ([20]), tells us that

$$K_2(k[t, t^{-1}]) \cong K_2(k) \oplus K_1(k).$$

Here the homomorphism $K_2(k[t, t^{-1}]) \to K_1(k)$ is the composite

$$K_2(k[t, t^{-1}]) \xrightarrow{\delta_t} K_2(k(t)) \xrightarrow{\delta_t} K_1(k)$$

where $\delta_t$ is the tame symbol associated to the $t$-adic valuation on $k(t)$.

It follows that

$$H_2(\text{SL}_n(k[t, t^{-1}]), \mathbb{Z}) \cong H_2(\text{SL}_n(k), \mathbb{Z}) \oplus K_1(k)$$

for all $n \geq 3$.

In section 5 (Theorem 5.1) below we prove an analogue of this for the case $n = 2$: Let $k$ be an infinite field of characteristic not equal to 2. Then

$$H_2(\text{SL}_2(k[t, t^{-1}]), \mathbb{Z}) \cong H_2(\text{SL}_2(k), \mathbb{Z}) \oplus K_1^{\text{MW}}(k).$$
Here $K_1^{MW}(k)$ is the first Milnor-Witt $K$-theory group of the field $k$ (see section [4]). There is a natural surjective map $K_1^{MW}(k) \to K_1(k) = k^\times$ whose kernel is the second power of the fundamental ideal in the Witt ring of $k$. Furthermore, for any infinite field $K$ there is a natural isomorphism $H_2(SL_2(k), \mathbb{Z}) \cong K_2^{MW}(K)$ of the second homology of $SL_2$ with the second Milnor-Witt $K$-group for a field $k$. The homomorphism $H_2(SL_2(k[t, t^{-1}]), \mathbb{Z}) \to K_1^{MW}(k)$ is the composite

$$H_2(SL_2(k[t, t^{-1}]), \mathbb{Z}) \longrightarrow H_2(SL_2(k(t)), \mathbb{Z}) \cong K_2^{MW}(k(t)) \overset{\delta_i}{\longrightarrow} K_1^{MW}(k)$$

where $\delta_i$ is a connecting homomorphism, analogous to the tame symbol, arising in Milnor-Witt $K$-theory.

In section [6] we describe some explicit computations which follow from this result. For example, when $k = \mathbb{Q}$, we derive a decomposition (Theorem [6.5])

$$H_2(SL_2(\mathbb{Q}[t, t^{-1}]), \mathbb{Z}) \cong (\mathbb{Z} \oplus \mathbb{Z}) \oplus \left( \bigoplus_p \mathbb{Z} \right) \oplus \left( \bigoplus_{p \text{ odd}} (\mathbb{Z}^p \oplus \mathbb{Z}/2) \right)$$

where $p$ runs over the set of prime numbers.

The third homology of the special linear groups of fields is also closely related to $K$-theory, at least when $n \geq 3$. Combining [24 Corollary 5.2] and [7 Theorem 4.7] it follows that

$$H_3(SL_n(k), \mathbb{Z}) \cong K_3(k)/((-1) \cdot K_2(k))$$

for all $n \geq 3$.

In particular,

$$H_3(SL_n(k), \mathbb{Z} \left[ \frac{1}{2} \right]) \cong K_3(k) \otimes \mathbb{Z} \left[ \frac{1}{2} \right]$$

for all $n \geq 3$.

By contrast, when $n = 2$, the relation between $H_3(SL_2(k), \mathbb{Z})$ and $K_3(k)$ is much more remote. There is a natural surjective map from $H_3(SL_2(k), \mathbb{Z})$ to $K_3^{\text{ind}}(k)$ but for general infinite fields the kernel may be very large, cf. [5 Theorem 5.1].

Nevertheless, recent computations (see [4]) suggest a natural candidate for the functor

$$\frac{H_3(SL_2(k[t, t^{-1}]), \mathbb{Z})}{H_3(SL_2(k), \mathbb{Z})},$$

at least when we replace $\mathbb{Z}$ by $\mathbb{Z} \left[ \frac{1}{2} \right]$: Namely, associated functorially to any field $K$ there is a group $\mathcal{RP}_1(K)$, the \textit{refined scissors congruence group} (for details see section [7] below), and a natural homomorphism $H_3(SL_2(K), \mathbb{Z}) \to \mathcal{RP}_1(K)$. If the field $K$ has a discrete valuation $v$ and residue field $k$ there is a natural specialization homomorphism

$$\delta_{\pi} : \mathcal{RP}_1(K) \otimes \mathbb{Z} \left[ \frac{1}{2} \right] \to \mathcal{RP}_1(k) \otimes \mathbb{Z} \left[ \frac{1}{2} \right]$$

associated to a choice of uniformizer $\pi$. The refined scissors congruence groups arise naturally in the calculation of the third homology of $SL_2$ of fields and local rings. Indeed, for a quadratically closed field $k$, $\mathcal{RP}_1(k)$ is naturally isomorphic to the scissors congruence group $\mathcal{P}(k)$. For more general fields $k$ the kernel of the natural map $\mathcal{RP}_1(k) \to \mathcal{P}(k)$ is not trivial but can often be expressed in terms of the scissors congruence groups of appropriate residue fields (see [5], [4] and section [7] below for more precise statements.).

The main result of section [8] (Theorem [8.1]) is the existence of a natural isomorphism

$$H_3(SL_2(k[t, t^{-1}]), \mathbb{Z} \left[ \frac{1}{2} \right]) \cong H_3(SL_2(k), \mathbb{Z} \left[ \frac{1}{2} \right]) \oplus (\mathcal{RP}_1(k) \otimes \mathbb{Z} \left[ \frac{1}{2} \right])$$
for any infinite field $k$. Here, analogously to the case of $H_2$, the homomorphism 
$H_3(\text{SL}_2(k[t, t^{-1}]), \mathbb{Z}\left[\frac{1}{2}\right]) \to \mathcal{R}\mathcal{P}_1(k) \otimes \mathbb{Z}\left[\frac{1}{2}\right]$ is the composite

$$
H_3(\text{SL}_2(k[t, t^{-1}]), \mathbb{Z}\left[\frac{1}{2}\right]) \longrightarrow H_3(\text{SL}_2(k(t)), \mathbb{Z}\left[\frac{1}{2}\right]) \longrightarrow \mathcal{R}\mathcal{P}_1(k(t)) \otimes \mathbb{Z}\left[\frac{1}{2}\right] \delta_i \to \mathcal{R}\mathcal{P}_1(k) \otimes \mathbb{Z}\left[\frac{1}{2}\right].
$$

Again, in section 10 we describe some explicit computations which follow from this result. In particular, in the case $k = \mathbb{Q}$ there is a short exact sequence (Corollary 10.5)

$$0 \to \left(\mathcal{R}\mathcal{B}_0(\mathbb{Q})^\otimes\right) \otimes \mathbb{Z}\left[\frac{1}{2}\right] \to H_3(\text{SL}_2(\mathbb{Q}[t, t^{-1}]), \mathbb{Z}\left[\frac{1}{2}\right]) \to \mathbb{Z}/3 \oplus \mathbb{Z}/3 \oplus V \to 0.$$

where $\mathcal{R}\mathcal{B}_0(\mathbb{Q})$ is an infinite torsion abelian group and $V$ is a free $\mathbb{Z}\left[\frac{1}{2}\right]$-module of countably infinite rank.

**Remark 1.1.** We should, perhaps, say a little more to justify the assertion that Theorem 8.1 is ‘an unstable analogue of the fundamental theorem of $K$-theory’. The functor $\mathcal{R}\mathcal{P}_1(k)$ is no longer a $K$-theory functor. Indeed, the proof of Theorem 10.7 below shows that the natural map $\mathcal{R}\mathcal{P}_1(k) \to K_2^M(k)$ is the zero map.

Nevertheless, the analogies with $K_2$-theory are suggestive. As noted above, the functor $\mathcal{R}\mathcal{P}_1(K)$ admits residue homomorphisms associated to a valuation. Furthermore, there is some evidence that these homomorphisms fit into localization sequences: In [4], the author has shown that for certain families of discretely valued fields $K$ with residue field $k$ and valuation ring $\mathcal{O}$ there is a natural exact localization sequence

$$0 \to H_3(\text{SL}_2(\mathcal{O}), \mathbb{Z}\left[\frac{1}{2}\right]) \to H_3(\text{SL}_2(K), \mathbb{Z}\left[\frac{1}{2}\right]) \to \mathcal{R}\mathcal{P}_1(k) \otimes \mathbb{Z}\left[\frac{1}{2}\right] \to 0.$$

**Remark 1.2.** A key ingredient in our calculations is Knudson’s homotopy invariance theorem, cf.[9] Theorem 3.4]: If $k$ is an infinite field, then the inclusion $\text{SL}_2(k) \to \text{SL}_2(k[t])$ induces an isomorphism on integral homology

$$H_n(\text{SL}_2(k), \mathbb{Z}) \cong H_n(\text{SL}_2(k[t]), \mathbb{Z}).$$

However, the example of Krstić and McCool, cf. [12], shows that even for $H_1$ this result does not extend to more general ground rings $k$. In particular, the corresponding statement for the ring of polynomials in two or more variables is not true.

**Remark 1.3.** The restriction to the case of infinite fields in this article is due the same restriction in Knudson’s results on homotopy invariance for $\text{SL}_2$ and his related Theorem 3.3 below.

However, in [11] Knudson also proves homotopy invariance for the homology of $\text{GL}_2$ over a finite field $k$. It seems likely that the results proved in that article can be extended to the case of $\text{SL}_2$, at least when the field $k$ is sufficiently large. If so, the results below should also extend to finite fields with sufficiently many elements.

**Remark 1.4.** It is natural to ask about the possibility of the extension of the main results of this article to higher-dimensional homology of $\text{SL}_2$. Recent work of M. Wendt suggests that this should be the case, although there is much work yet to be done.

The ring of Laurent polynomials $k[t, t^{-1}]$ is, of course, just the coordinate ring of the curve $\mathbb{P}^1(k) \setminus \{0, \infty\}$. In a recent preprint ([26]), Wendt studies the homology of $\text{SL}_2(k[C])$ where $C$ is a curve obtained from a smooth projective curve $\bar{C}$ over the algebraically closed field $k$ by removing a finite number of points. His calculations, when specialized to the case $C = \mathbb{P}^1(k) \setminus \{0, \infty\}$ suggest that the functors $\mathcal{R}\mathcal{P}_n(k)$ introduced in his paper should be naturally isomorphic to the quotients

$$\frac{H_n(\text{SL}_2(k[t, t^{-1}]), \mathbb{Z})}{H_n(\text{SL}_2(k), \mathbb{Z})},$$
I augmentation ideal, notation from the theory of quadratic forms, the element \( \langle \) of square classes of units of \( A \) follows that each of the groups \( H_n \) determines a determinant of a scalar matrix, the action of a morphism of groups and for the induced map on homology of these groups.

The calculations in section \([10]\) below depend very much on the simple presentation and algebraic properties of the refined scissors congruence groups \( \mathcal{R} \mathcal{P}_1(k) \). It is not yet known whether the groups \( \mathcal{R} \mathcal{P}_3(k) \) are equally tractable.

Remark 1.5. In Section \([11]\) below, we consider the analogue for \( GL_2 \) of the main results of this article. It is a straightforward matter to deduce the structure of the low-dimensional homology of \( GL_2(k[t, t^{-1}]) \) – at least over \( \mathbb{Z}[\frac{1}{2}] \) – from that of \( SL_2(k[t, t^{-1}]) \), via the Hochschild-Serre spectral sequence relating the two.

We carry out these computations in this final section of the article, proving that, for an infinite field \( k \) of characteristic not 2 and for \( r \leq 3 \), there are isomorphisms

\[
H_4(GL_2(k[t, t^{-1}]), \mathbb{Z}[\frac{1}{2}]) \cong H_4(GL_2(k), \mathbb{Z}[\frac{1}{2}]) \oplus H_{-1}(GL_2(k), \mathbb{Z}[\frac{1}{2}]).
\]

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2. Preliminaries and notation

For a commutative ring \( A \), the short exact sequence

\[
1 \to SL_2(A) \to GL_2(A) \to A^\times \to 1
\]

gives rise to a natural action of \( A^\times \) on \( H_*(SL_2(A), \mathbb{Z}) \): given \( a \in A^\times \), choose \( M = M(a) \in GL_2(A) \) with determinant \( a \). Conjugation by \( M \) is an automorphism of \( SL_2(A) \) whose induced action on \( H_*(SL_2(A), \mathbb{Z}) \) depends only on \( a \) and not on the choice of \( M \). Furthermore, since \( a^2 \) is the determinant of a scalar matrix, the action of \( a^2 \) on \( H_*(SL_2(A), \mathbb{Z}) \) is trivial for any \( a \in A^\times \). It follows that each of the groups \( H_n(SL_2(A), \mathbb{Z}) \) is naturally a module over the group ring

\[
R_A := \mathbb{Z}[A^\times/(A^\times)^2]
\]

of square classes of units of \( A \).

For \( a \in A^\times \), the square class of \( a \) will be denoted \( \langle a \rangle \in R_A \). Furthermore, still borrowing our notation from the theory of quadratic forms, the element \( \langle a \rangle - 1 \in R_A \) will be denoted \( \langle a \rangle \). The augmentation ideal, \( I_A \), in \( R_A \) is the ideal generated by all of the elements \( \langle a \rangle \).

For an abelian group \( M \) we will let \( M[\frac{1}{2}] \) denote the \( \mathbb{Z}[\frac{1}{2}] \)-module \( M \otimes \mathbb{Z}[\frac{1}{2}] \).

Where no confusion is likely to arise, we will tend to use the same symbol both for a homomorphism of groups and for the induced map on homology of these groups.

3. The Mayer-Vietoris sequence

Let \( k \) be an infinite field and let \( G = SL_2(k[t, t^{-1}]) \). We denote the matrix

\[
\begin{bmatrix}
t & 0 \\
0 & 1
\end{bmatrix} \in GL_2(k[t, t^{-1}]) \subset GL_2(k(t))
\]

by \( A(t) \). We denote by \( C \), the conjugation map \( g \mapsto A(t)^{-1}gA(t) = g^{A(t)} \) on both \( SL_2(k(t)) \) and \( G \).

Since \( \det(A(t)) = t \), the action of \( A(t) \) by conjugation on \( G \) induces on \( H_*(G, \mathbb{Z}) \) multiplication by \( \langle t \rangle \in R_{k[t, t^{-1}]} \).
Let $G_1$ denote the subgroup $\text{SL}_2(k[t])$ of $G$ and let

$$G_2 := C_r(G_1) = \left\{ \begin{array}{ccc} a & r^{-1}b & \vspace{1pt} \\ tc & d & \end{array} : a, b, c, d \in k[t], \ ad - bc = 1 \right\}.$$ 

Let

$$\Gamma = G_1 \cap G_2 = \left\{ \begin{array}{ccc} a & b & \vspace{1pt} \\ tc & d & \end{array} : a, b, c, d \in k[t], \ ad - tbc = 1 \right\}.$$ 

K. Knudson ([8]) observes that since $k[t, t^{-1}]$ is a dense subring of the complete discretely valued field $k((t))$, Serre’s theory of trees, cf. [21] Chapter II, Theorem 3], allows us to deduce:

**Theorem 3.1.** $G = \text{SL}_2(k[t, t^{-1}])$ is the sum of the subgroups $G_1$ and $G_2$ amalgamated along their common intersection $\Gamma$:

$$G = \text{SL}_2(k[t, t^{-1}]) = \text{SL}_2(k[t]) \star_{\Gamma} \text{SL}_2(k[t])^{A(i)} = G_1 \star_{\Gamma} G_2.$$ 

This amalgamated product decomposition gives us immediately a short exact sequence of $\mathbb{Z}[G]$-modules

$$0 \longrightarrow \mathbb{Z}[G/\Gamma] \overset{\alpha}{\longrightarrow} \mathbb{Z}[G/G_1] \oplus \mathbb{Z}[G/G_2] \overset{\beta}{\longrightarrow} \mathbb{Z} \longrightarrow 0.$$ 

where $\alpha$ is the map $g\Gamma \mapsto (gG_1, gG_2)$ and $\beta$ is the unique $\mathbb{Z}[G]$-homomorphism sending $(G_1, 0)$ to $-1$ and $(0, G_2)$ to $1$.

