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A NOTE ON MILNOR-WITT K-THEORY AND A THEOREM OF SUSLIN

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Abstract. We give a simple presentation of the additive Milnor-Witt K-theory groups $K_n^{\mathrm{MW}}(F)$ of the field F, for $n \geq 2$, in terms of the natural small set of generators. When $n = 2$, this specialises to a theorem of Suslin which essentially says that $K_2^{\mathrm{MW}}(F) \cong \mathrm{H}_2(\mathrm{Sp}(F), \mathbb{Z})$.

1. Introduction

In [7], Suslin proved that for an infinite field F, $\mathrm{H}_2(\mathrm{Sl}(2, F), \mathbb{Z})$ is isomorphic to the fibre product $K_2^{\mathrm{M}}(F) \times_{I^2/I^3} I^2(F)$, where $K_n^{\mathrm{M}}(F)$ is the n-th Milnor K-group of F and $I = I(F)$ is the ideal of even-dimensional forms in the Witt ring $W(F)$. The proof uses the Matsumoto-Moore presentation of the group $\mathrm{H}_2(\mathrm{Sp}(F), \mathbb{Z}) = \mathrm{H}_2(\mathrm{Sl}(2, F), \mathbb{Z})$ as well as the characterisation of the 2-torsion of $K_2^{\mathrm{M}}(F)$ as the set of all elements of the form $\{-1, a\}$. (More recently, Mazzoleni, [3], has given an alternative proof of this theorem which by-passes the theorem of Matsumoto-Moore.)

More recently, F. Morel has introduced the Milnor-Witt K-theory, $K_*^{\mathrm{MW}}(F)$ ([4], [5]). This is a graded algebra given by a simple presentation, due to Morel and M. Hopkins, from which the following properties are easily deduced: $K_n^{\mathrm{MW}}(F) \cong W(F)$ for all $n < 0$; $K_0^{\mathrm{MW}}(F) \cong GW(F)$, the Grothendieck-Witt ring of isometry classes of quadratic forms over F; there is an element η, of degree -1, such that $K_*^{\mathrm{MW}}(F)/\langle \eta \rangle \cong K_*^{\mathrm{M}}(F)$. The main result about Milnor-Witt K-theory is that it gives an exact description of certain operations in stable motivic homotopy theory ; namely there is a natural isomorphism of graded rings

$$K_*^{\mathrm{MW}}(F) \cong [\mathbb{S}^0, (\mathbb{G}_m)^*]$$

where $\mathbb{S}^0$ is the 'motivic' sphere spectrum, and $[\ ,\ ]$ denotes the group of morphisms in the stable $\mathbb{A}^1$-homotopy category ([4], section 6).

Morel has shown (see [5], for example) that, for all $n \geq 0$,

$$K_n^{\mathrm{MW}}(F) \cong K_n^{\mathrm{M}}(F) \times_{I^n/I^{n+1}} I^n(F).$$

In fact this result is essentially a reformulation of some of the main results of Arason and Elman, [1], on the powers of $I(F)$. Their work, in turn, relies heavily on the work of Voevodsky, Orlov and Vishik on the Milnor conjecture. In view of Morel's result, Suslin's theorem can be re-formulated as the statement that $\mathrm{H}_2(\mathrm{Sl}(2, F), \mathbb{Z}) \cong K_2^{\mathrm{MW}}(F)$, at least when F is infinite. Elsewhere ([6]), Morel has sketched a direct proof of this fact, using the machinery of $\mathbb{A}^1$-homotopy theory.

In this note, which is more elementary in nature than any of the references above, we prove that the Matsumoto-Moore relations give a simple presentation of the additive

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group $K_n^{\mathrm{MW}}(F)$, for all $n \geq 2$, in terms of the natural set of generators. When $n = 2$, this statement specializes to Suslin's theorem, as re-formulated above.

As another application of our main theorem, we give an abstract additive presentation of the group $I^n(F)$ with n-fold Pfister forms as generators. (Corollary 2.16).

