A NOTE ON MILNOR-WITT $K$-THEORY AND A THEOREM OF SUSLIN

KEVIN HUTCHINSON, LIQUN TAO

Abstract. We give a simple presentation of the additive Milnor-Witt $K$-theory groups $K_n^{MW}(F)$ of the field $F$, for $n \geq 2$, in terms of the natural small set of generators. When $n = 2$, this specialises to a theorem of Suslin which essentially says that $K_2^{MW}(F) \cong H_2(Sp(F), \mathbb{Z})$.

1. Introduction

In [7], Suslin proved that for an infinite field $F$, $H_2(\text{Sl}(2,F), \mathbb{Z})$ is isomorphic to the fibre product $K_2^M(F) \times_{I_2/F} I_2(F)$, where $K_n^M(F)$ is the $n$-th Milnor $K$-group of $F$ and $I = I(F)$ is the ideal of even-dimensional forms in the Witt ring $W(F)$. The proof uses the Matsumoto-Moore presentation of the group $H_2(\text{Sp}(F), \mathbb{Z}) = H_2(\text{Sl}(2,F), \mathbb{Z})$ as well as the characterisation of the 2-torsion of $K_2^M(F)$ as the set of all elements of the form $\{-1, a\}$. (More recently, Mazzoleni, [3], has given an alternative proof of this theorem which by-passes the theorem of Matsumoto-Moore.)

More recently, F. Morel has introduced the Milnor-Witt $K$-theory, $K_*^{MW}(F)$ ([4], [5]). This is a graded algebra given by a simple presentation, due to Morel and M. Hopkins, from which the following properties are easily deduced: $K_n^{MW}(F) \cong W(F)$ for all $n < 0$; $K_0^{MW}(F) \cong GW(F)$, the Grothendieck-Witt ring of isometry classes of quadratic forms over $F$; there is an element $\eta$, of degree $-1$, such that $K_*^{MW}(F)/\langle \eta \rangle \cong K_*^M(F)$. The main result about Milnor-Witt $K$-theory is that it gives an exact description of certain operations in stable motivic homotopy theory; namely there is a natural isomorphism of graded rings

$$K_*^{MW}(F) \cong [S^0, (\mathbb{G}_m)^*]$$

where $S^0$ is the ‘motivic’ sphere spectrum, and $[\ ,\ ]$ denotes the group of morphisms in the stable $\mathbb{A}^1$-homotopy category ([4], section 6).

Morel has shown (see [5], for example) that, for all $n \geq 0$,

$$K_n^{MW}(F) \cong K_n^M(F) \times_{I^n/I^{n+1}} I^n(F).$$

In fact this result is essentially a reformulation of some of the main results of Arason and Elman, [1], on the powers of $I(F)$. Their work, in turn, relies heavily on the work of Voevodsky, Orlov and Vishik on the Milnor conjecture. In view of Morel’s result, Suslin’s theorem can be re-formulated as the statement that $H_2(\text{Sl}(2,F), \mathbb{Z}) \cong K_2^{MW}(F)$, at least when $F$ is infinite. Elsewhere ([6]), Morel has sketched a direct proof of this fact, using the machinery of $\mathbb{A}^1$-homotopy theory.

In this note, which is more elementary in nature than any of the references above, we prove that the Matsumoto-Moore relations give a simple presentation of the additive

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group \(K_n^{\text{MW}}(F)\), for all \(n \geq 2\), in terms of the natural set of generators. When \(n = 2\), this statement specializes to Suslin’s theorem, as re-formulated above.

As another application of our main theorem, we give an abstract additive presentation of the group \(I^n(F)\) with \(n\)-fold Pfister forms as generators. (Corollary 2.16).

2. Milnor-Witt \(K\)-theory

**Definition 2.1** (Hopkins-Morel, [4]). The Milnor-Witt \(K\)-theory of the field \(F\) is the graded associative ring \(K_n^{\text{MW}}(F)\) generated by the symbols \([u], u \in F^\times\), of degree +1 and one symbol \(\eta\) of degree −1 subject to the following relations:

1. For each \(a \in F^\times \setminus \{1\}\), \([a] \cdot [1 - a] = 0\).
2. For each \(a, b \in F^\times\), \([ab] = [a] + [b] + \eta[a][b]\).
3. For each \(u \in F^\times\), \([u]\eta = \eta[u]\).
4. \(\eta^2[-1] + 2\eta = 0\).