The associated long exact sequence in homology – the Mayer-Vietoris sequence of the amalgamated product – takes the form

$$\cdots \overset{\delta}{\longrightarrow} H_i(\Gamma, \mathbb{Z}) \overset{\alpha}{\longrightarrow} H_i(G_1, \mathbb{Z}) \oplus H_i(G_2, \mathbb{Z}) \overset{\beta}{\longrightarrow} H_i(\text{SL}_2(k[t, t^{-1}]), \mathbb{Z}) \overset{\delta}{\longrightarrow} \cdots$$

Implicit here are the isomorphisms of Shapiro’s Lemma: If $H$ is a subgroup of $K$, then the inclusion $\mathbb{Z} \rightarrow \mathbb{Z}[G/H]$ of $\mathbb{Z}[H]$-modules induces an isomorphism $H_* (H, \mathbb{Z}) \cong H_* (G, \mathbb{Z}[G/H])$.

Known results allow us to simplify further some terms in this sequence. To begin with we have the following homotopy-invariance property of the homology of $\text{SL}_2(k)$ (see [10, Theorem 4.3.1]):

**Theorem 3.2.** Let $k$ be an infinite field. Then the inclusion $k \rightarrow k[t]$ induces an isomorphism on integral homology

$$H_* (\text{SL}_2(k), \mathbb{Z}) \cong H_* (\text{SL}_2(k[t]), \mathbb{Z}).$$

We let $\iota : k^\times \rightarrow \text{SL}_2(k)$ denote the embedding

$$a \mapsto \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix}.$$ 

The image of $\iota$ lies in the subgroup

$$\Gamma \cap \text{SL}_2(k) = B(k) := \left\{ \begin{array}{ccc} a & b & \vspace{1pt} \\ 0 & a^{-1} & \end{array} : a \in k^\times, b \in k \right\}.$$ 

Furthermore, we have ([10, Theorem 4.4.1])

**Theorem 3.3.** When the field $k$ is infinite, $\iota$ induces isomorphisms on integral homology

$$H_* (k^\times, \mathbb{Z}) \cong H_* (B, \mathbb{Z}) \cong H_* (\Gamma, \mathbb{Z}).$$
We let \( j \) denote the natural inclusion \( SL_2(k) \to SL_2(k[t, t^{-1}]) \) (and the map induced on homology by this inclusion).

With all of these identifications, together with the fact that \( C_t \) induces an isomorphism from \( G_1 \) to \( G_2 \), the Mayer-Vietoris sequence takes the form

\[
\cdots \to H_i(k^x, \mathbb{Z}) \xrightarrow{\alpha} H_i(SL_2(k), \mathbb{Z}) \oplus H_i(SL_2(k), \mathbb{Z}) \xrightarrow{\beta} H_i(SL_2(k[t, t^{-1}]), \mathbb{Z}) \xrightarrow{\delta} \cdots
\]

where \( \alpha(z) = (\alpha(z), \alpha(z)) \) and \( \beta(z_1, z_2) = \langle t \rangle j(z_2) - j(z_1) \).

As Knudson points out, the existence of this exact sequence immediately implies that

\[
H_1(SL_2(k[t, t^{-1}]), \mathbb{Z}) = 0
\]

for any infinite field \( k \). This fact is originally due to P. Cohn, [1].

We will use the following observation at many points below:

**Lemma 3.4.** The homomorphism \( j : SL_2(k) \to SL_2(k[t, t^{-1}]) \) induces split injections

\[
j : H_i(SL_2(k), \mathbb{Z}) \to H_i(SL_2(k[t, t^{-1}]), \mathbb{Z})\]

for all \( i \geq 0 \).

**Proof.** The inclusion of rings \( k \to k[t, t^{-1}] \) is split by the homomorphism \( k[t, t^{-1}] \to k \) sending \( t \) to \( 1 \). \( \square \)

### 4. Milnor-Witt \( K \)-theory and the second homology of \( SL_2 \)

In this section all fields are of characteristic different from 2.

The theorem of Matsumoto and Moore gives an explicit presentation of \( H_2(SL_2(k), \mathbb{Z}) \) for an infinite field \( k \). In [23], A. Suslin showed that this implies a natural isomorphism

\[
H_2(SL_2(k), \mathbb{Z}) \cong \hat{I}(k) \times F(k) / F(k) K_2^M(k)
\]

where \( I^n(k) \) denotes the \( n \)-th power of the fundamental ideal of the Witt ring \( W(k) \) of the field \( k \).

We can now recognize the group on the right-hand side of this isomorphism as the second Milnor-Witt \( K \)-group of the field \( k \). The Milnor-Witt \( K \)-theory of a field is the graded ring \( K_{\bullet}^{MW}(F) \) with the following presentation (due to F. Morel and M. Hopkins, see [16]):

Generators: [\( a \)], \( a \in F^\times \), in dimension 1 and a further generator \( \eta \) in dimension \( -1 \).

Relations:

(a) \( [ab] = [a] + [b] + \eta \cdot [a] \cdot [b] \) for all \( a, b \in F^\times \)

(b) \( [a] \cdot [1 - a] = 0 \) for all \( a \in F^\times \setminus \{1\} \)

(c) \( \eta \cdot [a] = [a] \cdot \eta \) for all \( a \in F^\times \)

(d) \( \eta \cdot h = 0 \), where \( h = \eta \cdot [-1] + 2 \in K_0^{MW}(F) \).

Clearly there is a unique surjective homomorphism of graded rings \( K_{\bullet}^{MW}(F) \to K_{\bullet}^M(F) \) sending \([a]\) to \([a]\) and inducing an isomorphism

\[
\frac{K_{\bullet}^{MW}(F)}{\langle \eta \rangle} \cong K_{\bullet}^M(F)
\]

(where \( K_n^M(F) := 0 \) for \( n < 0 \)).

Furthermore, there is a natural surjective homomorphism of graded rings \( K_{\bullet}^{MW}(F) \to I^*(F) \) sending \([a]\) to \( \langle [a] \rangle \) and \( \eta \) to \( \eta \). Morel shows that there is an induced isomorphism of graded rings

\[
\frac{K_{\bullet}^{MW}(F)}{\langle h \rangle} \cong I^*(F)
\]

(where \( I^n(F) := W(F) \) for \( n \leq 0 \) and \( \eta \in I^{-1}(F) = W(F) \) is the element 1).
The main structure theorem on Milnor-Witt $K$-theory is the following theorem of Morel:

**Theorem 4.1** (Morel, [17]). For a field $F$ of characteristic not 2, the commutative square of graded rings

$$
\begin{array}{c}
K^\text{MW}_\ast(F) \\
\downarrow
\end{array} \quad \begin{array}{c}
K^\ast(F)
\end{array}
\begin{array}{c}
I^\ast(F) \\
\downarrow
\end{array} \quad \begin{array}{c}
I^\ast(F)/I^{\ast+1}(F)
\end{array}
$$

is cartesian.

Thus for each $n \in \mathbb{Z}$ we have an isomorphism

(2) \quad $K^\text{MW}_n(F) \cong K^\ast_n(F) \times_{I_n(F)/I_{n+1}(F)} I^n(F)$.

It follows that for all $n$ there is a natural short exact sequence

(3) \quad $0 \to I^{n+1}(F) \to K^\text{MW}_n(F) \to K^\ast_n(F) \to 0$

where the inclusion $I^{n+1}(F) \to K^\text{MW}_n(F)$ is given by

$$\langle a_1, \ldots, a_{n+1} \rangle \mapsto \eta[a_1] \cdots [a_{n+1}].$$

Similarly, for $n \geq 0$, there is a short exact sequence

$$0 \to 2K^\ast_n(F) \to K^\text{MW}_n(F) \to I^n(F) \to 0$$

where the inclusion $2K^\ast_n(F) \to K^\text{MW}_n(F)$ is given (for $n \geq 1$) by

$$2[a_1, \ldots, a_n] \mapsto h[a_1] \cdots [a_n].$$

For $n \geq 2$ we have

$$h[a_1][a_2] \cdots [a_n] = ([a_1][a_2] - [a_2][a_1])[a_3] \cdots [a_n] = [a_1^2][a_2] \cdots [a_n].$$

In particular, by (1) and (2), there is a natural isomorphism

$$H_2(\text{SL}_2(k), \mathbb{Z}) \cong K^\text{MW}_2(k).$$

Indeed, the resulting diagram

$$
\begin{array}{c}
H_2(\text{SL}_2(k), \mathbb{Z}) \cong K^\text{MW}_2(k)
\end{array}
\begin{array}{c}
H_2(\text{SL}_3(k), \mathbb{Z}) \cong K^M_2(k) = K_2(k)
\end{array}
$$

commutes.

Note also that the case $n = 0$ of Theorem 4.1 gives a natural isomorphism of rings

$$K^\text{MW}_0(F) \cong \text{GW}(F), 1 + \eta[a] \leftrightarrow \langle a \rangle$$

where $\text{GW}(F) \cong \mathbb{Z} \times_{\mathbb{Z}/2} W(F)$ is the Grothendieck-Witt ring of the field $F$. In particular, the groups $K^\text{MW}_n(F)$ are $\text{GW}(F)$-module for all $n$. It follows that these groups admit the structure of $R_F$-modules since there is a natural surjection of rings $R_F \to \text{GW}(F)$ (sending $\langle a \rangle$ to $\langle a \rangle$).
5. The second homology of $\text{SL}_2(k[t, t^{-1}])$

In this section $k$ is an infinite field of characteristic not equal to 2.

The goal of this section is to prove:

**Theorem 5.1.** Let $k$ be an infinite field of characteristic not equal to 2. Then

$$H_2(\text{SL}_2(k[t, t^{-1}]), \mathbb{Z}) \cong H_2(\text{SL}_2(k), \mathbb{Z}) \oplus K_1^{\text{MW}}(k).$$

The proof will take up the rest of the section. We will begin by analysing the maps in the appropriate segment of the Mayer-Vietoris long exact sequence, especially the connecting homomorphism, $\delta$. This analysis, combined with known results about Milnor-Witt $K$-theory, allows us essentially to reduce the proof of [5.1] to the calculation of the value of $\delta$ on certain elements of $H_2(\text{SL}_2(k[t, t^{-1}]), \mathbb{Z})$ (Proposition 5.2 below).

The Mayer-Vietoris sequence gives us an exact sequence

$$H_2(k^x, \mathbb{Z}) \longrightarrow H_2(\text{SL}_2(k), \mathbb{Z}) \oplus H_2(\text{SL}_2(k), \mathbb{Z}) \longrightarrow H_2(\text{SL}_2(k[t, t^{-1}]), \mathbb{Z}) \longrightarrow H_1(k^x, \mathbb{Z}) \longrightarrow 0.$$ 

However, since the homomorphism $j : H_1(\text{SL}_2(k), \mathbb{Z}) \rightarrow H_1(\text{SL}_2(k[t, t^{-1}]), \mathbb{Z})$ is a split injection, we obtain an induced exact sequence

$$H_2(k^x, \mathbb{Z}) \xrightarrow{\iota} H_2(\text{SL}_2(k), \mathbb{Z}) \xrightarrow{\beta} H_2(\text{SL}_2(k[t, t^{-1}]), \mathbb{Z}) \xrightarrow{\delta} H_1(k^x, \mathbb{Z}) \longrightarrow 0$$

where $\beta(z) = \langle t \rangle j(z)$.

Recall that there is a natural isomorphism $K_2^{\text{MW}}(k) \cong H_2(\text{SL}_2(k), \mathbb{Z})$. Given $a, b \in k^x$, we will let $[a, b]$ denote the element of $H_2(\text{SL}_2(k), \mathbb{Z})$ corresponding, under this isomorphism, to $[a][b] \in K_2^{\text{MW}}(k)$.

Furthermore, there is a natural isomorphism $k^x \wedge k^x \cong H_2(k^x, \mathbb{Z})$ sending $a \wedge b$ to the homology class represented by $([ab] - [ba]) \otimes 1$ in the bar resolution and, by the calculations of Mazzoleni ([14]), these isomorphisms fit into a commutative diagram

$$k^x \wedge k^x \xrightarrow{\bar{i}} K_2^{\text{MW}}(k)$$

$$\downarrow \cong \downarrow \cong$$

$$H_2(k^x, \mathbb{Z}) \xrightarrow{\iota} H_2(\text{SL}_2(k), \mathbb{Z})$$

where $\bar{i}(a \wedge b) = [a][b] - [b][a] = [a^2][b] = [a][b^2]$.

Furthermore, Mazzoleni has shown that the image of $\bar{i}$ is isomorphic to $2 \cdot K_2^{\text{MW}}(k) \subset K_2^{\text{MW}}(k)$, via

$$\bar{i}(a \wedge b) = [a^2][b] \leftrightarrow [a^2][b] = 2[a][b].$$

It follows that the cokernel of $\bar{i}$ is isomorphic to $I^2(k)$, the isomorphism being induced by the homomorphism

$$K_2^{\text{MW}}(k) \twoheadrightarrow I^2(k), \quad [a][b] \mapsto \langle a \rangle \langle b \rangle.$$

Putting all of this together, we have a natural short exact sequence

$$0 \longrightarrow I^2(k) \longrightarrow H_2(\text{SL}_2(k[t, t^{-1}]), \mathbb{Z}) \xrightarrow{H_2(\text{SL}_2(k), \mathbb{Z})} k^x \longrightarrow 0.$$ 

Morel ([18 Theorem 2.15]) shows that if $F$ is a field with discrete valuation $v$, residue field $k$ and corresponding uniformizer $\pi$, there is a well-defined residue homomorphism (depending on the choice of uniformizer) $\delta_\pi^\sharp = \delta_\pi : K_2^{\text{MW}}(F) \rightarrow K_1^{\text{MW}}(k)$ with the properties:
(1) $\delta_v[\pi][a] = [\bar{a}]$ whenever $v(a) = 0$, and
(2) $\delta_v[a][b] = 0$ if $v(a) = v(b) = 0$.

Now let $\Delta : H_2(\text{SL}_2(k[t, t^{-1}]), \mathbb{Z}) \to K_{1}^{\text{MW}}(k)$ denote the composite

$$H_2(\text{SL}_2(k[t, t^{-1}]), \mathbb{Z}) \xrightarrow{\cong} H_2(\text{SL}_2(k(t)), \mathbb{Z}) \xrightarrow{\delta_t} K_2^{\text{MW}}(k(t)) \xrightarrow{\delta_t} K_1^{\text{MW}}(k).$$

From property (2) of $\delta_\pi$ (with $\pi = t$) it follows that the composite map

$$K_2^{\text{MW}}(k) \xrightarrow{\delta_t} K_2^{\text{MW}}(k(t)) \xrightarrow{\delta_t} K_1^{\text{MW}}(k)$$

is the zero map. Thus $\Delta(j(z)) = 0$ for all $z \in H_2(\text{SL}_2(k), \mathbb{Z})$ and hence $\Delta$ induces a well-defined homomorphism

$$\tilde{\Delta} : \frac{H_2(\text{SL}_2(k[t, t^{-1}]), \mathbb{Z})}{H_2(\text{SL}_2(k), \mathbb{Z})} \to K_1^{\text{MW}}(k).$$

To complete the proof of Theorem 5.1 we show that $\tilde{\Delta}$ is an isomorphism.

From our work above, there is a natural injective map

$$I^2(k) \to \frac{H_2(\text{SL}_2(k[t, t^{-1}]), \mathbb{Z})}{H_2(\text{SL}_2(k), \mathbb{Z})}$$

given by $\langle\langle a \rangle\rangle \langle\langle b \rangle\rangle \mapsto \langle t \rangle j([a, b])$.