2. Milnor-Witt K-theory

Definition 2.1 (Hopkins-Morel, [4])**.** The Milnor-Witt K-theory of the field F is the graded associative ring $K_*^{\mathrm{MW}}(F)$ generated by the symbols $[u]$, $u \in F^\times$, of degree $+1$ and one symbol η of degree -1 subject to the following relations:

1. For each $a \in F^\times \setminus \{1\}$, $[a] \cdot [1-a] = 0$.
2. For each $a, b \in F^\times$, $[ab] = [a] + [b] + \eta[a][b]$.
3. For each $u \in F^\times$, $[u]\eta = \eta[u]$.
4. $\eta^2[-1] + 2\eta = 0$.

The following result is easily deduced ([6], Lemma 2.4):

Lemma 2.2. *For all $n \in \mathbb{Z}$, $K_n^{\mathrm{MW}}(F)$ has the following presentation as an additive group: It is generated by the elements $\eta^m[a_1]\cdots[a_r]$, $m \geq 0$, $r = n + m \geq 0$ subject to the following relations:*

(1) $\eta^m[a_1]\cdots[a_r] = 0$ *if* $r \geq 2$ *and* $a_{i-1} + a_i = 1$ *for some* $i \geq 2$.
(2) $\eta^m[a_1]\cdots[a_{i-1}][a_ia_i']\cdots[a_r] = \eta^m[a_1]\cdots[a_{i-1}][a_i']\cdots[a_r] + \eta^m[a_1]\cdots[a_{i-1}][a_i]\cdots[a_r] + \eta^{m+1}[a_1]\cdots[a_{i-1}][a_i][a_i']\cdots[a_r]$
(3) $\eta^{m+2}[a_1]\cdots[a_{i-1}][-1][a_{i+1}]\cdots[a_{r+2}] + 2\eta^{m+1}[a_1]\cdots[a_{i-1}][a_{i+1}]\cdots[a_{r+2}] = 0$

However, in view of the relation $\eta[a_1][a_2] = [a_1a_2] - [a_1] - [a_2]$, it is clear that $K_n^{\mathrm{MW}}(F)$ can be generated by the elements $[a_1]\cdots[a_n]$ whenever $n \geq 1$. Our main theorem is a presentation of $K_n^{\mathrm{MW}}(F)$ in terms of these generators when $n \geq 2$.

The theorem of Matsumoto and Moore ([2]), for the case of the symplectic group $\mathrm{Sp}(F)$, gives a presentation of the group $\mathrm{H}_2(\mathrm{Sp}(F), \mathbb{Z})$. It has the following form: The generators are symbols $\langle a_1, a_1\rangle$, $a_i \in F^\times$, subject to the relations:

(i) $\langle a_1, a_2\rangle = 0$ if $a_i = 1$ for some i
(ii) $\langle a_1, a_2\rangle = \langle a_2^{-1}, a_1\rangle$
(iii) $\langle a_1, a_2a_2'\rangle + \langle a_2, a_2'\rangle = \langle a_1a_2, a_2'\rangle + \langle a_1, \alpha_2\rangle$
(iv) $\langle a_1, a_2\rangle = \langle a_1, -a_1a_2\rangle$
(v) $\langle a_1, a_2\rangle = \langle a_1, (1-a_1)a_2\rangle$

This motivates the following (provisional) definition:

Definition 2.3. Let $n \geq 2$. For a field F, $K_n^{\mathrm{MM}}(F)$ (MM is for 'Matsumoto-Moore') will denote the additive group which has the following presentation: the generators are $\langle a_1, \ldots, a_n\rangle$, $a_i \in F^\times$ subject to the following relations:

(i) $\langle a_1, \ldots, a_n\rangle = 0$ if $a_i = 1$ for some i
(ii) $\langle a_1, \ldots, a_{i-1}, a_i, \ldots, a_n\rangle = \langle a_1, \ldots, a_i^{-1}, a_{i-1}, \ldots, a_n\rangle$
(iii) $\langle a_1, \ldots, a_{n-1}, a_na_n'\rangle + \langle a_1, \ldots, a_n, a_n'\rangle = \langle a_1, \ldots, a_{n-1}a_n, a_n'\rangle + \langle a_1, \ldots, a_{n-1}, a_n\rangle$
(iv) $\langle a_1, \ldots, a_{n-1}, a_n\rangle = \langle a_1, \ldots, a_{n-1}, -a_{n-1}a_n\rangle$
(v) $\langle a_1, \ldots, a_{n-1}, a_n\rangle = \langle a_1, \ldots, a_{n-1}, (1-a_{n-1})a_n\rangle$

Remark 2.4. In particular, $K_2^{\mathrm{MM}}(F) \cong \mathrm{H}_2(\mathrm{Sp}(F), \mathbb{Z})$ $(= \mathrm{H}_2(\mathrm{Sl}(2, F), \mathbb{Z})$ if F is infinite or at least sufficiently large) by the theorem of Matsumoto-Moore.

Observe that, using relation (ii) together with (iii), (iv) and (v), we easily deduce the following relations in $K_n^{\mathrm{MM}}(F)$:

(*iii*)′ $\langle a_1,\dots,a_{i-1},a_ia_i',\dots,a_n\rangle + \langle a_1,\dots,a_i,a_i',\dots,a_n\rangle = \langle a_1,\dots,a_{i-1}a_i,a_i',\dots,a_n\rangle + \langle a_1,\dots,a_{i-1},a_i,\dots,a_n\rangle$

(*iv*)′ $\langle a_1,\dots,a_{i-1},a_i,\dots,a_n\rangle = \langle a_1,\dots,a_{i-1},-a_{i-1}a_i,\dots,a_n\rangle$

(*v*)′ $\langle a_1,\dots,a_{i-1},a_i,\dots,a_n\rangle = \langle a_1,\dots,a_{i-1},(1-a_{i-1})a_i,\dots,a_n\rangle$

Theorem 2.5. *$K_n^{\mathrm{MW}}(F) \cong K_n^{\mathrm{MM}}(F)$ for all $n \geq 2$ via an isomorphism sending $[a_1]\cdots[a_n]$ to $\langle a_1,\dots,a_n\rangle$.*

Proof. The theorem follows from Lemmas 2.8 and 2.15 below. □

Corollary 2.6. *For all infinite fields F, $K_2^{\mathrm{MW}}(F) \cong \mathrm{H}_2(\mathrm{Sl}(2,F),\mathbb{Z})$.*

Lemma 2.7. *The relations $[a][-a] = 0$ and $[a][b] = [b^{-1}][a]$ hold in $K_*^{\mathrm{MW}}(F)$ for all $a, b \in F^\times$.*

Proof. Using the identity

$$-a(1-a^{-1}) = 1-a$$

together with **2.** gives

$$[-a] = [1-a] - [1-a^{-1}] - \eta[-a][1-a^{-1}].$$

Hence

$$\begin{aligned}
[a][-a] &= -[a](1+\eta[-a])[1-a^{-1}] && \text{(using } \mathbf{1.}\text{)}\\
&= -(1+\eta[-a])[a][1-a^{-1}] && \text{(since } \eta[u][v]=\eta[v][u] \text{ by } \mathbf{2.}\text{)}\\
&= (1+\eta[-a])(1+\eta[a])[a^{-1}][1-a^{-1}]) && \text{(since } [a]=(1+\eta[a])[a^{-1}] \text{ by } \mathbf{2.}\text{)}\\
&= 0 && \text{(by } \mathbf{1.}\text{)}
\end{aligned}$$

We will need the identity

$$\eta([a]+[-a])[-1] = -2[-1]$$

obtained by letting $x = a$ and $x = -a$ in the identity $[x] = [-1] + [-x] + \eta[-x][-1]$ and adding.