The following result is easily deduced ([6], Lemma 2.4):

**Lemma 2.2.** For all \(n \in \mathbb{Z}\), \(K_n^{\text{MW}}(F)\) has the following presentation as an additive group:

It is generated by the elements \(\eta^m[a_1] \cdots [a_r], m \geq 0, r = n + m \geq 0\) subject to the following relations:

1. \(\eta^m[a_1] \cdots [a_r] = 0\) if \(r \geq 2\) and \(a_{i-1} + a_i = 1\) for some \(i \geq 2\).
2. \(\eta^m[a_1] \cdots [a_{i-1}][a_i][a_{i-1}] \cdots [a_r] + \eta^m[a_1] \cdots [a_{i-1}][a_i] \cdots [a_r] + \eta^m[1][a_i][a_{i-1}] \cdots [a_r] = 0\).
3. \(\eta^{m+1}[a_1] \cdots [a_{i-1}][a_i] \cdots [a_{i+1}] \cdots [a_r] + 2\eta^{m+1}[a_1] \cdots [a_{i-1}][a_i] \cdots [a_r] = 0\).

However, in view of the relation \(\eta[a_1][a_2] = [a_1a_2] - [a_1] - [a_2]\), it is clear that \(K_n^{\text{MW}}(F)\) can be generated by the elements \([a_1] \cdots [a_n]\) whenever \(n \geq 1\). Our main theorem is a presentation of \(K_n^{\text{MW}}(F)\) in terms of these generators when \(n \geq 2\).

The theorem of Matsumoto and Moore ([2]), for the case of the symplectic group \(\text{Sp}(F)\), gives a presentation of the group \(H_2(\text{Sp}(F), \mathbb{Z})\). It has the following form: The generators are symbols \(\langle a_1, a_2 \rangle\), \(a_i \in F^\times\), subject to the relations:

(i) \(\langle a_1, a_2 \rangle = 0\) if \(a_i = 1\) for some \(i\)
(ii) \(\langle a_1, a_2 \rangle = \langle a_2^{-1}, a_1 \rangle\)
(iii) \(\langle a_1, a_2a_2' \rangle + \langle a_2, a_2' \rangle = \langle a_1a_2, a_2' \rangle + \langle a_1, a_2 \rangle\)
(iv) \(\langle a_1, a_2 \rangle = \langle a_1, -a_1a_2 \rangle\)
(v) \(\langle a_1, a_2 \rangle = \langle a_1, (1 - a_1)a_2 \rangle\)

This motivates the following (provisional) definition:

**Definition 2.3.** Let \(n \geq 2\). For a field \(F\), \(K_n^{\text{MM}}(F)\) (MM is for ‘Matsumoto-Moore’) will denote the additive group which has the following presentation: the generators are \(\langle a_1, \ldots, a_n \rangle, a_i \in F^\times\) subject to the following relations:

(i) \(\langle a_1, \ldots, a_n \rangle = 0\) if \(a_i = 1\) for some \(i\)
(ii) \(\langle a_1, \ldots, a_{i-1}, a_i, \ldots, a_n \rangle = \langle a_1, \ldots, a_{i-1}, a_i^{-1}, a_i, \ldots, a_n \rangle\)
(iii) \(\langle a_1, \ldots, a_{n-1}, a_na_n' \rangle + \langle a_1, \ldots, a_{n-1}, a_n' \rangle = \langle a_1, \ldots, a_{n-1}a_n, a_n' \rangle + \langle a_1, \ldots, a_{n-1}, a_n \rangle\)
(iv) \(\langle a_1, \ldots, a_{n-1}, a_n \rangle = \langle a_1, \ldots, a_{n-1}, -a_{n-1}a_n \rangle\)
(v) \(\langle a_1, \ldots, a_{n-1}, a_n \rangle = \langle a_1, \ldots, a_{n-1}, (1 - a_{n-1})a_n \rangle\)

**Remark 2.4.** In particular, \(K_2^{\text{MM}}(F) \cong H_2(\text{Sp}(F), \mathbb{Z}) = H_2(\text{Sl}(2, F), \mathbb{Z})\) if \(F\) is infinite or at least sufficiently large) by the theorem of Matsumoto-Moore.
Observe that, using relation (ii) together with (iii), (iv) and (v), we easily deduce the following relations in \( K_{\text{MM}}^n(F) \):

\[
(iii)' \langle a_1, \ldots, a_{i-1}, a_i a_i', \ldots, a_n \rangle + \langle a_1, \ldots, a_i, a_i', \ldots, a_n \rangle = \langle a_1, \ldots, a_{i-1} a_i a_i', \ldots, a_n \rangle + \langle a_1, \ldots, a_{i-1}, a_i, \ldots, a_n \rangle
\]
\[
(iv)' \langle a_1, \ldots, a_{i-1}, a_i, \ldots, a_n \rangle = \langle a_1, \ldots, a_{i-1}, -a_{i-1} a_i, \ldots, a_n \rangle
\]
\[
(v)' \langle a_1, \ldots, a_{i-1}, a_i, \ldots, a_n \rangle = \langle a_1, \ldots, a_{i-1}, (1 - a_{i-1}) a_i, \ldots, a_n \rangle
\]