On the other hand, there is also a natural short exact sequence

$$0 \to I^2(k) \to K_1^{\text{MW}}(k) \xrightarrow{\pi} k^\times \to 0$$

where the inclusion $I^2(k) \to K_1^{\text{MW}}(k)$ is given by $\langle\langle a \rangle\rangle \langle\langle b \rangle\rangle \mapsto \eta[a][b] = [ab] - [a] - [b]$ by (3) above.

Now, from the definition of $\tilde{\Delta}$ and the properties of $\delta_t$, and recalling that $\langle t \rangle = \eta[t] + 1 \in K_0^{\text{MW}}(k(t))$, we have

$$\tilde{\Delta}(\langle t \rangle j([a, b])) = \delta_t(\langle t \rangle [a][b])$$

$$= \delta_t(\eta[t][a][b] + [a][b])$$

$$= \delta_t([t][a][b]) \quad \text{since } \delta_t([a][b]) = 0$$

$$= \eta[a][b] \quad \text{since } \eta[a][b] = [ab] - [a] - [b] \text{ in } K_2^{\text{MW}}(k(t)).$$

We deduce immediately that there is a natural map of short exact sequences (defining the map $\tilde{\Delta}$)

$$0 \to I^2(k) \to \frac{H_2(\text{SL}_2(k[t, t^{-1}]), \mathbb{Z})}{H_2(\text{SL}_2(k), \mathbb{Z})} \to k^\times \to 0$$

We now complete the argument by showing that $\tilde{\Delta} = \tilde{\Delta}_k$ is the identity map on $k^\times$. 

Homology of $\text{SL}_2$ of Laurent polynomials
We begin by observing that it is enough to show that $\tilde{\Delta}_k(a) = a$ for all $a \in (k^\times)^2$ (and for all fields $k$), since, for any $a \in k^\times$, the diagram

$$
\begin{array}{ccc}
  k^\times & \longrightarrow & k(\sqrt{a})^\times \\
  \downarrow & & \downarrow \\
  k^\times & \longrightarrow & k(\sqrt{a})^\times
\end{array}
$$

commutes, and hence $\tilde{\Delta}_k(a) = \tilde{\Delta}_k(\sqrt{a}) = a \in k^\times \subset k(\sqrt{a})^\times$.

Recall that $\delta$ denotes the connecting homomorphism $H_2(\text{SL}_2(k[t, t^{-1}]), \mathbb{Z}) \rightarrow H_1(k^\times, \mathbb{Z}) \cong k^\times$ of the Mayer-Vietoris sequence.

For a ring $A$, we let $\tau_A$ denote the composite map

$$A^\times \wedge A^\times \longrightarrow H_2(A^\times, \mathbb{Z}) \longrightarrow H_2(\text{SL}_2(A), \mathbb{Z})$$

where $\iota$ denotes the homomorphism

$$A^\times \rightarrow \text{SL}_2(A), a \mapsto \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix}$$

(as well as the map induced on homology by this homomorphism).

**Proposition 5.2.** Let $A = k[t, t^{-1}]$ and let $a \in k^\times$. Then $\delta(\tau_A(t \wedge a)) = a^2$ in $k^\times$.

**Proof.** Let $F_\bullet(G)$ denote the (right) bar resolution of the group $G = \text{SL}_2(k[t, t^{-1}])$. The Mayer-Vietoris sequence is derived from the long exact homology sequence of the exact sequence of complexes

$$0 \rightarrow F_\bullet(G) \otimes_{\mathbb{Z}[G]} \mathbb{Z}[G/\Gamma] \rightarrow F_\bullet(G) \otimes_{\mathbb{Z}[G]} (\mathbb{Z}[G/G_1] \oplus \mathbb{Z}[G/G_2]) \rightarrow F_\bullet(G) \otimes_{\mathbb{Z}[G]} \mathbb{Z} \rightarrow 0.$$

The homology class $\tau_A(t \wedge a) \in H_2(G, \mathbb{Z})$ is represented by the cycle

$$([\iota(t)]([a]) - [\iota(a)]([t])) \otimes 1 \in F_2(G) \otimes_{\mathbb{Z}[G]} \mathbb{Z}.$$

This lifts to

$$([\iota(t)]([a]) - [\iota(a)]([t])) \otimes (1 \cdot G_1, 0) \in F_2(G) \otimes_{\mathbb{Z}[G]} (\mathbb{Z}[G/G_1] \oplus \mathbb{Z}[G/G_2]).$$

The boundary homomorphism sends this element to

$$[\iota(a)] \otimes (\iota(t) \cdot G_1 - 1 \cdot G_1, 0) \in F_1(G) \otimes_{\mathbb{Z}[G]} (\mathbb{Z}[G/G_1] \oplus \mathbb{Z}[G/G_2]).$$

Now let

$$w = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \in G_1 = \text{SL}_2(k[t]).$$

Observe that

$$w \cdot \iota(t) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} t & 0 \\ 0 & t^{-1} \end{bmatrix} = \begin{bmatrix} 0 & -t^{-1} \\ t & 0 \end{bmatrix} = C_t(w) \in G_2$$

and hence $\iota(t) = w^{-1} C_t(w)$ with $w^{-1} \in G_1, C_t(w) \in G_2$.

It follows that $([\iota(t) \cdot G_1 - 1 \cdot G_1, 0]$ is the image of $w^{-1} \cdot (C_t(w) \Gamma - \Gamma) \in \mathbb{Z}[G/\Gamma]$ under the map $
\mathbb{Z}[G/\Gamma] \rightarrow \mathbb{Z}[G/G_1] \oplus \mathbb{Z}[G/G_2]$. 

Thus $\delta(\tau_A(t \wedge a))$ is represented by the cycle

$$[\iota(a)] \otimes w^{-1} \cdot (C_t(w) \Gamma - \Gamma) = ([\iota(a)]([t] - [\iota(a)] w^{-1}) \otimes \Gamma \in F_1(G) \otimes_{\mathbb{Z}[G]} \mathbb{Z}[G/\Gamma].$$
This, in turn, is the image of
\[
\left( (\iota(a))\iota(t) - [\iota(a)]w^{-1} \right) \otimes 1 \in F_1(G) \otimes_{\mathbb{Z}[\Gamma]} \mathbb{Z}
\]
under the natural homology isomorphism \( F_\ast(G) \otimes_{\mathbb{Z}[\Gamma]} \mathbb{Z} \to F_\ast(G) \otimes_{\mathbb{Z}[G]} \mathbb{Z}/G[\Gamma] \).

For a group \( H \), we let \( C_\ast(H) \) denote the (right) homogeneous resolution of \( H \). Then the isomorphism \( F_\ast(H) \to C_\ast(H) \) is obtained by the map of right \( \mathbb{Z}[H] \)-complexes
\[
[h_n] \cdots [h_1] \mapsto (h_n \cdot h_{n-1} \cdots h_1, h_{n-1} \cdots h_1, \ldots, h_1, 1).
\]
Thus \( \left( (\iota(a))\iota(t) - [\iota(a)]w^{-1} \right) \otimes 1 \in F_1(G) \otimes_{\mathbb{Z}[\Gamma]} \mathbb{Z} \) corresponds to
\[
\left( (\iota(at), \iota(t)) - (\iota(a)w^{-1}, w^{-1}) \right) \otimes 1 \in C_1(G) \otimes_{\mathbb{Z}[\Gamma]} \mathbb{Z}.
\]
To construct an augmentation preserving map of \( \mathbb{Z}[\Gamma] \)-resolutions of \( \mathbb{Z} \) from \( C_\ast(G) \) to \( C_\ast(\Gamma) \) we choose a set-theoretic section \( s : G/\Gamma \to G \) and send \( (g_n, \ldots, g_0) \) to \( (\overline{g}_n, \ldots, \overline{g}_0) \), where \( \overline{g} := s(g\Gamma)^{-1}g \in \Gamma \).

To be more specific, we choose a section \( s \) satisfying \( s(\iota(a)w^{-1}G) = w^{-1} \) and \( s(\iota(at)\Gamma) = \iota(t) \) for all \( a \in k^\times \).

Thus the element \( (\iota(at), \iota(t)) - (\iota(a)w^{-1}, w^{-1}) \in C_1(G) \) maps to \( (\iota(a), 1) - (\iota(a^{-1}), 1) \in C_1(\Gamma) \) (since \( \iota(at)w^{-1} = \iota(a^{-1}) \) in \( G \)).

Finally the homology class
\[
\left( (\iota(a), 1) - (\iota(a^{-1}), 1) \right) \otimes 1 \in C_1(\Gamma) \otimes_{\mathbb{Z}[\Gamma]} \mathbb{Z}
\]
corresponds to \( \iota(a) \cdot \iota(a^{-1})^{-1} = \iota(a^2) \) under the isomorphism \( H_1(\Gamma, \mathbb{Z}) \cong \Gamma_{ab} \) and hence to \( a^2 \in k^\times \cong \Gamma_{ab} \).

**Corollary 5.3.** For any field \( k \), we have \( \hat{\Delta}_k(a^2) = a^2 \) for all \( a \in k^\times \).

**Proof.** By Proposition 5.2, \( a^2 \in k^\times \) is the image, under \( \delta \), of the element of \( H_2(\text{SL}_2(k[t, t^{-1}]), \mathbb{Z}) \) represented by \( \tau_A(t \wedge a) \). Therefore, it is enough to verify that \( \Delta(\tau_A(t \wedge a)) = [a^2] \in K^\text{MW}_1(k) \):

The image of \( \tau_A(t \wedge a) \) in \( K^\text{MW}_2(k(t)) \) is \( \bar{\iota}(t \wedge a) = [t][a^2] \).

Hence, from the definitions,
\[
\hat{\Delta}_k(a^2) = \Delta(\tau_A(t \wedge a)) = \delta([t][a^2]) = [a^2]
\]
as required.

**6. Some Examples and Special Cases**

Theorem 5.1 tells us that if \( k \) is an infinite field of characteristic not equal to 2, then
\[
H_2(\text{SL}_2(k[t, t^{-1}]), \mathbb{Z}) \cong H_2(\text{SL}_2(k), \mathbb{Z}) \oplus K^\text{MW}_1(k) \cong K^\text{MW}_2(k) \oplus K^\text{MW}_1(k).
\]

This allows us to calculate these homology groups for certain families of fields for which the Milnor \( K \)-theory and Witt rings are known.

For a global field \( F \), we will let \( \Omega = \Omega(F) \) be the set of real embeddings of \( F \). We denote by \( K^\ast(F)_+ \) the kernel of the (split) surjection \( K^\ast(F) \to \bigoplus_{\iota \in \Omega} \mu_2 \) induced by the Hilbert symbols associated to each of the real embeddings.

**Proposition 6.1.** Let \( F \) be a global field. Then there is a natural split exact sequence
\[
0 \to K^\ast(F)_+ \to K^\text{MW}_2(F) \to \bigoplus_{\iota \in \Omega} \mathbb{Z} \to 0.
\]
Proof. We begin by recalling from quadratic form theory that for all $n \geq 1$, $I^n(\mathbb{R}) \cong \mathbb{Z}$ with generator $\langle -1 \rangle \cdots \langle -1 \rangle$.

For any real embedding $\sigma : F \to \mathbb{R}$, let $T_\sigma$ denote the composite homomorphism

$$K_2^\text{MW}(F) \to K_2^\text{MW}(\mathbb{R}) \to I^2(\mathbb{R}) \cong \mathbb{Z}.$$ 

Thus $T_\sigma([a][b]) = \langle \text{sgn}(\sigma(a)) \rangle \langle \text{sgn}(\sigma(b)) \rangle$ where

$$\text{sgn}(x) = \begin{cases} 
1, & x > 0 \\
-1, & x < 0. 
\end{cases}$$

Let

$$T = \bigoplus_{\sigma \in \Omega} T_\sigma : K_2^\text{MW}(F) \to \bigoplus_{\sigma \in \Omega} I^2(\mathbb{R}),$$

and let $K_2^\text{MW}(F)_+ := \text{Ker}(T)$.

Thus we obtain a commutative diagram with exact rows and columns

$$
\begin{array}{c}
0 \\
\downarrow \\
I^3(F) \\
\downarrow T \\
\bigoplus_{\sigma \in \Omega} I^2(\mathbb{R}) \\
\downarrow T \\
K_2^\text{MW}(F)_+ \\
\downarrow T \\
K_2(F) \\
\downarrow \bigoplus_{\sigma \in \Omega} \mu_2 \\
0 \\
\end{array}
$$

where $I^2(\mathbb{R}) \to \mu_2$ is the unique homomorphism sending $\langle -1 \rangle \langle -1 \rangle$ to $-1$.

Now quadratic form theory (see [13], Chapter VI, Corollary 3.9, for example) tells us that, for global fields $F$, the map $T : I^3(F) \to \bigoplus_{\sigma \in \Omega} I^3(\mathbb{R})$ is an isomorphism. The result follows immediately. \hfill $\square$

Since $K_2(\mathbb{Q})_+ \cong \bigoplus_{p \text{ odd}} \mathbb{F}_p^\times$ we immediately deduce:

**Corollary 6.2.** $H_2(\text{SL}_2(\mathbb{Q}), \mathbb{Z}) \cong K_2^\text{MW}(\mathbb{Q}) \cong \mathbb{Z} \oplus \left( \bigoplus_{p \text{ odd}} \mathbb{F}_p^\times \right)$

The situation for $K_1^\text{MW}(F)$ is slightly more complicated:

Again, we set

$$K_1^\text{MW}(F)_+ = \text{Ker}(K_1^\text{MW}(F) \to \bigoplus_{\sigma \in \Omega} I(\mathbb{R}))$$

and

$$K_1(F)_+ = F_+ = \text{Ker}(F^\times \to \bigoplus_{\sigma \in \Omega} \mu_2),$$

the latter map being induced by the sgn homomorphisms.

Let $k_2(F) := K_2(F) \otimes \mathbb{Z}/2$ and let $k_2(F)_+ := \text{Ker}(k_2(F) \to \bigoplus_{\sigma \in \Omega} \mu_2)$.

**Proposition 6.3.** Let $F$ be a global field. There is a split short exact sequence

$$0 \to K_1^\text{MW}(F)_+ \to K_1^\text{MW}(F) \to \bigoplus_{\sigma \in \Omega} I(\mathbb{R}) \to 0.$$ 

Furthermore, there is a short exact sequence

$$0 \to k_2(F)_+ \to K_1^\text{MW}(F)_+ \to F_+ \to 0.$$
which is split if $\Omega \neq 0$.

In particular, if $F$ is a number field admitting a real embedding, as an additive group,

$$K_1^{\text{MW}}(F) \cong \bigoplus_{\sigma \in \Omega} \mathbb{Z} \oplus k_2(F)_+ \oplus F_+^\times.$$

**Proof.** Let $\mathcal{T} := \text{Ker}(I^2(F) \to \bigoplus_{\sigma \in \Omega} I^2(\mathbb{R}))$. Then there is a commutative diagram with exact rows and columns

\[
\begin{array}{cccc}
0 & 0 & & \\
\downarrow & & & \\
\hat{I}^3(F) & \xrightarrow{\mathcal{T}} & \bigoplus_{\sigma \in \Omega} I^3(\mathbb{R}) & \\
\downarrow & & & \\
0 & \mathcal{T} & \hat{I}^2(F) & \bigoplus_{\sigma \in \Omega} I^2(\mathbb{R}) & 0 \\
\downarrow & & & & \\
0 & k_2(F)_+ & k_2(F) & \bigoplus_{\sigma \in \Omega} \mu_2 & 0 \\
\downarrow & & & & \\
0 & & 0 & & \\
\end{array}
\]

It follows that $\mathcal{T} \cong k_2(F)_+$.