Observe also that $[a]^2 = [a][-1]$ for all $a \in F^\times$ (let $x = a$ above and use $[a][-a] = 0$).

Now, for any $a, b \in F^\times$ we have

$$\begin{aligned}
0 &= [ab][-ab]\\
&= ([a]+[b]+\eta[a][b])([-a]+[b]+\eta[-a][b])\\
&= [a][b]+[b][-a]+\eta([a]+[-a])[b]^2+[b]^2\\
&= [a][b]+[b][-a]+\eta([a]+[-a])[-1][b]+[b][-1]\\
&= [a][b]+[b][-a]-[b][-1]\\
&= [a][b]+[b]([-a]-[-1])\\
&= [a][b]+[b][a](1+\eta[-1])
\end{aligned}$$

and hence

$$[a][b] = -[b][a](1+\eta[-1]) = -(1+\eta[-1])[b][a].$$

Thus

$$\begin{aligned}
[a][b]-[b^{-1}][a] &= -\left((1+\eta[-1])[b]+[b^{-1}]\right)[a]\\
&= -([b]+[b^{-1}]+\eta[-1][b])[a]\\
&= -([b]+[b^{-1}]+\eta[b^{-1}][b])[a] = -[1][a] = 0
\end{aligned}$$

(where we have used $[-1][b]=[b^{-1}][b]$ which follows from $[-1]=[-b]+[b^{-1}]+\eta[-b][b^{-1}]$ and $[b][-b]=0$).

□

Lemma 2.8. *Let $n\geq 2$. The map ϕ which sends the element $\langle a_1,\ldots,a_n\rangle$ of $K_n^{\mathrm{MM}}(F)$ to $[a_1]\cdots[a_n]$ in $K_n^{\mathrm{MW}}(F)$ extends uniquely to a well-defined epimorphism of groups.*

Proof. Well-definedness is the issue; we must prove that relations (i)–(v) are preserved by ϕ.

Relation (i): This follows from the identity $[1]=0$ in $K_*^{\mathrm{MW}}(F)$ (since $[1]=2[-1]+\eta[-1]^2$ by **2.** and hence $\eta[1]=0$ by **4.** and thus $[1]=2[1]$by **2.** again).

Relation (ii): This follows immediately from the relation $[a_{i-1}][a_i]=[a_i^{-1}][a_{i-1}]$ (Lemma 2.7).

Relation (iii): We have

$$\begin{aligned}
[a_{n-1}][a_na_n']+[a_n][a_n'] &= [a_{n-1}]([a_n]+[a_n']+\eta[a_n][a_n'])+[a_n][a_n']\\
&= ([a_{n-1}]+[a_n]+\eta[a_{n-1}][a_n])[a_n']+[a_{n-1}][a_n]\\
&= [a_{n-1}a_n][a_n']+[a_{n-1}][a_n]
\end{aligned}$$

Relation (iv): We have $[a_{n-1}][-a_{n-1}a_n]=[a_{n-1}]([-a_{n-1}]+[a_n]+\eta[-a_{n-1}][a_n])=[a_{n-1}][a_n]$ by Lemma 2.7.

Relation (v): Similarly, $[a_{n-1}][(1-a_{n-1})a_n]=[a_{n-1}][a_n]$ using **1.** and **2.**

□

Lemma 2.9. *Let $n\geq 2$. For $a_1,\ldots,a_n,x\in F^\times$ let*

$$\rho_x(\langle a_1,\ldots,a_n\rangle):=\langle a_1,\ldots,a_nx\rangle-\langle a_1,\ldots,a_n\rangle-\langle a_1,\ldots,x\rangle.$$

Then ρ_x extends uniquely to an endomorphism of $K_n^{\mathrm{MM}}(F)$.

Proof. We must prove that ρ_x preserves defining relations (i)-(v).

Relation (i) is clear.