**Theorem 2.5.** \( K_{\text{MW}}^n(F) \cong K_{\text{MM}}^n(F) \) for all \( n \geq 2 \) via an isomorphism sending \([a_1] \cdots [a_n]\) to \( \langle a_1, \ldots, a_n \rangle \).

**Proof.** The theorem follows from Lemmas 2.8 and 2.15 below.

\[
\square
\]

**Corollary 2.6.** For all infinite fields \( F \), \( K_{\text{MW}}^2(F) \cong H_2(\text{Sl}(2, F), \mathbb{Z}) \).

**Lemma 2.7.** The relations \([a][-a] = 0\) and \([a][b] = [b^{-1}][a]\) hold in \( K_*^\text{MW}(F) \) for all \( a, b \in F^\times \).

**Proof.** Using the identity

\[-a(1 - a^{-1}) = 1 - a\]

together with 2. gives

\([a][-a] = [1 - a] - [1 - a^{-1}] - \eta([-a][1 - a^{-1}]).\]

Hence

\[
[a][-a] = -[a](1 + \eta([-a])[1 - a^{-1}] \quad \text{(using 1.)}
\]
\[
= -(1 + \eta([-a])[a][1 - a^{-1}] \quad \text{(since } \eta(u)[v] = \eta[v][u] \text{ by 2.)}
\]
\[
= (1 + \eta([-a])(1 + \eta[a])[a^{-1}][1 - a^{-1}] \quad \text{(since } [a] = (1 + \eta[a])[a^{-1}] \text{ by 2.)}
\]
\[
= 0 \quad \text{(by 1.)}
\]

We will need the identity

\[
\eta([a] + [-a])[-1] = -2[-1]
\]

obtained by letting \( x = a \) and \( x = -a \) in the identity \([x] = [-1] + [-x] + \eta[-x][-1]\) and adding.

Observe also that \([a]^2 = [a][-1]\) for all \( a \in F^\times \) (let \( x = a \) above and use \([a][-a] = 0\)).

Now, for any \( a, b \in F^\times \) we have

\[
0 = [ab][-ab]
\]
\[
= ([a] + [b] + \eta[a][b])([-a] + [b] + \eta[-a][b])
\]
\[
= [a][b] + [b][-a] + \eta([a] + [-a])[b] + [b]^2
\]
\[
= [a][b] + [b][-a] + \eta([a] + [-a])[1][b] + [b][1]
\]
\[
= [a][b] + [b][-a] - [b][-1]
\]
\[
= [a][b] + [b]([-a] - [-1])
\]
\[
= [a][b] + [b][a](1 + \eta[-1])
\]

and hence

\[
[a][b] = -[b][a](1 + \eta[-1]) = -(1 + \eta[-1])[b][a].
\]
Thus
\[
[a][b] - [b^{-1}][a] = -((1 + \eta[-1])[b] + [b^{-1}])[a]
\]
\[
= -([b] + [b^{-1}] + \eta[-1][b])[a]
\]
\[
= -([b] + [b^{-1}] + \eta[b^{-1}][b])[a] = -[1][a] = 0
\]
(where we have used \([-1][b] = [b^{-1}][b]\) which follows from \([-1] = [-b] + [b^{-1}] + \eta[-b][b^{-1}]
and \([b][b] = 0\)).

\[\square\]

**Lemma 2.8.** Let \(n \geq 2\). The map \(\phi\) which sends the element \(\langle a_1, \ldots, a_n \rangle\) of \(K_n^{\text{MM}}(F)\) to
\([a_1] \cdots [a_n]\) in \(K_n^{\text{MW}}(F)\) extends uniquely to a well-defined epimorphism of groups.