Therefore, we obtain a commutative diagram with exact rows and columns

\[
\begin{array}{cccc}
0 & 0 & 0 & \\
\downarrow & & & \\
0 & k_2(F)_+ & \hat{I}^2(F) & \bigoplus_{\sigma \in \Omega} I^2(\mathbb{R}) & \\
\downarrow & & & & \\
0 & K_1^{\text{MW}}(F)_+ & K_1^{\text{MW}}(F) & \bigoplus_{\sigma \in \Omega} I(\mathbb{R}) & 0 \\
\downarrow & & & & \\
0 & F_+^\times & F_+^\times & \bigoplus_{\sigma \in \Omega} \mu_2 & 0 \\
\downarrow & & & & \\
0 & & 0 & & \\
\end{array}
\]

Finally, if $\Omega \neq 0$, then $F_+^\times$ is a free abelian group and hence the sequence

$$0 \to k_2(F)_+ \to K_1^{\text{MW}}(F)_+ \to F_+^\times \to 0$$

is split. \qed

Since $\mathbb{Q}_+^\times \cong \bigoplus_p \mathbb{Z}$ and $k_2(\mathbb{Q})_+ \cong \bigoplus_{p \text{ odd}} \mathbb{Z}/2$ we deduce:

**Corollary 6.4.** $K_1^{\text{MW}}(\mathbb{Q}) \cong \mathbb{Z} \oplus \left( \bigoplus_p \mathbb{Z} \right) \oplus \left( \bigoplus_{p \text{ odd}} \mathbb{Z}/2 \right)$.

Putting this together with the result of section 5 gives:
Theorem 6.5.
\[ H_2(\text{SL}_2(\mathbb{Q}[t, t^{-1}]), \mathbb{Z}) \cong (\mathbb{Z} \oplus \mathbb{Z}) \oplus \left( \bigoplus_{\mathbb{P}} \mathbb{Z} \right) \oplus \left( \bigoplus_{\mathbb{P} \text{ odd}} (\mathbb{F}_p^\times \oplus \mathbb{Z}/2) \right). \]

The results of section 5 also tell us about the stabilization homomorphism from \( H_2(\text{SL}_2(k[t, t^{-1}]), \mathbb{Z}) \) to \( H_2(\text{SL}(k[t, t^{-1}]), \mathbb{Z}) \):

Proposition 6.6. Let \( k \) be an infinite field of characteristic not equal to 2. Then the natural stabilization homomorphism
\[ H_2(\text{SL}_2(k[t, t^{-1}]), \mathbb{Z}) \to H_2(\text{SL}(k[t, t^{-1}]), \mathbb{Z}) = K_2(k[t, t^{-1}]) \]
is surjective with kernel isomorphic to \( I_3^3(k) \oplus I_2^2(k) \).

Proof. By the Fundamental Theorem of Algebraic K-theory (see [25, V.6]) there is a natural split exact sequence
\[ 0 \to K_2(k) \to K_2(k[t, t^{-1}]) \to K_1(k) \to 0. \]

Let \( \mathcal{K} \) denote the kernel of the map
\[ H_2(\text{SL}_2(k[t, t^{-1}]), \mathbb{Z}) \to H_2(\text{SL}(k[t, t^{-1}]), \mathbb{Z}). \]

Thus we get a natural commutative diagram:

\[
\begin{array}{c c c c}
0 & 0 & 0 & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
0 & \to & I^3(k) & \to \mathcal{K} & \to I^2(k) & \to 0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
0 & \to & K_2^\text{MW}(k) & \to H_2(\text{SL}_2(k[t, t^{-1}]), \mathbb{Z}) & \to K_1^\text{MW}(k) & \to 0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
0 & \to & K_2(k) & \to H_2(\text{SL}(k[t, t^{-1}]), \mathbb{Z}) & \to K_1(k) & \to 0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
0 & 0 & 0 & 0 \\
\end{array}
\]

The second arrows in each of the bottom two rows are split (compatibly) by the map \( k[t, t^{-1}] \to k \) sending \( t \) to 1. Thus there is an induced splitting of the top row. \( \Box \)

Corollary 6.7. (1) Let \( k \) be a quadratically closed field of characteristic not equal to 2. Then the natural stabilization homomorphism
\[ H_2(\text{SL}_2(k[t, t^{-1}]), \mathbb{Z}) \to H_2(\text{SL}(k[t, t^{-1}]), \mathbb{Z}) \]
is an isomorphism.
(2) Let \( k \) be an infinite field of characteristic not equal to 2 such that \( I_3^3(k) = 0 \). Then there is a natural short exact sequence
\[ 0 \to k_2(k) \to H_2(\text{SL}_2(k[t, t^{-1}]), \mathbb{Z}) \to H_2(\text{SL}(k[t, t^{-1}]), \mathbb{Z}) \to 0. \]

Proof. (1) When \( k \) is quadratically closed \( I^3(k) = 0 \) for all \( n \geq 1 \).
(2) In this case

\[ I^2(k) = I^3(k) / I^3(k) \cong k_2(k) \]

by a result of Milnor ([15, Theorem 4.1]).

\[ \square \]

Remark 6.8. The condition \( I^3(k) = 0 \) is satisfied, for example, by global fields of positive characteristic and by totally imaginary number fields \( k \) (and for these fields \( k_2(k) = K_2(k) \otimes \mathbb{Z}/2\mathbb{Z} \) is infinite). More generally, this condition is satisfied by fields of cohomological dimension 2 or less.

7. Scissors Congruence Groups and the third homology of \( \text{SL}_2 \)

7.1. The classical scissors congruence group. Let \( k \) be a field with at least four elements. The scissors congruence group, \( P(k) \), of \( k \) is the \( \mathbb{Z} \)-module with generators \([a], a \in k^\times \), subject to the relations \([1] = 0 \) and

\[ [x] - [y] + \left[ \frac{y}{x} \right] - \left[ \frac{1 - x^{-1}}{1 - y^{-1}} \right] + \left[ \frac{1 - x}{1 - y} \right], \quad x, y \neq 1. \]

Remark 7.1. When \( k = \mathbb{C} \), the minus-eigenspace of \( P(\mathbb{C}) \) for the action of complex conjugation can be naturally identified with the scissors congruence group of polyhedra in 3-dimensional hyperbolic space ([3]). This is the origin of the name. Furthermore, the plus-eigenspace is closely related to the three-dimensional spherical scissors congruence group ([2, Theorem 5.15]).

For finite fields we have the following calculation ([6, Lemma 7.4]):

**Theorem 7.2.** Let \( n' \) denote the odd part of the integer \( n \). Let \( \overline{\mathbb{F}}_q \) denote the finite field with \( q \) elements. Then

\[ P(\overline{\mathbb{F}}_q) \left[ \frac{1}{2} \right] \cong \mathbb{Z}/(q + 1)'. \]

We let

\[ S_2^k (k^\times) := \frac{k^\times \otimes_\mathbb{Z} k^\times}{\langle x \otimes y + y \otimes x | x, y \in k^\times \rangle}, \]

the second (graded) symmetric power. We let \( x \circ y \) denote the image of \( x \otimes y \) in \( S_2^k (k^\times) \).

The Bloch group, \( B(k) \), of the field \( k \) is the kernel of the map

\[ \lambda : P(k) \rightarrow S_2^k (k^\times), \quad [a] \mapsto a \circ (1 - a). \]

By a result of Suslin ([24]), the Bloch group of a field is very closely related to the indecomposable \( K_3 \) of the field:

**Theorem 7.3.** For any field \( k \) there is a natural short exact sequence

\[ 0 \rightarrow \text{tor}(\mu_k, \mu_k) \rightarrow K_3^{\text{ind}}(k) \rightarrow B(k) \rightarrow 0. \]

Here \( \text{tor}(\mu_k, \mu_k) = \text{tor}(\mu_k, \mu_k) \) when \( \text{char}(k) = 2 \) and otherwise is the nontrivial extension of \( \text{tor}(\mu_k, \mu_k) \) by \( \mathbb{Z}/2 \).
### 7.2. The refined scissors congruence group

The refined scissors congruence group, $\mathcal{RP}(k)$, of $k$ is the $R_k$-module with generators $[a]$, $a \in k^\times$, subject to the relations $[1] = 0$ and

$$[x] - [y] + \langle x \rangle \left\{ \frac{y}{x} \right\} - \langle x^{-1} - 1 \rangle \left\{ \frac{1 - x^{-1}}{1 - y^{-1}} \right\} + \langle 1 - x \rangle \left\{ \frac{1 - x}{1 - y} \right\} , \quad x, y \neq 1.$$

We endow $S_2^2(k^\times)$ with the trivial $R_k$-module structure.

As in [6], we let $\lambda_1 : \mathcal{RP}(k) \to R_k$ be the $R_k$-module homomorphism sending $[a]$ to $\langle a \rangle \langle 1 - a \rangle$ and let $\lambda_2 : \mathcal{RP}(k) \to S_2^2(k^\times)$ be the $R_k$-homomorphism sending $[a]$ to $a \circ (1 - a)$.

Furthermore, let $\Lambda := (\lambda_1, \lambda_2) : \mathcal{RP}(k) \to R_k \oplus S_2^2(k^\times)$.

The refined Bloch group of $k$, $\mathcal{RB}(k)$, is the kernel of $\Lambda$.

It has the same relation to $H_3(\text{SL}_2(k), \mathbb{Z})$ as $\mathcal{B}(k)$ does to $K_3^{\text{ind}}(k)$:

**Theorem 7.4.** ([6, Theorem 4.3]) Let $k$ be a field with at least 29 elements. There is a natural complex

$$0 \to \text{tor}(\mu_k, \mu_k) \to H_3(\text{SL}_2(k), \mathbb{Z}) \to \mathcal{RB}(k) \to 0$$

which is exact except possibly at the middle term where the homology is annihilated by 4.

The natural map $\mathcal{RP}(k) \to \mathcal{P}(k)$ induces a homomorphism $\mathcal{RB}(k) \to \mathcal{B}(k)$. We denote the kernel of this map by $\mathcal{RB}_0(k)$.

Here we collect some of the relevant facts about $\mathcal{RB}_0(k)$:

**Theorem 7.5.** Let $k$ be a field with at least 4 elements.

1. The map $\mathcal{RB}(k) \to \mathcal{B}(k)$ is surjective. Hence there is a natural short exact sequence of $R_k$-modules

   $$0 \to \mathcal{RB}_0(k) \to \mathcal{RB}(k) \to \mathcal{B}(k) \to 0.$$

   Furthermore, if $k$ has finitely many square classes, after tensoring with $\mathbb{Z} \left[ \frac{1}{2} \right]$ this sequence is split.

2. If $k$ has at least 29 elements there is a natural short exact sequence of $R_k$-modules

   $$0 \to \mathcal{RB}_0(k) \left[ \frac{1}{2} \right] \to H_3(\text{SL}_2(k), \mathbb{Z} \left[ \frac{1}{2} \right]) \to K_3^{\text{ind}}(k) \left[ \frac{1}{2} \right] \to 0.$$

   Furthermore, if $k$ has finitely many square classes, this sequence is split.

3. For any infinite field $k$ there is a natural exact sequence

   $$0 \to \mathcal{RB}_0(k) \left[ \frac{1}{2} \right] \to H_3(\text{SL}_2(k), \mathbb{Z} \left[ \frac{1}{2} \right]) \to H_3(\text{SL}_n(k), \mathbb{Z} \left[ \frac{1}{2} \right]) \to K_3^M(k) \left[ \frac{1}{2} \right] \to 0$$

   for any $n \geq 3$.

4. If $k$ is finite or real-closed or quadratically closed, then $\mathcal{RB}_0(k) = 0$.

5. Suppose that $k$ is a local field with finite residue field $\bar{k}$ of odd order. If $\mathbb{Q}_3 \subset k$ we suppose that $[k : \mathbb{Q}_3]$ is odd. Then there is a natural isomorphism

   $$\mathcal{RB}_0(k) \left[ \frac{1}{2} \right] \cong \mathcal{P}(\bar{k}) \left[ \frac{1}{2} \right].$$

6. Suppose that $k$ is the field of fractions of a unique factorization domain $A$. Let $\mathcal{P}$ be a set of representatives of the association classes of prime elements of $A$. Then there is a natural surjective homomorphism

   $$\mathcal{RB}_0(k) \left[ \frac{1}{2} \right] \twoheadrightarrow \bigoplus_{p \in \mathcal{P}} \mathcal{P}(\bar{k}_p) \left[ \frac{1}{2} \right].$$

**Proof.**

1. This is [6, Corollary 5.1].
(2) This is [6, Lemma 5.2].

The statement about the splitting of the sequences follows from the fact that \( R\mathcal{B}_0(k) \left[ \frac{1}{2} \right] = I_k \mathcal{R}\mathcal{B}(k) \left[ \frac{1}{2} \right] \), together with the fact that if \( k^\times/(k^\times)^2 \) is finite and if \( M \) is any \( R_k \left[ \frac{1}{2} \right] \)-module, then the sequence

\[
0 \to I_kM \to M \to M/I_kM \to 0
\]

is naturally split.

(3) This follows from [7, Theorem 4.7] and [6, Lemma 5.2].

(4) For quadratically closed fields the result is immediate from the definition, since \( R\mathcal{B}(k) = \mathcal{B}(k) \) in this case.

For real closed fields this is the result of Parry and Sah [19].

For finite fields, this is [6, Lemma 7.1].

(5) This is [5, Theorem 6.19].

(6) This is [5, Theorem 5.1].

\[\square\]

### 7.3. The module \( R\mathcal{P}_1(k) \)

The \( R_k \)-module

\[ R\mathcal{P}_1(k) := \text{Ker}(\lambda_1 : R\mathcal{P}(k) \to R_k) \]

will play a key role in our calculations below.

Note that \( \mathcal{R}\mathcal{B}(k) \) is the kernel of \( \lambda_2 : R\mathcal{P}_1(k) \to S_2^2(k^\times) \).

**Lemma 7.6.** There is a natural short exact sequence of \( R_k \)-modules

\[
0 \to \mathcal{R}\mathcal{B}_0(k) \left[ \frac{1}{2} \right] \to R\mathcal{P}_1(k) \left[ \frac{1}{2} \right] \to P(k) \left[ \frac{1}{2} \right] \to 0
\]

which is split if \( k \) has finitely many square classes.

**Proof.** For \( x \in k^\times \) we let \( \psi_1(x) \) denote the element \( [x] +\langle -1\rangle [x^{-1}] \in R\mathcal{P}(k) \).

An elementary calculation (see, for example, [5, Lemma 3.3]) shows that for any \( 1 \neq x \in k^\times \)

\[
r(x) := ((-1) + 1) [x] + \langle 1 - x \rangle \psi_1(x) \in \text{Ker}(\lambda_1) = R\mathcal{P}_1(k).
\]

But \( \lambda_2(r(x)) = 2\lambda([x]) \).

Let

\[ R(k) := \text{Im}(\lambda : P(k) \to S_2^2(k^\times)) \text{ and } R_1(k) := \text{Im}(\lambda_2 : R\mathcal{P}_1(k) \to S_2^2(k^\times)). \]

Then \( R_1(k) \subset R(k) \) and the quotient, \( C(k) \), is annihilated by 2.

The statement now follows by applying the snake lemma to the map of short exact sequences

\[
\begin{array}{cccccc}
0 & \longrightarrow & \mathcal{R}\mathcal{B}(k) & \longrightarrow & R\mathcal{P}_1(k) & \longrightarrow & R_1(k) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{B}(k) & \longrightarrow & P(k) & \longrightarrow & R(k) & \longrightarrow & 0
\end{array}
\]

and tensoring with \( \mathbb{Z} \left[ \frac{1}{2} \right] \). \[\square\]

We let \( K_1^{(1)} \) denote the \( R_k \)-submodule of \( R\mathcal{P}(k) \) generated by the elements \( \psi_1(x), x \in k^\times \).