Relation (ii): When $i<n$ in (ii), the result is clear. For the case $i=n$, we find:

$$\begin{aligned}
\rho_x(\langle a_1,\ldots,a_{n-1},a_n\rangle) &= \langle a_1,\ldots,a_{n-1},xa_n\rangle-\langle a_1,\ldots,a_{n-1},a_n\rangle-\langle a_1,\ldots,a_{n-1},x\rangle\\
&= (\langle a_1,\ldots,a_{n-1}x,a_n\rangle+\langle a_1,\ldots,a_{n-1},x\rangle-\langle a_1,\ldots,x,a_n\rangle)\\
&\quad -\langle a_1,\ldots,a_{n-1},a_n\rangle-\langle a_1,\ldots,a_{n-1},x\rangle \text{ (using (iii))}\\
&= \langle a_1,\ldots,a_{n-1}x,a_n\rangle-\langle a_1,\ldots,x,a_n\rangle-\langle a_1,\ldots,a_{n-1},a_n\rangle\\
&= \langle a_1,\ldots,a_n^{-1},a_{n-1}x\rangle-\langle a_1,\ldots,a_n^{-1},a_{n-1}\rangle-\langle a_1,\ldots,a_n^{-1},x\rangle\\
&= \rho_x(\langle a_1,\ldots,a_n^{-1},a_{n-1}\rangle).
\end{aligned}$$

Relation (iii):

$$\begin{aligned}
& \rho_x(\langle a_1,\ldots,a_{n-1},a_na_n'\rangle)+\rho_x(\langle a_1,\ldots,a_n,a_n'\rangle)\\
= & \langle a_1,\ldots,a_{n-1},a_n(a_n'x)\rangle-\langle a_1,\ldots,a_{n-1},x\rangle-\langle a_1,\ldots,a_{n-1},a_na_n'\rangle\\
& +\langle a_1,\ldots,a_n,a_n'x\rangle-\langle a_1,\ldots,a_n,x\rangle-\langle a_1,\ldots,a_n,a_n'\rangle\\
= & (\langle a_1,\ldots,a_{n-1},a_n(a_n'x)\rangle+\langle a_1,\ldots,a_n,a_n'x\rangle)-(\langle a_1,\ldots,a_{n-1},a_na_n'\rangle+\langle a_1,\ldots,a_n,a_n'\rangle)\\
& -\langle a_1,\ldots,a_{n-1},x\rangle-\langle a_1,\ldots,a_n,x\rangle\\
= & (\langle a_1,\ldots,a_{n-1}a_n,a_n'x\rangle+\langle a_1,\ldots,a_{n-1},a_n\rangle)-(\langle a_1,\ldots,a_{n-1}a_n,a_n'\rangle+\langle a_1,\ldots,a_{n-1},a_n\rangle)\\
& -\langle a_1,\ldots,a_{n-1},x\rangle-\langle a_1,\ldots,a_n,x\rangle \qquad \text{(using (iii) again)}\\
= & \langle a_1,\ldots,a_{n-1}a_n,a_n'x\rangle-\langle a_1,\ldots,a_{n-1}a_n,a_n'\rangle-\langle a_1,\ldots,a_{n-1},x\rangle-\langle a_1,\ldots,a_n,x\rangle\\
= & \langle a_1,\ldots,a_{n-1}a_n,a_n'x\rangle-\langle a_1,\ldots,a_{n-1}a_n,a_n'\rangle-\langle a_1,\ldots,a_{n-1},x\rangle\\
& -(\langle a_1,\ldots,a_{n-1}a_n,x\rangle+\langle a_1,\ldots,a_{n-1},a_n\rangle-\langle a_1,\ldots,a_{n-1},a_nx\rangle)\\
= & (\langle a_1,\ldots,a_{n-1}a_n,a_n'x\rangle-\langle a_1,\ldots,a_{n-1}a_n,a_n'\rangle-\langle a_1,\ldots,a_{n-1}a_n,x\rangle)\\
& +(\langle a_1,\ldots,a_{n-1},a_nx\rangle-\langle a_1,\ldots,a_{n-1},x\rangle-\langle a_1,\ldots,a_{n-1},a_n\rangle)\\
= & \rho_x(\langle a_1,\ldots,a_{n-1}a_n,a_n'\rangle)+\rho_x(\langle a_1,\ldots,a_{n-1},a_n\rangle)
\end{aligned}$$

Relations (iv) and (v) are immediate. □

For $n\geq 2$, we will denote the element $\rho_x(\langle a_1,\ldots,a_n\rangle)$ in $K_n^{\mathrm{MM}}(F)$ by $[a_1,\ldots,a_n,x]$.

Lemma 2.10. *Let $n\geq 2$. For any permutation, σ, of $\{1,\ldots,n+1\}$, $[a_1,\ldots,a_{n+1}]=[a_{\sigma(1)},\ldots,a_{\sigma(n+1)}]$.*

Proof. It is immediate from the definition that $[a_1,\ldots,a_n,a_{n+1}]=[a_1,\ldots,a_{n+1},a_n]$; i.e the result is true when σ is the transposition $(n\ n+1)$. From this it follows that

$$\begin{aligned}
[a_1,\ldots,a_{n-1},a_n,x] &= \rho_x(\langle a_1,\ldots,a_n\rangle)\\
&= \rho_{a_n}(\langle a_1,\ldots,a_{n-1},x\rangle)\\
&= \rho_{a_n}(\langle a_1,\ldots,x^{-1},a_{n-1}\rangle)\\
&= [a_1,\ldots,x^{-1},a_{n-1},a_n]=[a_1,\ldots,x^{-1},a_n,a_{n-1}]\\
&= \rho_x(\langle a_1,\ldots,a_n,a_{n-1}\rangle)\\
&= [a_1,\ldots,a_n,a_{n-1},x]
\end{aligned}$$

This fact, together with relation (ii), now implies that

$$\begin{aligned}
[a_1,\ldots,a_{i-1},a_i,\ldots,a_n,x] &= \rho_x(\langle a_1,\ldots,a_{i-1},a_i,\ldots,a_n\rangle)\\
&= \rho_x(\langle a_1,\ldots,a_i,a_{i-1},\ldots,a_n\rangle)=[a_1,\ldots,a_{i-1},a_i,\ldots,a_n,x]
\end{aligned}$$

proving the lemma. □

Lemma 2.11. *For all $x,y\in F^\times$, $\rho_x\rho_y=\rho_y\rho_x$.*

Proof. Let $a_1,\ldots,a_n\in F^\times$. Then

$$\rho_y(\langle a_1,\ldots,a_n\rangle)=[a_1,\ldots,a_n,y] \quad = \quad [y,a_1,\ldots,a_n]$$

and hence

$$\begin{aligned} & \rho_x(\rho_y(\langle a_1,\ldots,a_n\rangle)) = \rho_x([y,a_1,\ldots,a_n]) \\ = \; & \rho_x(\langle y,a_1,\ldots,a_{n-1}a_n\rangle) - \rho_x(\langle y,a_1,\ldots,a_{n-1}\rangle) - \rho_x(\langle y,a_1,\ldots,a_{n-2},a_n\rangle) \\ = \; & [y,\ldots,a_{n-1}a_n,x] - [y,a_1,\ldots,a_{n-1},x] - [y,a_1,\ldots,a_n,x] \\ = \; & [x,\ldots,a_{n-1}a_n,y] - [x,a_1,\ldots,a_{n-1},y] - [x,a_1,\ldots,a_n,y] \\ = \; & \rho_y(\rho_x(\langle a_1,\ldots,a_n\rangle)). \end{aligned}$$

□

More generally, we define elements $[a_1,\ldots,a_r] \in K_n^{\mathrm{MM}}(F)$ $(r > n)$ recursively by the formula

$$[a_1,\ldots,a_{r+1}] := \rho_{a_{r+1}}([a_1,\ldots,a_r]).$$

We will also use the notation $[a_1,\ldots,a_n] := \langle a_1,\ldots,a_n\rangle$ (i.e., when $r = n$).