Let \( \overline{R\mathcal{P}}(k) \) denote the \( R_k \)-module \( R\mathcal{P}(k)/K_1^{(1)} \). Let \( \overline{R\mathcal{P}_1}(k) \) denote the image of the map \( R\mathcal{P}_1(k) \to \overline{R\mathcal{P}}(k) \).

In [5, Lemma 3.3] it is shown that \( R\mathcal{P}_1(k) \cap K_1^{(1)} \) is precisely the torsion subgroup of \( K_1^{(1)} \) and is annihilated by 4. We deduce:
Lemma 7.7. The natural map $\mathcal{RP}_1(k) \to \mathcal{RP}_1(k)$ induces an isomorphism

$$\mathcal{RP}_1(k)\left[\frac{1}{2}\right] \cong \mathcal{RP}_1(k)\left[\frac{1}{2}\right].$$

7.4. Specialization homomorphisms. Suppose given a field $F$ with valuation $\nu : F^\times \to \Gamma$. Let $O_\nu$ be the corresponding valuation ring, let $U_\nu$ the group of units of $O_\nu$ and let $k$ be the residue field. Given $a \in U_\nu$, we denote by $\bar{a}$ its image in $k$.

Given an $R_\nu$-module $M$, we let $M_\nu$ denote the $R_\nu$-module

$$R_\nu \otimes_{\mathbb{Z}[U_\nu/U_\nu^2]} M.$$

The following is [5 Theorem 4.9]:

Theorem 7.8. There is a natural $R_\nu$-module homomorphism $S_\nu : \mathcal{RP}(F) \to \mathcal{RP}(k)_F$ which sends $[a]$ to $1 \otimes [\bar{a}]$ when $a \in U_\nu$.

It can be easily verified that $S_\nu$ restricts to a map $S_\nu : \mathcal{RP}_1(F) \to \mathcal{RP}_1(k)_F$ (see [4 Section 5.2]).

When $\nu$ is a discrete value and $\pi$ is a uniformizer, then there is a $\mathbb{Z}[U_\nu/U_\nu^2]$-module decomposition $R_\nu \cong \mathbb{Z}[U_\nu/U_\nu^2] \oplus \langle \pi \rangle \cdot \mathbb{Z}[U_\nu/U_\nu^2]$ where $\langle \pi \rangle \in F^\times/(F^\times)^2$ is the square class of $\pi$. Thus for any $R_\nu$-module $M$, there is a $\mathbb{Z}[U_\nu/U_\nu^2]$-decomposition (depending on the choice of $\pi$)

$$M_\nu \cong M \oplus \langle \pi \rangle \cdot M.$$

We let $\rho_\pi : M_\nu \to M$ denote the resulting $\mathbb{Z}[U_\nu/U_\nu^2]$-map

$$M_\nu \to \langle \pi \rangle \cdot M \cong M$$

arising from projection on the second factor.

When $\nu$ is a discrete value and $\pi$ is a uniformizer, we let $\delta_\pi$ denote the composite

$$\mathcal{RP}_1(F) \xrightarrow{S_\nu} \mathcal{RP}_1(k)_F \xrightarrow{\rho_\pi} \mathcal{RP}_1(k).$$

8. The third homology of $\text{SL}_2(k[t, t^{-1}])$

In this section we prove the following result:

Theorem 8.1. Let $k$ be an infinite field. Then there is a natural isomorphism

$$H_3(\text{SL}_2(k[t, t^{-1}]), \mathbb{Z}\left[\frac{1}{2}\right]) \cong H_3(\text{SL}_2(k), \mathbb{Z}\left[\frac{1}{2}\right]) \oplus \mathcal{RP}_1(k)\left[\frac{1}{2}\right].$$

8.1. Overview. The proof of Theorem 8.1 resembles that of Theorem 5.1. We again analyse the terms and maps in the Mayer-Vietoris sequences. Moving from second to third homology, and working now over the base ring $\mathbb{Z}\left[\frac{1}{2}\right]$, the role of $K_1^{\text{MW}}(F)$ is now taken by $\mathcal{RP}_1(F)$, that of $P^2(F)$ by $\mathcal{RB}(F)$ and $F^\times$ is replaced by $P(F)$, when $F = k$ or $k(t)$.

The key technical step in the proof is Lemma 8.5 below. This plays the same role for the third homology as Proposition 5.2 above plays for the second homology. The proof essentially requires us to compute the connecting homomorphism in the Mayer-Vietoris sequence and to relate it to the natural homomorphism $H_3(\text{SL}_2(k(t)), \mathbb{Z}) \to \mathcal{RB}(k(t))$. This latter homomorphism arises naturally as an edge homomorphism for a spectral sequence computing the homology of $\text{SL}_2(F)$ of a field $F$ from its action on ordered tuples of distinct points of $\mathbb{P}^1(F)$. This is the spectral sequence of a double complex. It is convenient, therefore, to use the corresponding total complex, $T_\ast(\text{SL}_2(F), \mathbb{Z})$, for the calculation of the connecting homomorphism. We begin by describing this complex in the next section.
8.2. The complexes $T_*(G, M)$. Let $G$ be a group and $L_*$ a nonnegative acyclic complex of $\mathbb{Z}[G]$-modules augmented over $\mathbb{Z}$ via a map $\epsilon : L_0 \to \mathbb{Z}$; i.e. $\epsilon$ induces a weak equivalence of complexes of $\mathbb{Z}[G]$-modules $L_* \simeq \mathbb{Z}[0]$.

If we suppose further that each of the $\mathbb{Z}$-modules $L_n$ is free then for any abelian group $M$, $\epsilon \otimes \text{Id}_M$ induces a weak equivalence of complexes

$$L_* \otimes_{\mathbb{Z}} M \simeq \mathbb{Z}[0] \otimes_{\mathbb{Z}} M = M[0].$$

If $M$ is furthermore a $\mathbb{Z}[G]$-module, then this is a weak equivalence of complexes of $\mathbb{Z}[G]$-modules if $G$ acts diagonally on $L_* \otimes_{\mathbb{Z}} M$. It follows that there is an induced isomorphism of homology groups

$$H_n(G, L_* \otimes_{\mathbb{Z}} M) \simeq H_n(G, M).$$

Here the left-hand term is the hyperhomology of $G$ with coefficients in the complex $L_* \otimes_{\mathbb{Z}} M$. This, by definition, is the homology of the total complex $T_*(G, M)$ associated to the double complex $D_{p,q}(G, M) = F_p(G) \otimes_{\mathbb{Z}[G]} (L_q \otimes_{\mathbb{Z}} M)$.

Associated to the double complex $D_{p,q}(G, M)$ is a spectral sequence of the form

$$E^1_{p,q}(G, M) = H_p(G, L_q \otimes M) \Rightarrow H_{p+q}(G, M).$$

We make the following observations about this construction:

1. Functoriality: Given a map of pairs $(H, M') \to (G, M)$, and endowing $L_*$ with the structure of a $\mathbb{Z}[H]$-complex via the map $H \to G$, we obtain a natural map of complexes $D_{p,q}(H, M') \to D_{p,q}(G, M)$ and $T_*(H, M') \to T_*(G, M)$.

2. The functor $M \mapsto L_* \otimes M$ from $\mathbb{Z}[G]$-modules to $\mathbb{Z}[G]$-complexes is exact since the $L_q$ are $\mathbb{Z}$-free. It follows that the functors $M \mapsto \{D_{p,q}(G, M)\}_{p,q}$ and $M \mapsto T_*(G, M)$ are also exact, since the $F_p(G)$ are free $\mathbb{Z}[G]$-modules.

3. The natural map $D_{**,0}(G, M) \to T_*(G, M)$ is a map of complexes for which the resulting edge homomorphism on homology $H_*(G, L_0 \otimes M) \to H_*(G, M)$ coincides with that induced by $\epsilon \otimes \text{Id}_M$.

4. Given subgroups $H \subset K \subset G$, there are natural composite maps of double complexes

$$D_{p,q}(H, \mathbb{Z}) \to D_{p,q}(H, \mathbb{Z}[G/K]) \to D_{p,q}(G, \mathbb{Z}[G/K]).$$

Here, the first map is induced from the inclusion of $\mathbb{Z}[H]$-modules

$$\mathbb{Z} \to \mathbb{Z}[G/K], 1 \mapsto K.$$

Thus it sends an element of the form $z \otimes \ell \in D_{p,q}(H, \mathbb{Z}) = F_p(H) \otimes L_q$ to $z \otimes (\ell \otimes K) \in D_{p,q}(H, \mathbb{Z}[G/K]) = F_p(H) \otimes (L_q \otimes \mathbb{Z}[G/K])$. For convenience, we will use the following notation: If $w \in D_{p,q}(H, \mathbb{Z})$ we will let $w \otimes K$ denote its image in $D_{p,q}(H, \mathbb{Z}[G/K])$ or $D_{p,q}(G, \mathbb{Z}[G/K])$.

In this article, the relevant example is the case where $G = \text{SL}_2(k)$ and $L_q$ is the free abelian group of $(q + 1)$-tuples of distinct elements of $\mathbb{P}^1(k)$, with the usual action of $\text{SL}_2(k)$ on $\mathbb{P}^1(k)$. The boundary map $d_q : L_{q+1} \to L_q$ is the standard simplicial boundary map.

We will require the following facts about the associated spectral sequence

$$E^1_{p,q} := E^1_{p,q}(\text{SL}_2(k), \mathbb{Z}) \Rightarrow H_{p+q}(\text{SL}_2(k), \mathbb{Z}) :$$

**Lemma 8.2.** The map

$$F_p(k^\times) \otimes_{\mathbb{Z}[k^\times]} \mathbb{Z} \to F_p(\text{SL}_2(k)) \otimes_{\mathbb{Z}[\text{SL}_2(k)]} L_0 = D_{p,0}$$

$$z \otimes 1 \mapsto z \otimes (\infty)$$
induces an isomorphism

\[ H_p(k^\times, \mathbb{Z}) \cong E^1_{p,0} = H_p(SL_2(k), L_0). \]

With this identification, the edge homomorphism

\[ H_p(k^\times, \mathbb{Z}) \cong E^1_{p,0} \rightarrow E'^\infty_{p,0} \rightarrow H_p(SL_2(k), \mathbb{Z}) \]

is identified with the map \( i \). In particular, \( E'^\infty_{p,0} \cong H_p(k^\times, \mathbb{Z})/\text{Ker}(i) \).

**Proof.** \( L_0 \) is a transitive permutation module over \( SL_2(k) \) and the stabilizer of \( (\infty) \) is the subgroup \( B = B(k) \) of upper-triangular matrices. By Shapiro’s Lemma it follows that the map of complexes

\[ F_\bullet(B) \otimes_{\mathbb{Z}[B]} \mathbb{Z} \rightarrow D_\bullet, \quad z \otimes 1 \mapsto z \otimes (\infty) \]

induces an isomorphism on homology \( H_\bullet(B, \mathbb{Z}) \cong H_\bullet(SL_2(k), L_0) \). However, for any infinite field \( k \), the natural inclusion \( k^\times \rightarrow B \) induces an isomorphism on homology \( H_\bullet(k^\times, \mathbb{Z}) \cong H_\bullet(B, \mathbb{Z}) \) by Theorem [3, 3] above. \( \square \)

**Lemma 8.3.**

1. \( 2 \cdot E^\infty_{2,1} = 0 = 2 \cdot E^\infty_{1,2} \) and hence \( E^\infty_{2,1}\left[\frac{1}{2}\right] = 0 = E^\infty_{1,2}\left[\frac{1}{2}\right] \)

2. \( E^3_{0,\frac{1}{2}} \cong \mathcal{R}P^1(k)\left[\frac{1}{2}\right] \).

3. \( E^3_{2,0}\left[\frac{1}{2}\right] \cong H_2(k^\times, \mathbb{Z}\left[\frac{1}{2}\right]) \cong (k^\times \wedge k^\times)\left[\frac{1}{2}\right] \) and if we let \( \rho \) denote the isomorphism

\[ (k^\times \wedge k^\times)\left[\frac{1}{2}\right] \rightarrow S^2(k^\times)\left[\frac{1}{2}\right], \quad a \wedge b \mapsto 2(a \circ b) \]

then we have an equality of maps

\[ \rho \circ d^3 = \lambda_2 : \mathcal{R}P_1(k)\left[\frac{1}{2}\right] = E^3_{0,3} \rightarrow S^2(k^\times)\left[\frac{1}{2}\right]. \]

The proofs of these facts can be found in section 4 of [6] (where low-dimensional terms of the spectral sequence associated to \( D_{p,q}(SL_2(k), \mathbb{Z}) \) are calculated).

In particular, we have the following:

**Lemma 8.4.** \( E^\infty_{3,0}\left[\frac{1}{2}\right] \cong \mathcal{R}B(k)\left[\frac{1}{2}\right] \) and the inclusion \( k^\times \rightarrow SL_2(k) \) gives rise to a natural exact sequence

\[ H_3(k^\times, \mathbb{Z}\left[\frac{1}{2}\right]) \rightarrow H_3(SL_2(k), \mathbb{Z}\left[\frac{1}{2}\right]) \rightarrow \mathcal{R}B(k)\left[\frac{1}{2}\right] \rightarrow 0. \]

**8.3. The Mayer-Vietoris sequence again.** When \( i = 3 \) in the Mayer-Vietoris sequence, we obtain an exact sequence

\[ H_3(k^\times, \mathbb{Z}) \rightarrow H_3(SL_2(k), \mathbb{Z}) \oplus H_3(SL_2(k), \mathbb{Z}) \rightarrow H_3(SL_2(k[t, t^{-1}]), \mathbb{Z}) \rightarrow \delta \rightarrow H_2(k^\times, \mathbb{Z}) \rightarrow \cdots \]

Thus, tensoring with \( \mathbb{Z}\left[\frac{1}{2}\right] \) and using Lemma 8.4 we obtain a natural exact sequence

\[ 0 \rightarrow \mathcal{R}B(k)\left[\frac{1}{2}\right] \rightarrow H_3(SL_2(k[t, t^{-1}]), \mathbb{Z}\left[\frac{1}{2}\right]) \rightarrow \delta \rightarrow H_2(k^\times, \mathbb{Z}\left[\frac{1}{2}\right]) \rightarrow H_2(SL_2(k), \mathbb{Z}\left[\frac{1}{2}\right]) \rightarrow \cdots \]

Here the map \( \beta \) is calculated as follows: Given \( x \in \mathcal{R}B(k)\left[\frac{1}{2}\right] \), choose \( z \in H_3(SL_2(k), \mathbb{Z}\left[\frac{1}{2}\right]) \) mapping to \( x \). Then \( \beta(x) \) is the class of \( (t) \cdot j(z) \) in

\[ \frac{H_3(SL_2(k[t, t^{-1}]), \mathbb{Z}\left[\frac{1}{2}\right])}{H_3(SL_2(k), \mathbb{Z}\left[\frac{1}{2}\right])}. \]
8.4. The map \( H_3(\text{SL}_2(k[t, t^{-1}]), \mathbb{Z}\left[\frac{1}{2}\right]) \rightarrow \overline{RP}_1(k)\left[\frac{1}{2}\right] \). Recall from section 7 that for any field \( F \) there is a natural injection \( R\mathcal{B}(F) \rightarrow R\mathcal{P}_1(F) \) and a natural surjection \( R\mathcal{P}_1(F) \rightarrow \overline{RP}_1(F) \).

Now let \( \Delta : H_3(\text{SL}_2(k[t, t^{-1}]), \mathbb{Z}) \rightarrow \overline{RP}_1(k) \) denote the following composition of maps:

\[
\begin{array}{c}
H_3(\text{SL}_2(k[t, t^{-1}]), \mathbb{Z}) \rightarrow H_3(\text{SL}_2(k(t)), \mathbb{Z}) \rightarrow R\mathcal{B}(k(t)) \rightarrow \overline{RP}_1(k(t)) \rightarrow \overline{RP}_1(k).
\end{array}
\]

The composite map

\[
R\mathcal{P}_1(k) \rightarrow R\mathcal{P}_1(k(t)) \xrightarrow{S_v} \overline{RP}_1(k) = \overline{RP}_1(k) \oplus \langle t \rangle \cdot \overline{RP}_1(k)
\]

is just the projection on the first factor (since \( S_v([a]) = 1 \otimes [\bar{a}] \) when \( a \in U_v \), by Theorem 7.8) and hence the composite

\[
R\mathcal{P}_1(k) \rightarrow \overline{RP}_1(k(t)) \xrightarrow{\delta} \overline{RP}_1(k)
\]

is the zero map.