Corollary 2.12. *Fix $n \geq 2$. For all $r > n$ and for all permutations, σ, of $\{1,\ldots,r\}$*

$$[a_1,\ldots,a_r] = [a_{\sigma(1)},\ldots,a_{\sigma(r)}].$$

Proof. We use induction on r. The case $r = n+1$ has already been proved.

For permutations of $\{1,\ldots,r+1\}$ which fix $r+1$, the result holds by induction since

$$[a_1,\ldots,a_r,a_{r+1}] = \rho_{a_{r+1}}([a_1,\ldots,a_r]).$$

On the other hand, when $r > n$, the result holds for the permutation $\sigma = (r\ r+1)$ since

$$[a_1,\ldots,a_r,a_{r+1}] = \rho_{a_{r+1}}\left(\rho_{a_r}([a_1,\ldots,a_{r-1}])\right) = \rho_{a_r}\left(\rho_{a_{r+1}}([a_1,\ldots,a_{r-1}])\right) = [a_1,\ldots,a_{r+1},a_r].$$

□

Remark 2.13. Observe that it follows that the relations (i)-(v) extend to the symbols $[a_1,\ldots,a_r]$ $(r \geq n)$ since we can always permute the key entries to before the n-th position and then use the fact that $[a_1,\ldots,a_r] = \phi(\langle a_1,\ldots,a_n\rangle)$ for an appropriate endomorphism ϕ. Furthermore, property (ii) and symmetry (Corollary 2.12) imply that $[a_1,\ldots,a_i,\ldots,a_r] = [a_1,\ldots,a_i^{-1},\ldots,a_r]$ for any i.

Corollary 2.14. *Let $n \geq 2$. If $r > n$ and if $a_1,\ldots,a_r \in F^\times$ then $[a_1,\ldots,a_r] = 0$ if a_i is a square in $F^\times$ for some $i \leq r$.*

Proof. By symmetry we can suppose that $i > 1$. Suppose that $a_i = b_i^2$. We thus have

$$\begin{aligned} [a_1,\ldots,a_{i-1},b_i^2,\ldots] &= [a_1,\ldots,a_{i-1}b_i,b_i,\ldots] + [a_1,\ldots,a_{i-1},b_i,\ldots] - [a_1,\ldots,b_i,b_i,\ldots] \\ &= [a_1,\ldots,a_{i-1}b_i,b_i^{-1},\ldots] + [a_1,\ldots,a_{i-1},b_i,\ldots] - [a_1,\ldots,b_i,b_i^{-1},\ldots] \\ &= [a_1,\ldots,a_{i-1},b_i\cdot b_i^{-1},\ldots] = [a_1,\ldots,a_{i-1},1,\ldots] = 0. \end{aligned}$$

□

Lemma 2.15. *Let $n \geq 2$. There is a unique epimorphism $\lambda : K_n^{\mathrm{MW}}(F) \to K_n^{\mathrm{MM}}(F)$ satisfying*

$$\lambda(\eta^m[a_1]\cdots[a_r]) = [a_1,\ldots,a_r] \qquad (r = n+m)$$

Proof. We must show that λ preserves relations (1)-(3) of Lemma 2.2.

Relation (1) follows from (i) and (v) (see Remark 2.13).

Relation (2): We must prove that, for $r \geq n$ and $i \leq r$,

$$[a_1, \ldots, a_i a_i', \ldots, a_r] = [a_1, \ldots, a_i, \ldots, a_r] + [a_1, \ldots, a_i', \ldots, a_r] + [a_1, \ldots, a_i, a_i', \ldots, a_r].$$

By symmetry we can assume $i \leq n$ and thus we reduce to the key case $r = n$; i.e. we must prove

$$\langle a_1, \ldots, a_i a_i', \ldots, a_n \rangle = \langle a_1, \ldots, a_i, \ldots, a_n \rangle + \langle a_1, \ldots, a_i', \ldots, a_n \rangle + [a_1, \ldots, a_i, a_i', \ldots, a_n].$$

By property (ii), and by symmetry, we can assume $i = n$. The identity is now just the definition of $[a_1, \ldots, a_n, a_n']$.

Relation (3): We must prove that for $r \geq n$

$$[a_1, \ldots, a_{i-1}, -1, a_{i+1}, \ldots, a_{r+2}] = -2[a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_{r+2}].$$

By symmetry, we can suppose that $r = n$ and $i = n + 1$. So we must show that $[a_1, \ldots, a_n, -1, a_{n+2}] = -2[a_1, \ldots, a_n, a_{n+2}]$. Now

$$\begin{aligned} [a_1, \ldots, a_n, -1] &= [a_1, \ldots, a_n, a_n] \qquad \text{(by (iv))} \\ &= \langle a_1, \ldots, a_n^2 \rangle - 2\langle a_1, \ldots, a_n \rangle \end{aligned}$$

and thus

$$[a_1, \ldots, a_n, -1, a_{n+2}] \;=\; [a_1, \ldots, a_n^2, a_{n+2}] - 2[a_1, \ldots, a_n, a_{n+2}] = -2[a_1, \ldots, a_n, a_{n+2}]$$

by Corollary 2.14. □

As an application, we derive a simple additive presentation of the ideals $I^n(F)$, $n \geq 2$, in the Witt Ring of a field F :

Corollary 2.16. *For any field F, let $I(F)$ be the ideal of even-dimensional forms in the Witt Ring, $W(F)$, of the field F. As an additive group, $I^n(F) = I(F)^n$ has the following abstract presentation:*

It is generated by the classes of Pfister forms $<< a_1, \ldots, a_n >>$, $a_i \in F^\times$ subject to the following relations:

(i) $<< a_1, \ldots, a_n >>= 0$ *if a_i is a square for some i*
(ii) $<< a_1, \ldots, a_{i-1}, a_i, \ldots, a_n >>=<< a_1, \ldots, a_i, a_{i-1}, \ldots, a_n >>$
(iii) $<< a_1, \ldots, a_{n-1}, a_n a_n' >> + << a_1, \ldots, a_n, a_n' >>=<< a_1, \ldots, a_{n-1} a_n, a_n' >> + << a_1, \ldots, a_{n-1}, a_n >>$
(iv) $<< a_1, \ldots, a_{n-1}, a_n >>=<< a_1, \ldots, a_{n-1}, (1 - a_{n-1}) a_n >>$

Proof. Morel's theorem ([5], Théorème 5.3), shows that there is an exact sequence

$$0 \to K_n^{\mathrm{M}}(F)^2 \to K_n^{\mathrm{MW}}(F) \to I^n(F) \to 0$$

where the first (nontrivial) homomorphism maps $\{a_1, \ldots, a_n\}^2 = \{a_1, \ldots, a_i^2, \ldots, a_n\}$ to $[a_1] \cdots [a_i^2] \cdots [a_n]$ (for any i) and the next homomorphism sends $[a_1] \cdots [a_n]$ to the class of the Pfister form $<< a_1, \ldots, a_n >>$. Combining this with Theorem 2.5 give the result, since the identity $-a = (1 - a)/(1 - a^{-1})$ shows that (i) and (iv) imply the identity $<< a_1, \ldots, a_{n-1}, a_n >>=<< a_1, \ldots, a_{n-1}, -a_{n-1} a_n >>$. □

Remark 2.17. Compare this with the presentation of $I^n(F)$ given by Arason and Elman ([1], Theorem 3.1). Of course, Corollary 2.16 – like the result of Arason and Elman - requires the proof of the Milnor conjecture (since it is needed for Morel's theorem), and conversely easily implies the Milnor conjecture.

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