From the commutativity of the diagram

\[
\begin{array}{ccc}
H_3(\text{SL}_2(k), \mathbb{Z}) & \xrightarrow{j} & R\mathcal{B}(k) \\
\downarrow & & \downarrow \\
H_3(\text{SL}_2(k[t, t^{-1}]), \mathbb{Z}) & \xrightarrow{\iota} & H_3(\text{SL}_2(k(t)), \mathbb{Z}) \rightarrow R\mathcal{B}(k(t))
\end{array}
\]

it follows that \( \Delta(j(z)) = 0 \) for all \( z \in H_3(\text{SL}_2(k), \mathbb{Z}) \) and hence there is an induced map

\[
\tilde{\Delta} : \frac{H_3(\text{SL}_2(k[t, t^{-1}]), \mathbb{Z})}{H_3(\text{SL}_2(k), \mathbb{Z})} \rightarrow \overline{RP}_1(k).
\]

We will show that, at least after tensoring with \( \mathbb{Z}\left[\frac{1}{2}\right] \), \( \tilde{\Delta} \) is an isomorphism.

We recall that the map

\[
k^\times \wedge k^\times \cong H_2(k^\times, \mathbb{Z}) \xrightarrow{\iota} H_2(\text{SL}_2(k), \mathbb{Z}) \cong K_2^{MW}(k)
\]

factors as

\[
k^\times \wedge k^\times \xrightarrow{\bar{\sigma}} 2 \cdot K_2^M(k) \rightarrow K_2^{MW}(k)
\]

where \( \bar{\sigma}(a \wedge b) = 2[a][b] \). In particular,

\[
\text{Ker}(\iota) = \text{Ker}(\bar{\sigma}) = \text{Im}(\delta : H_3(\text{SL}_2(k[t, t^{-1}]), \mathbb{Z}) \rightarrow H_2(k^\times, \mathbb{Z})).
\]

(In fact, this is, of course, the subgroup of \( k^\times \wedge k^\times \) generated by terms of the form \((1 - a) \wedge a \), where \( a \neq 0, 1 \).)

The square

\[
\begin{array}{ccc}
k^\times \wedge k^\times & \xrightarrow{\bar{\sigma}} & 2 \cdot K_2^M(k) \\
\downarrow \rho & & \downarrow \delta \\
S_2\left( k^\times \right) & \xrightarrow{\sigma} & K_2^M(k)
\end{array}
\]

– where \( \sigma \) is the symbol map \( a \circ b \mapsto \{a\}[b] \) – commutes, and the vertical maps become isomorphisms upon tensoring with \( \mathbb{Z}\left[\frac{1}{2}\right] \).

Thus \( \rho \) induces an isomorphism

\[
\text{Ker}(\bar{\sigma})\left[\frac{1}{2}\right] \cong \text{Ker}(\sigma)\left[\frac{1}{2}\right].
\]
We recall that by Matsumoto’s presentation of $K_2$ of fields we have an exact sequence
\[
\mathcal{P}(k) \xrightarrow{\lambda} S_2^2(k^\times) \xrightarrow{\sigma} K_2(k) \to 0.
\]
Furthermore, the natural map
\[
\mathcal{RP}_1(k) \left[ \frac{1}{2} \right] \to \mathcal{P}(k) \left[ \frac{1}{2} \right]
\]
is surjective by Lemma 7.6. Thus there is an induced exact sequence
\[
\widehat{\mathcal{RP}}_1(k) \left[ \frac{1}{2} \right] \cong \mathcal{RP}_1(k) \left[ \frac{1}{2} \right] \xrightarrow{\lambda_2} S_2^2(k^\times) \left[ \frac{1}{2} \right] \xrightarrow{\sigma} K_2(k) \left[ \frac{1}{2} \right] \to 0.
\]
Since $\mathcal{RB}(k) = \ker(\lambda_2 : \mathcal{RP}_1(k) \to S_2^2(k^\times))$, we thus have a diagram with exact rows
\[
0 \to \mathcal{RB}(k) \left[ \frac{1}{2} \right] \xrightarrow{\beta} H_3(\text{SL}_2(k[t, t^{-1}]), \mathbb{Z} \left[ \frac{1}{2} \right]) \xrightarrow{\delta} \ker(\overline{\delta}) \left[ \frac{1}{2} \right] \to 0
\]
and
\[
0 \to \mathcal{RB}(k) \left[ \frac{1}{2} \right] \xrightarrow{\lambda} \mathcal{RP}_1(k) \left[ \frac{1}{2} \right] \xrightarrow{\lambda_2} \ker(\sigma) \left[ \frac{1}{2} \right] \to 0.
\]

To prove that $\tilde{\lambda}$ is an isomorphism, it is enough to show that both squares in this diagram commute.

Let $x \in \mathcal{RB}(k) \left[ \frac{1}{2} \right]$. Let $z \in H_3(\text{SL}_2(k), \mathbb{Z} \left[ \frac{1}{2} \right])$ map to $x$. Then $\beta(x)$ is represented by the element $\langle t \rangle \cdot j(z) \in H_3(\text{SL}_2(k[t, t^{-1}]), \mathbb{Z} \left[ \frac{1}{2} \right])$. This element maps to $\langle t \rangle \cdot x \in \widehat{\mathcal{RP}}_1(k(t)) \left[ \frac{1}{2} \right]$ under the composite
\[
H_3(\text{SL}_2(k[t, t^{-1}]), \mathbb{Z} \left[ \frac{1}{2} \right]) \to H_3(\text{SL}_2(k(t)), \mathbb{Z} \left[ \frac{1}{2} \right]) \to \mathcal{RB}(k(t)) \left[ \frac{1}{2} \right] \to \widehat{\mathcal{RP}}_1(k(t)) \left[ \frac{1}{2} \right].
\]

Since $S_1(t) x = \langle t \rangle \otimes x \in \widehat{\mathcal{RP}}_1(k(t))$ by Theorem 7.8, we have
\[
\tilde{\lambda}(\beta(x)) = \tilde{\lambda}(\langle t \rangle \cdot j(z)) = \delta_1(\langle t \rangle \otimes x) = x \in \widehat{\mathcal{RP}}_1(k) \left[ \frac{1}{2} \right] = \mathcal{RP}_1(k) \left[ \frac{1}{2} \right].
\]

This shows that the left-hand square commutes.

Finally, the commutativity of the right-hand square follows from

**Lemma 8.5.** For any infinite field $k$, the diagram
\[
\begin{array}{ccc}
H_3(\text{SL}_2(k[t, t^{-1}]), \mathbb{Z} \left[ \frac{1}{2} \right]) & \xrightarrow{\delta} & (k^\times \wedge k^\times) \left[ \frac{1}{2} \right] \\
\downarrow \Lambda & & \downarrow \Lambda \\
\mathcal{RP}_1(k) \left[ \frac{1}{2} \right] & \xrightarrow{\lambda_2} & S_2^2(k^\times) \left[ \frac{1}{2} \right]
\end{array}
\]
commutes.

**Proof.** Recall that the Mayer-Vietoris exact sequence associated to the amalgamated product decomposition
\[
G = \text{SL}_2(k[t, t^{-1}]) = \text{SL}_2(k[t]) \star \Gamma \text{SL}_2(k[t])^{A(t)} = G_1 \star \Gamma G_2
\]
is the long exact homology sequence associated to the short exact sequence of $\mathbb{Z}[G]$-modules
\[
0 \to \mathbb{Z}[G/\Gamma] \to \mathbb{Z}[G/G_1] \oplus \mathbb{Z}[G/G_2] \to \mathbb{Z} \to 0.
\]
This is therefore the long exact sequence resulting from the short exact sequence of complexes
\[
0 \to T(G, \mathbb{Z}[G/G_1]) \to T(G, \mathbb{Z}[G/G_1]) \oplus T(G, \mathbb{Z}[G/G_2]) \to T(G, \mathbb{Z}) \to 0
\]
This corresponds to the class $dx = \tilde{z}$.

Let $\tilde{z} = (0, 0, z \otimes (\infty))$ be the corresponding element of

$$T_2(k^x, \mathbb{Z}[\frac{1}{2}]) = \bigoplus_{p+q=2} D_{p,q}(k^x, \mathbb{Z}[\frac{1}{2}]) = \bigoplus_{p=0}^2 \left( F_p(k^x) \otimes \mathbb{Z} \right) \left( L_{2-p}(\mathbb{Z}[\frac{1}{2}]) \right).$$

Since $w$ maps to 0 in $H_2(G(1), \mathbb{Z}[\frac{1}{2}])$, it follows that the image of $\tilde{z}$ in $T_2(G(1), \mathbb{Z}[\frac{1}{2}])$ is a boundary. Thus there exist $x_p \in D_{p,3-p}(G(1), \mathbb{Z}[\frac{1}{2}])$, $p = 0, \ldots, 3$ such that $x = (x_0, x_1, x_2, x_3) \in T_3(G(1), \mathbb{Z}[\frac{1}{2}])$ satisfies $dx = \tilde{z}$.

We spell out what this means. Let $d^h$ be the horizontal differential on $D_{p,q}$, induced from the differential on $F_q(G(1))$, and let $d^v$ be the vertical differential, induced from the differential of $L^*$. Then the total differential on $T_{p,q}(G, \mathbb{Z}[\frac{1}{2}])$ is $d = d^h + (-1)^p d^v$. Thus we have

$$(0, 0, z \otimes (\infty)) = \tilde{z} = dx = (d^h(x_1) + d^v(x_0), d^h(x_2) = d^v(x_1), d^h(x_3) + d^v(x_2)).$$

Since $d^h(x_0) = d^h(-x_1)$, $d^h(x_0) = 0$ and $x_0$ represents an element of $E^2_{0,3}$. Since $d^v(-x_1) = d^h(-x_2)$, $d^v(x_0) = 0$ and hence $x_0$ represents an element, $\alpha$ say, of $E^3_{0,3}(G, \mathbb{Z}[\frac{1}{2}]) \cong \mathcal{RP}_1(k)[\frac{1}{2}]$.

Now $d^3(\alpha)$ is represented by

$$d^v(x_2) = z \otimes (\infty) + d^h(-x_3)$$

and hence $d^3(\alpha)$ is the homology class represented by $z \otimes (\infty)$ in $E^3_{2,0}(G, \mathbb{Z}[\frac{1}{2}]) = H_2(G(1), L_0)[\frac{1}{2}]$.

This corresponds to the class $w \in H_2(k^x, \mathbb{Z}[\frac{1}{2}])$ under the isomorphism $H_2(k^x, \mathbb{Z}[\frac{1}{2}]) \cong H_2(G(1), L_0)[\frac{1}{2}]$.

So $d^3(\alpha) = w$ with this identification.

Recall that $C_t : G_1 \to G_2$ denotes the isomorphism given by conjugating by $A(t)$: $C_t(g) = A(t)^{-1} g A(t) = g^{A(t)}$. Observe that $C_t$ induces the identity map of the diagonal subgroup $k^x$, since $A(t)$ commutes with other diagonal matrices.

Thus $C_t(x) = (C_t(x_0), C_t(x_1), C_t(x_2), C_t(x_3)) \in T_3(G_2, \mathbb{Z}[\frac{1}{2}])$. Then $C_t(x_0)$ represents an element, $C_t(\alpha)$, of $E^3_{0,3} = C_t(\mathcal{RP}_1) \left( \frac{1}{2} \right)$ and $d^3(C_t(\alpha)) = w$ in $E^3_{2,0} = H_2(k^x, \mathbb{Z}[\frac{1}{2}])$.

Now the cycle $\tilde{z} \in T_2(k^x, \mathbb{Z}[\frac{1}{2}])$ maps to the cycle $z \otimes \Gamma \in T_2(G, \mathbb{Z}[G/\Gamma][\frac{1}{2}])$, and this cycle in turn represents the class $w \in H_2(k^x, \mathbb{Z}[\frac{1}{2}]) \cong H_2(G, \mathbb{Z}[\frac{1}{2}])$.

Under the map

$$T_2(G, \mathbb{Z}[G/\Gamma][\frac{1}{2}]) \to T_2(G, \mathbb{Z}[G/G_1][\frac{1}{2}]) \oplus T_2(G, \mathbb{Z}[G/G_2][\frac{1}{2}])$$

$\tilde{z} \otimes \Gamma$ maps to $(\tilde{z} \otimes G_1, \tilde{z} \otimes G_2)$. By the calculations above, this is the boundary of

$$(x \otimes G_1, C_t(x) \otimes G_2) \in T_3(G, \mathbb{Z}[G/G_1][\frac{1}{2}]) \oplus T_3(G, \mathbb{Z}[G/G_2][\frac{1}{2}]).$$

This element in turn maps to the cycle $W = C_t(x) - x \in T_3(G, \mathbb{Z}[\frac{1}{2}])$ under the map

$$T_3(G, \mathbb{Z}[G/G_1][\frac{1}{2}]) \oplus T_3(G, \mathbb{Z}[G/G_2][\frac{1}{2}]) \to T_3(G, \mathbb{Z}[\frac{1}{2}]).$$

By construction, the cycle $W$ represents an element of $H_3(G, \mathbb{Z}[\frac{1}{2}])$ which maps to $w \in H_2(k^x, \mathbb{Z}[\frac{1}{2}])$ under the connecting homomorphism of the Mayer-Vietoris sequence of the amalgamated product.
9. Eigenspace decompositions

The decompositions of Theorems 5.1 and 8.1 are decompositions of $\mathbb{R}_k$-modules but are not $R_{k[t,t^{-1}]}$-module decompositions. Indeed it is clear from the Mayer-Vietoris exact sequence that the image of $\lambda_3^{+}(SL_2(k), \mathbb{Z}) \rightarrow H_3^{+}(SL_2(k[t,t^{-1}]), \mathbb{Z})$ under the natural split inclusion is not invariant under the action of the square class $\langle t \rangle$. However, there is a natural decomposition of $H_3^{+}(SL_2(k[t,t^{-1}]), \mathbb{Z}[\frac{1}{2}])$ as a $R_{k[t,t^{-1}]}$-module:

If $M$ is a $\mathbb{Z}[\frac{1}{2}]$-module on which $\langle t \rangle$ acts as an involution, we let

$$e_+ = \frac{\langle t \rangle + 1}{2}, e_- = \frac{\langle t \rangle - 1}{2} \in \text{End}(M)$$

and let

$$M_+ = e_+ M = \{ m \in M \mid \langle t \rangle m = m \} \text{ and } M_- = e_- M = \{ m \in M \mid \langle t \rangle m = -m \}.$$}

Thus $M = M_+ + M_- \cong M_+ \oplus M_-$. Furthermore, we note that if $M$ is a $R_{k[t,t^{-1}]}$-module and if $N$ is a $R_k$-submodule, then $e_+ N$ and $e_- N$ are $R_{k[t,t^{-1}]}$-submodules of $M$.

**Theorem 9.1.** Let $k$ be an infinite field. For convenience, identify $H_3^{+}(SL_2(k), \mathbb{Z}[\frac{1}{2}])$ with its image in $H_3^{+}(SL_2(k[t,t^{-1}]), \mathbb{Z}[\frac{1}{2}])$ under the $R_k$-embedding $j$.

Then

$$H_3^{+}(SL_2(k[t,t^{-1}]), \mathbb{Z}[\frac{1}{2}])_+ = e_+ H_3^{+}(SL_2(k), \mathbb{Z}[\frac{1}{2}]) \cong H_3^{+}(SL_2(k), \mathbb{Z}[\frac{1}{2}])$$

and

$$H_3^{+}(SL_2(k[t,t^{-1}]), \mathbb{Z}[\frac{1}{2}])_- \cong R_{P_1}(k)[\frac{1}{2}].$$

**Proof.** Certainly $e_+ H_3^{+}(SL_2(k), \mathbb{Z}[\frac{1}{2}]) \subset H_3^{+}(SL_2(k[t,t^{-1}]), \mathbb{Z}[\frac{1}{2}])$, and the composite

$$H_3^{+}(SL_2(k), \mathbb{Z}[\frac{1}{2}]) \xrightarrow{e_+} e_+ H_3^{+}(SL_2(k), \mathbb{Z}[\frac{1}{2}]) \xrightarrow{\text{inc}} H_3^{+}(SL_2(k[t,t^{-1}]), \mathbb{Z}[\frac{1}{2}]) \xrightarrow{\iota^{-1}} H_3^{+}(SL_2(k), \mathbb{Z}[\frac{1}{2}])$$

is the identity map.

On the other hand, if $\alpha \in R_{P_1}(k)$, the construction in the proof of Lemma 8.5 gives a cycle $W$ of the form $C_{\ell}(x) - x \in T_3^{+}(SL_2(k[t,t^{-1}]), \mathbb{Z}[\frac{1}{2}])$ representing a homology class $[W] \in H_3^{+}(SL_2(k[t,t^{-1}]), \mathbb{Z}[\frac{1}{2}])$ which satisfies $\Delta([W]) = \alpha$ in $R_{P_1}(k)[\frac{1}{2}]$. Thus, by construction, $[W] \in H_3^{+}(SL_2(k[t,t^{-1}]), \mathbb{Z}[\frac{1}{2}])_-$.

It follows that the map $\Delta$ induces a surjection

$$H_3^{+}(SL_2(k[t,t^{-1}]), \mathbb{Z}[\frac{1}{2}])_- \twoheadrightarrow R_{P_1}(k)[\frac{1}{2}].$$
On the other hand, clearly $H_3(\text{SL}_2(k), \mathbb{Z}[\frac{1}{2}])$ lies in the kernel of the projection

$$H_3(\text{SL}_2(k[t, t^{-1}]), \mathbb{Z}[\frac{1}{2}]) \to H_3(\text{SL}_2(k), \mathbb{Z}[\frac{1}{2}])$$

induced from the map sending $t$ to 1.

**Remark 9.2.** It follows from this proof that the map

$$\mathcal{RP}_1(k) \left[ \frac{1}{2} \right] \to H_3(\text{SL}_2(k[t, t^{-1}]), \mathbb{Z}[\frac{1}{2}])$$

sending $\alpha$ to the class $[W]$ gives a well-defined splitting of $\Delta$.

**Remark 9.3.** Similar arguments for $n = 2$ give an analogous eigenspace decomposition over $\mathbb{Z}[\frac{1}{2}]$:

$$H_2(\text{SL}_2(k[t, t^{-1}]), \mathbb{Z}[\frac{1}{2}])_+ = e_+ H_2(\text{SL}_2(k), \mathbb{Z}[\frac{1}{2}]) \cong K_{2}^{\operatorname{MW}}(k)$$

and

$$H_2(\text{SL}_2(k[t, t^{-1}]), \mathbb{Z}[\frac{1}{2}])_- \cong K_{1}^{\operatorname{MW}}(k) \left[ \frac{1}{2} \right]$$

for any infinite field $k$ of characteristic different from 2.

### 10. Some Examples and Special Cases

Theorem 8.1 tells us that

$$H_3(\text{SL}_2(k[t, t^{-1}]), \mathbb{Z}[\frac{1}{2}]) \cong H_3(\text{SL}_2(k), \mathbb{Z}[\frac{1}{2}]) \oplus \mathcal{RP}_1(k) \left[ \frac{1}{2} \right]$$

for any infinite field $k$.

By Theorem 7.5(2) and Lemma 7.6, there are natural short exact sequences

$$0 \to \mathcal{RB}_0(k) \left[ \frac{1}{2} \right] \to H_3(\text{SL}_2(k), \mathbb{Z}[\frac{1}{2}]) \to K_3^{\operatorname{ind}}(k) \left[ \frac{1}{2} \right] \to 0$$

and

$$0 \to \mathcal{RB}_0(k) \left[ \frac{1}{2} \right] \to \mathcal{RP}_1(k) \left[ \frac{1}{2} \right] \to \mathcal{P}(k) \left[ \frac{1}{2} \right] \to 0.$$ 

Thus we have:

**Proposition 10.1.** Let $k$ be an infinite field. Then there is a natural short exact sequence of $\mathbb{R}_k$-modules

$$0 \longrightarrow \mathcal{RB}_0(k) \oplus \mathcal{RP}_1(k) \left[ \frac{1}{2} \right] \longrightarrow H_3(\text{SL}_2(k[t, t^{-1}]), \mathbb{Z}[\frac{1}{2}]) \longrightarrow K_3^{\operatorname{ind}}(k) \left[ \frac{1}{2} \right] \oplus \mathcal{P}(k) \left[ \frac{1}{2} \right] \longrightarrow 0.$$ 

By Theorem 7.5(4), we deduce:

**Corollary 10.2.** Let $k$ be a quadratically closed or real-closed field. Then there is a natural isomorphism

$$H_3(\text{SL}_2(k[t, t^{-1}]), \mathbb{Z}[\frac{1}{2}]) \cong K_3^{\operatorname{ind}}(k) \left[ \frac{1}{2} \right] \oplus \mathcal{P}(k) \left[ \frac{1}{2} \right].$$

By Theorem 7.5(5) we have:

**Corollary 10.3.** Let $k$ be a local field with finite residue field $\bar{k}$ of odd order. If $\mathbb{Q}_3 \subset k$ suppose that $[k : \mathbb{Q}_3]$ is odd. Then there is a natural (split) short exact sequence of the form

$$0 \to \mathcal{P}(\bar{k}) \left[ \frac{1}{2} \right] \oplus \mathcal{RP}_1(k) \left[ \frac{1}{2} \right] \rightarrow H_3(\text{SL}_2(k[t, t^{-1}]), \mathbb{Z}[\frac{1}{2}]) \rightarrow K_3^{\operatorname{ind}}(k) \left[ \frac{1}{2} \right] \oplus \mathcal{P}(k) \left[ \frac{1}{2} \right] \to 0.$$ 

**Example 10.4.** In particular, if $p \geq 3$ is prime there is a natural decomposition

$$H_3(\text{SL}_2(\mathbb{Q}_p[t, t^{-1}]), \mathbb{Z}[\frac{1}{2}]) \cong K_3^{\operatorname{ind}}(\mathbb{Q}_p) \left[ \frac{1}{2} \right] \oplus \mathcal{P}(\mathbb{Q}_p) \left[ \frac{1}{2} \right] \oplus \mathcal{P}(\mathbb{F}_p) \left[ \frac{1}{2} \right] \oplus \mathcal{RP}_1(k) \left[ \frac{1}{2} \right].$$
We consider the case \( k = \mathbb{Q} \). By Theorem 7.5(6) there is a surjective map
\[
\mathcal{R}B_0(\mathbb{Q}) \left[ \frac{1}{2} \right] \longrightarrow \bigoplus_{p \text{ prime}} \mathcal{P}(\mathbb{F}_p) \left[ \frac{1}{2} \right] \cong \bigoplus_{p \text{ prime}} \mathbb{Z}/(p + 1)'.
\]
It is an open question whether this map is an isomorphism. We note, however, that Theorem 6.1 of [8] implies that \( \mathcal{R}B_0(\mathbb{Q}) \) is a torsion group.

Furthermore, we have \( K^\text{ind}_3(\mathbb{Q}) \cong \mathbb{Z}/24 \) and \( \mathcal{B}(\mathbb{Q}) \cong \mathbb{Z}/6 \) so that
\[
K^\text{ind}_3(\mathbb{Q}) \left[ \frac{1}{2} \right] \cong \mathcal{B}(\mathbb{Q}) \left[ \frac{1}{2} \right] \cong \mathbb{Z}/3.
\]
From the exact sequence
\[
0 \rightarrow \mathcal{B}(\mathbb{Q}) \rightarrow \mathcal{P}(\mathbb{Q}) \rightarrow S_2^2(\mathbb{Q}^\times) \rightarrow K_2(\mathbb{Q}) \rightarrow 0
\]
and the fact that \( K_2(\mathbb{Q}) \) is a torsion group, we deduce that
\[
\mathcal{P}(\mathbb{Q}) \left[ \frac{1}{2} \right] \cong \mathbb{Z}/3 \oplus V
\]
where \( V \) is a free \( \mathbb{Z} \left[ \frac{1}{2} \right] \)-module of countably infinite rank. Combining this with Proposition 10.1 we deduce:

**Corollary 10.5.** There is a short exact sequence
\[
0 \rightarrow \mathcal{R}B_0(\mathbb{Q}) \left[ \frac{1}{2} \right] \oplus \mathcal{P}(\mathbb{Q}) \left[ \frac{1}{2} \right] \rightarrow H_3(\text{SL}_2(\mathbb{Q}[t, t^{-1}]), \mathbb{Z} \left[ \frac{1}{2} \right]) \rightarrow \mathbb{Z}/3 \oplus \mathbb{Z}/3 \oplus V \rightarrow 0.
\]
where \( \mathcal{R}B_0(\mathbb{Q}) \) is a torsion abelian group which is not finitely generated and \( V \) is a free \( \mathbb{Z} \left[ \frac{1}{2} \right] \)-module of countably infinite rank.

Finally, we use the results of section 8quire to calculate the kernel and cokernel of the stabilization map from \( H_3(\text{SL}_2(k[t, t^{-1}]), \mathbb{Z} \left[ \frac{1}{2} \right]) \) to \( H_3(\text{SL}(k[t, t^{-1}]), \mathbb{Z} \left[ \frac{1}{2} \right]) \).

First we require the following observation about the stabilization homomorphism for \( \text{SL}_n \):

**Lemma 10.6.** Let \( A \) be a commutative ring. For any \( k \geq 0 \), the stabilization map \( H_k(\text{SL}_n(A), \mathbb{Z}) \rightarrow H_k(\text{SL}_{n+1}(A), \mathbb{Z}) \) is a homomorphism of \( \mathbb{Z}[A^\times] \)-modules.

The image of this homomorphism is a trivial \( \mathbb{Z}[A^\times] \)-module.

**Proof.** Let \( u \in A^\times \). The action of \( u \) on \( H_3(\text{SL}_r(A), \mathbb{Z}) \), for any \( r \), is induced from conjugation by the diagonal matrix \( \text{diag}(u, 1, \ldots, 1) \). This commutes with the stabilization homomorphism
\[
A \mapsto \begin{bmatrix} A & 0 \\ 0 & 1 \end{bmatrix} \in \text{SL}_{n+1}(A).
\]
However, \( u^r \) acts trivially on \( H_k(\text{SL}_r(A), \mathbb{Z}) \) since it acts (also) via conjugation by the central scalar matrix \( \text{diag}(u, \ldots, u) \). Thus \( u^r \) and \( u^{r+1} \) both act trivially on the image of map of the lemma. It follows that \( u \) acts trivially on this image. \( \square \)

**Theorem 10.7.** Let \( k \) be an infinite field. There is a natural exact sequence
\[
0 \longrightarrow \mathcal{R}B_0(k) \left[ \frac{1}{2} \right] \oplus \mathcal{R}P_1(k) \left[ \frac{1}{2} \right] \longrightarrow H_3(\text{SL}_2(k[t, t^{-1}]), \mathbb{Z} \left[ \frac{1}{2} \right]) \longrightarrow H_3(\text{SL}(k[t, t^{-1}]), \mathbb{Z} \left[ \frac{1}{2} \right]) \longrightarrow K^M_3(k) \left[ \frac{1}{2} \right] \oplus K^M_2(k) \left[ \frac{1}{2} \right] \longrightarrow 0.
\]
**Proof.** For any ring $A$, Suslin has shown \cite{24} that the Hurewicz homomorphism $K_3(A) \to H_3(\text{GL}(A), \mathbb{Z})$ induces an isomorphism

$$K_3(A) \\ {\{−1\} \cdot K_2(A)} \cong H_3(\text{SL}(A), \mathbb{Z}).$$

Since the element $\{−1\} \in K_1(A)$ has order 2, it follows that there is an induced isomorphism $K_3(A) \{\frac{1}{2}\} \cong H_3(\text{SL}(A), \mathbb{Z})$.

In particular, for an infinite field $k$

$$H_3(\text{SL}(k[t, r^{-1}]), \mathbb{Z}) \{\frac{1}{2}\} \cong K_3(k[t, r^{-1}]) \{\frac{1}{2}\} \cong K_3(k) \{\frac{1}{2}\} \oplus K_2(k) \{\frac{1}{2}\} \cong H_3(\text{SL}(k), \mathbb{Z}) \{\frac{1}{2}\} \oplus H_2(\text{SL}(k), \mathbb{Z}) \{\frac{1}{2}\}).$$

Thus the stabilization map induces a map of short exact sequences with compatible splittings

$$0 \longrightarrow H_3(\text{SL}_2(k), \mathbb{Z}) \{\frac{1}{2}\} \longrightarrow H_3(\text{SL}_2(k[t, r^{-1}]), \mathbb{Z}) \{\frac{1}{2}\} \longrightarrow \mathcal{R}P_1(k) \{\frac{1}{2}\} \longrightarrow 0$$

$$0 \longrightarrow H_3(\text{SL}(k), \mathbb{Z}) \{\frac{1}{2}\} \longrightarrow H_3(\text{SL}(k[t, r^{-1}]), \mathbb{Z}) \{\frac{1}{2}\} \longrightarrow K_2(k) \{\frac{1}{2}\} \longrightarrow 0$$

By Theorem \ref{7.5}(3), the kernel of the stabilization map $H_3(\text{SL}_2(k), \mathbb{Z}) \{\frac{1}{2}\} \to H_3(\text{SL}(k), \mathbb{Z}) \{\frac{1}{2}\})$ is isomorphic to $\mathcal{R}B_0(k) \{\frac{1}{2}\}$ and the cokernel is isomorphic to $K_3^M(k) \{\frac{1}{2}\}$.

We conclude by showing that the map $\mathcal{R}P_1(k) \{\frac{1}{2}\} \to K_2(k) \{\frac{1}{2}\}$ in the diagram above is the zero map:

By the proof of Theorem \ref{9.1} given $\alpha \in \mathcal{R}P_1(k) \{\frac{1}{2}\}$ there exists $W \in H_3(\text{SL}_2(k[t, r^{-1}]), \mathbb{Z}) \{\frac{1}{2}\})$, satisfying $\Delta(W) = \alpha$. But $H_3(\text{SL}_2(k[t, r^{-1}]), \mathbb{Z}) \{\frac{1}{2}\})$, is in the kernel of the stabilization homomorphism by Lemma \ref{10.6}. Therefore $\alpha$ maps to 0 in $K_2(k) \{\frac{1}{2}\}$ as claimed.

\[\square\]

**Corollary 10.8.** Let $k$ be a quadratically closed or a real-closed field. Then the kernel of the stabilization map

$$H_3(\text{SL}_2(k[t, r^{-1}]), \mathbb{Z}) \{\frac{1}{2}\}) \to H_3(\text{SL}(k[t, r^{-1}]), \mathbb{Z}) \{\frac{1}{2}\})$$

is naturally isomorphic to $\mathcal{P}(k) \{\frac{1}{2}\}$.

11. **The homology of GL$_2$ of Laurent polynomials**

In this section, we derive the structure of $H_r(\text{GL}_2(k[t, r^{-1}]), \mathbb{Z})$ when $n \leq 3$, at least over $\mathbb{Z} \{\frac{1}{2}\}$, from our knowledge of the structure of $H_r(\text{SL}_2(k[t, r^{-1}]), \mathbb{Z}) (r \leq n)$ by using the Hochschild-Serre spectral sequence relating the two.

For any ring $A$ the short exact sequence of groups

$$1 \longrightarrow \text{SL}_2(A) \longrightarrow \text{GL}_2(A) \longrightarrow \det A^\times \longrightarrow 1$$

gives a spectral sequence

$$E^2_{p,q} = H_p(A^\times, H_q(\text{SL}_2(A), \mathbb{Z})) \Rightarrow H_{p+q}(\text{GL}_2(A), \mathbb{Z}).$$
Since the determinant map $\det$ has a splitting, the induced map on homology $H_\bullet(\text{GL}_2(A), \mathbb{Z}) \to H_\bullet(A^\times, \mathbb{Z})$ is a split surjection. In particular, in the spectral sequence we have
\[
E_{2,0}^2 = H_n(A^\times, \mathbb{Z}) = E_{n,0}^\infty
\]
and all higher differentials leaving the base are zero.

We let
\[
H_0^\bullet(\text{GL}_2(A), \mathbb{Z}) := \text{Ker}(\det : H_\bullet(\text{GL}_2(A), \mathbb{Z}) \to H_\bullet(A^\times, \mathbb{Z})).
\]
Thus there is a natural split short exact sequence
\[
0 \to H_0^\bullet(\text{GL}_2(A), \mathbb{Z}) \to H_\bullet(\text{GL}_2(A), \mathbb{Z}) \to H_\bullet(A^\times, \mathbb{Z}) \to 0.
\]

For the homology of $A^\times$ there is a natural version of the ‘fundamental theorem’:

**Lemma 11.1.** Let $k$ be a field. Then
\[
H_n(k[t, t^{-1}]^\times, \mathbb{Z}) \cong H_n(k^\times, \mathbb{Z}) \oplus H_{n-1}(k^\times, \mathbb{Z}).
\]

**Proof.** Since $k[t, t^{-1}]^\times \cong k^\times \times \mathbb{Z}$, the Künneth formula gives:
\[
H_n(k[t, t^{-1}], \mathbb{Z}) \cong H_n(k^\times \times \mathbb{Z}, \mathbb{Z})
\]
\[
\cong \oplus_{p+q=n} H_p(k^\times, \mathbb{Z}) \otimes H_q(\mathbb{Z}, \mathbb{Z})
\]
\[
\cong H_n(k^\times, \mathbb{Z}) \oplus H_{n-1}(k^\times, \mathbb{Z}).
\]

□

**Proposition 11.2.** Let $k$ be an infinite field. Then
\[
H_1(\text{GL}_2(k[t, t^{-1}]), \mathbb{Z}) \cong H_1(\text{GL}_2(k), \mathbb{Z}) \oplus \mathbb{Z}.
\]

**Proof.** Since $H_1(\text{SL}_2(k[t, t^{-1}]), \mathbb{Z}) = 0 = H_1(\text{SL}_2(k), \mathbb{Z})$, it follows that the determinant $H_1(\text{GL}_2(A), \mathbb{Z}) \to H_1(A^\times, \mathbb{Z})$ is an isomorphism when $A = k$ or $k[t, t^{-1}]$. We conclude by Lemma 11.1.

Before treating the second homology, we make some preliminary remarks. Let $R$ be a $\mathbb{Z}[\frac{1}{2}]$-algebra and let $M$ be an $R$-module with an automorphism $\sigma$ of order 2. Let
\[
e_+ = \frac{\sigma + 1}{2} \quad \text{and} \quad e_- = \frac{\sigma - 1}{2}
\]
be the corresponding idempotents in $\text{End}_R(M)$. We regard $M_+ := e_+ M$ and $M_- := e_- M$ canonically as both sub-modules and quotient modules of $M$.

**Lemma 11.3.** Let $C$ be an infinite cyclic group whose generator acts on $M$ as $\sigma$. Then
\[
H_\bullet(C, M) = \begin{cases} 
M_+, & i = 0, 1 \\
0, & i \geq 2.
\end{cases}
\]

**Proof.** $H_\bullet(C, M)$ is the homology of the complex
\[
M \xrightarrow{\sigma^{-1}} M
\]
concentrated in degrees 1 and 0. This is equal to the homology of
\[
M \xrightarrow{e_+} M
\]

□
**Corollary 11.4.** Let $G$ be any abelian group acting on $M$. Suppose that $G$ contains an infinite cyclic subgroup $C$ which acts on $M$ via $\sigma$. Then

$$H_i(G, M) \cong H_i(G, M_+)$$

for all $i$.

**Proof.** Certainly $H_i(C, M_+) = 0$ for all $i$ by Lemma [11.3]. The Hochschild-Serre spectral sequence for the extension

$$1 \to C \to G \to G/C \to 1$$

then implies that $H_i(G, M_+) = 0$ for all $i$. \hfill \Box

**Proposition 11.5.** Let $k$ be an infinite field of characteristic not equal to 2. Then

$$H_2(\text{GL}_2(k[t, t^{-1}]), \mathbb{Z}\left[\frac{1}{2}\right]) \cong H_2(\text{GL}_2(k), \mathbb{Z}\left[\frac{1}{2}\right]) \oplus k^x \left[\frac{1}{2}\right].$$

**Proof.** In view of Lemma [11.1] we need only show that the natural map $\text{GL}_2(k) \to \text{GL}_2(k[t, t^{-1}])$ induces an isomorphism

$$H^0_2(\text{GL}_2(k), \mathbb{Z}\left[\frac{1}{2}\right]) \cong H^0_2(\text{GL}_2(k[t, t^{-1}]), \mathbb{Z}\left[\frac{1}{2}\right]).$$

Now, since $H_1(\text{SL}_2(k[t, t^{-1}]), \mathbb{Z}) = 0 = H_1(\text{SL}_2(k), \mathbb{Z})$, the Hochschild-Serre sequence tells us that the inclusion $\text{SL}_2(A) \to \text{GL}_2(A)$ induces an isomorphism

$$H_0(A^x, H_2(\text{SL}_2(A), \mathbb{Z})) \cong H_0^2(\text{GL}_2(A), \mathbb{Z})$$

when $A = k$ or $k[t, t^{-1}]$.

We have $k[t, t^{-1}]^x \cong k^x \times \mathbb{Z}$, where the infinite cyclic factor is generated by $t$. Thus

$$H_0(k[t, t^{-1}]^x, H_2(\text{SL}_2(k[t, t^{-1}]), \mathbb{Z})) \cong H_0(k^x \times \mathbb{Z}, H_2(\text{SL}_2(k[t, t^{-1}]), \mathbb{Z}))$$

$$\cong H_0(k^x, H_0(\mathbb{Z}, H_2(\text{SL}_2(k[t, t^{-1}]), \mathbb{Z}))).$$

However, recall from section [9] that with respect to the action of $\sigma = \langle t \rangle$ on $H_2(\text{SL}_2(k[t, t^{-1}]), \mathbb{Z}\left[\frac{1}{2}\right])$ we have

$$H_2(\text{SL}_2(k[t, t^{-1}]), \mathbb{Z}\left[\frac{1}{2}\right]), \cong H_2(\text{SL}_2(k), \mathbb{Z}\left[\frac{1}{2}\right]).$$

By Lemma [11.3]

$$H_0(\mathbb{Z}, H_2(\text{SL}_2(k[t, t^{-1}]), \mathbb{Z})) \cong H_2(\text{SL}_2(k), \mathbb{Z}\left[\frac{1}{2}\right])$$

and hence

$$H^0_2(\text{GL}_2(k), \mathbb{Z}\left[\frac{1}{2}\right]) \cong H_0(k^x, H_2(\text{SL}_2(k), \mathbb{Z}\left[\frac{1}{2}\right]))$$

$$\cong H_0(k^x, H_0(\mathbb{Z}, H_2(\text{SL}_2(k[t, t^{-1}]), \mathbb{Z}))))$$

$$\cong H_0(k[t, t^{-1}]^x, H_2(\text{SL}_2(k[t, t^{-1}]), \mathbb{Z}))$$

$$\cong H^0_2(\text{GL}_2(k[t, t^{-1}]), \mathbb{Z}\left[\frac{1}{2}\right]).$$

\hfill \Box

**Proposition 11.6.** Let $k$ be an infinite field of characteristic not equal to 2. Then

$$H_3(\text{GL}_2(k[t, t^{-1}]), \mathbb{Z}\left[\frac{1}{2}\right]) \cong H_3(\text{GL}_2(k), \mathbb{Z}\left[\frac{1}{2}\right]) \oplus K^M_2(k)\left[\frac{1}{2}\right] \oplus H_2(k^x, \mathbb{Z}\left[\frac{1}{2}\right]).$$
PROOF. In view of Lemma [11.1] we need only show that the natural map \( \text{GL}_2(k) \to \text{GL}_2(k[t, t^{-1}]) \) induces an exact sequence of the form

\[
0 \to H^0_2(\text{GL}_2(k), \mathbb{Z} \left[ \frac{1}{2} \right]) \to H^0_2(\text{GL}_2(k[t, t^{-1}]), \mathbb{Z} \left[ \frac{1}{2} \right]) \to K_2^{MW}(k) \left[ \frac{1}{2} \right] \to 0.
\]

For \( A = k \) or \( k[t, t^{-1}] \), the Hochschild-Serre spectral sequence induces short exact sequences

\[
0 \to H_0(A^\times, H_3(\text{SL}_2(A), \mathbb{Z} \left[ \frac{1}{2} \right])) \to H_0^0(\text{GL}_2(k), \mathbb{Z} \left[ \frac{1}{2} \right]) \to H_1(A^\times, H_2(\text{SL}_2(A), \mathbb{Z} \left[ \frac{1}{2} \right])) \to 0.
\]

Thus there is a commutative diagram in which the rows are short exact sequences:

\[
\begin{array}{ccc}
H_0(k^\times, H_3(\text{SL}_2(k), \mathbb{Z} \left[ \frac{1}{2} \right])) & \to & H_0^0(\text{GL}_2(k), \mathbb{Z} \left[ \frac{1}{2} \right]) \\
\downarrow f & & \downarrow \\
H_0(k[t, t^{-1}]^\times, H_3(\text{SL}_2(k[t, t^{-1}]), \mathbb{Z} \left[ \frac{1}{2} \right])) & \to & H_0^0(\text{GL}_2(k[t, t^{-1}]), \mathbb{Z} \left[ \frac{1}{2} \right]) \\
\downarrow g & & \downarrow
\end{array}
\]

\[
H_1(k[t, t^{-1}]^\times, H_2(\text{SL}_2(k[t, t^{-1}]), \mathbb{Z} \left[ \frac{1}{2} \right])) \to H_1(k[t, t^{-1}]^\times, H_2(\text{SL}_2(k[t, t^{-1}]), \mathbb{Z} \left[ \frac{1}{2} \right]))
\]

We recall from section [9] that

\[
H_3(\text{SL}_2(k[t, t^{-1}]), \mathbb{Z} \left[ \frac{1}{2} \right]), \cong H_3(\text{SL}_2(k), \mathbb{Z} \left[ \frac{1}{2} \right])
\]

(isomorphism of \( R_k \)-modules). Thus

\[
H_0(k[t, t^{-1}]^\times, H_3(\text{SL}_2(k[t, t^{-1}]), \mathbb{Z} \left[ \frac{1}{2} \right])) \cong H_0(k^\times \times \mathbb{Z}, H_3(\text{SL}_2(k[t, t^{-1}]), \mathbb{Z} \left[ \frac{1}{2} \right]))
\]

\[
\cong H_0(k^\times, H_0(\mathbb{Z}, H_3(\text{SL}_2(k[t, t^{-1}]), \mathbb{Z} \left[ \frac{1}{2} \right])))
\]

\[
\cong H_0(k^\times, H_3(\text{SL}_2(k[t, t^{-1}]), \mathbb{Z} \left[ \frac{1}{2} \right]))
\]

\[
\cong H_0(k^\times, H_3(\text{SL}_2(k), \mathbb{Z} \left[ \frac{1}{2} \right]))
\]

and the map \( f \) is an isomorphism.

By Corollary [11.4] we have isomorphisms

\[
H_1(k[t, t^{-1}]^\times, H_2(\text{SL}_2(k[t, t^{-1}]), \mathbb{Z} \left[ \frac{1}{2} \right])) \cong H_1(k[t, t^{-1}]^\times, H_2(\text{SL}_2(k), \mathbb{Z} \left[ \frac{1}{2} \right]))
\]

\[
\cong H_1(k^\times \times \mathbb{Z}, H_2(\text{SL}_2(k), \mathbb{Z} \left[ \frac{1}{2} \right]))
\]

where the factor \( \mathbb{Z} \) acts trivially.

The Hochschild-Serre spectral sequence for the extension

\[
0 \to \mathbb{Z} \to k^\times \times \mathbb{Z} \to k^\times \to 1
\]

gives a short exact sequence

\[
0 \to H_1(k^\times, H_2(\text{SL}_2(k), \mathbb{Z} \left[ \frac{1}{2} \right])) \to H_1(k^\times \times \mathbb{Z}, H_2(\text{SL}_2(k), \mathbb{Z} \left[ \frac{1}{2} \right])) \to H_0(k^\times, H_2(\text{SL}_2(k), \mathbb{Z} \left[ \frac{1}{2} \right])) \to 0
\]

where the leftmost non-trivial map can be identified with the map \( g \).

Finally, we have

\[
H_0(k^\times, H_2(\text{SL}_2(k), \mathbb{Z} \left[ \frac{1}{2} \right])) \cong H_0(k^\times, K_2^{MW}(k) \left[ \frac{1}{2} \right]) \cong K_2^{MW}(k) \left[ \frac{1}{2} \right]
\]

as required. \( \square \)
Remark 11.7. It is well-known, from calculations of Suslin, [22], that $H^0(\text{GL}_2(k), \mathbb{Z}) \cong K^M_2(k)$ for any infinite field $k$. Thus

$$H_2(\text{GL}_2(k), \mathbb{Z}) \cong H^0_2(\text{GL}_2(k), \mathbb{Z}) \oplus H_2(k^\times, \mathbb{Z}) \cong K^M_2(k) \oplus H_2(k^\times, \mathbb{Z})$$

for any infinite field $k$.

In view of this, Proposition [11.6] can be reformulated to assert that

$$H_3(\text{GL}_2(k[t, t^{-1}]), \mathbb{Z}
\left[\frac{t}{2}\right]) \cong H_3(\text{GL}_2(k), \mathbb{Z}
\left[\frac{t}{2}\right]) \oplus H_2(\text{GL}_2(k), \mathbb{Z}
\left[\frac{t}{2}\right])$$

when $k$ is an infinite field of characteristic not equal to 2.

It is not at all clear to the author whether this pattern can be expected to persist for higher-dimensional homology.

Remark 11.8. Alternatively, there is a more direct approach to the calculation of $\text{GL}_2$ of Laurent polynomials. The method used for $\text{SL}_2$ above can be adapted, with some care, to the case of $\text{GL}_2$. The group $\text{GL}_2(k[t, t^{-1}])$ also acts on the Serre tree derived from the $t$-adic valuation on $k(t)$. However, unlike the case of $\text{SL}_2(k[t, t^{-1}])$, it contains matrices which invert the edge joining the classes of the lattices $t\mathcal{O} \oplus \mathcal{O}$ and $\mathcal{O} \oplus t\mathcal{O}$. If we pass instead to the barycentric subdivision of the Serre tree, the fundamental domain for the action of $\text{GL}_2(k[t, t^{-1}])$ is then an edge which is a half of the fundamental domain of the action of $\text{SL}_2(k[t, t^{-1}])$. This again leads to an amalgamated product decomposition of $\text{GL}_2(k[t, t^{-1}])$ – where the stabilizers of the vertices of the fundamental edge are no longer isomorphic – and hence to a Mayer-Vietoris sequence. The author is grateful to the referee for clarifying these matters.

References


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