**Proof.** Well-definedness is the issue; we must prove that relations (i)–(v) are preserved by \(\phi\).
Relation (i): This follows from the identity \([1] = 0\) in \(K_n^{\text{MW}}(F)\) (since \([1] = 2[-1] + \eta[-1]^2\)
by 2. and hence \(\eta[1] = 0\) by 4. and thus \([1] = 2[1]\) by 2. again).
Relation (ii): This follows immediately from the relation \([a_{i-1}][a_i] = [a_i^{-1}][a_{i-1}]\) (Lemma 2.7).
Relation (iii): We have
\[
[a_{n-1}][a_na'_n] + [a_n][a'_n] = [a_{n-1}][(a_n) + [a'_n] + \eta[a_n][a'_n]] + [a_n][a'_n]
\]
\[
= ([a_{n-1}] + [a_n] + \eta[a_{n-1}][a_n])[a'_n] + [a_{n-1}][a_n]
\]
\[
= [a_{n-1}a_n][a'_n] + [a_{n-1}][a_n]
\]
Relation (iv): We have \([a_{n-1}][-a_{n-1}a_n] = [a_{n-1}][(-a_{n-1}) + [a_n] + \eta[-a_{n-1}][a_n]] = [a_{n-1}][a_n]\)
by Lemma 2.7.
Relation (v): Similarly, \([a_{n-1}][(1 - a_{n-1})a_n] = [a_{n-1}][a_n]\) using 1. and 2.

\[\square\]

**Lemma 2.9.** Let \(n \geq 2\). For \(a_1, \ldots, a_n, x \in F^\times\) let
\[
\rho_x(\langle a_1, \ldots, a_n \rangle) := \langle a_1, \ldots, a_n x \rangle - \langle a_1, \ldots, a_n \rangle - \langle a_1, \ldots, x \rangle.
\]
Then \(\rho_x\) extends uniquely to an endomorphism of \(K_n^{\text{MM}}(F)\).

**Proof.** We must prove that \(\rho_x\) preserves defining relations (i)-(v).
Relation (i) is clear.
Relation (ii): When \(i < n\) in (ii), the result is clear. For the case \(i = n\), we find:
\[
\rho_x(\langle a_1, \ldots, a_{n-1}, a_n \rangle) = \langle a_1, \ldots, a_{n-1}, x a_n \rangle - \langle a_1, \ldots, a_{n-1}, a_n \rangle - \langle a_1, \ldots, a_{n-1}, x \rangle
\]
\[
= \langle a_1, \ldots, a_{n-1}, x a_n \rangle + \langle a_1, \ldots, a_{n-1}, x \rangle - \langle a_1, \ldots, x, a_n \rangle
\]
\[
- \langle a_1, \ldots, a_{n-1}, a_n \rangle - \langle a_1, \ldots, a_{n-1}, x \rangle (\text{using (iii)})
\]
\[
= \langle a_1, \ldots, a_{n-1}, x a_n \rangle - \langle a_1, \ldots, x, a_n \rangle - \langle a_1, \ldots, a_{n-1}, a_n \rangle
\]
\[
= \langle a_1, \ldots, a_{n-1}, x a_n \rangle - \langle a_1, \ldots, a_{n-1}, x \rangle - \langle a_1, \ldots, a_{n-1}, a_n \rangle
\]
\[
= \rho_x(\langle a_1, \ldots, a_{n-1}, a_n \rangle).
\]
Lemma 2.10.

Let $\sigma$ be a permutation of $\{1, \ldots, n\}$. This fact, together with relation (ii), now implies that

$$\rho_x((a_1, \ldots, a_{n-1}, a_n')) + \rho_x((a_1, \ldots, a_n, a')_n)$$

$$= \langle a_1, \ldots, a_{n-1}, a_n'x \rangle - \langle a_1, \ldots, a_{n-1}, x \rangle - \langle a_1, \ldots, a_{n-1}, a_n' \rangle$$

$$+ (a_1, \ldots, a_n, a'_n) - \langle a_1, \ldots, a_n, x \rangle - \langle a_1, \ldots, a_n, a'_n \rangle$$

$$= \langle 1, \ldots, a_{n-1}, a_n(a_n'x) + a_1, \ldots, a_n'x \rangle - \langle 1, \ldots, a_{n-1}, a_n' \rangle + \langle a_1, \ldots, a_n, a'_n \rangle$$

$$- \langle a_1, \ldots, a_{n-1}, x \rangle - \langle a_1, \ldots, a_n, x \rangle$$

Relations (iv) and (v) are immediate.

For $n \geq 2$, we will denote the element $\rho_x((a_1, \ldots, a_n))$ in $K^{\text{MM}}_n(F)$ by $[a_1, \ldots, a_n, x]$.

Lemma 2.10. Let $n \geq 2$. For any permutation, $\sigma$, of $\{1, \ldots, n+1\}$, $[a_1, \ldots, a_{n+1}] = [a_{\sigma(1)}, \ldots, a_{\sigma(n+1)}]$.

Proof. It is immediate from the definition that $[a_1, \ldots, a_n, a_{n+1}] = [a_1, \ldots, a_{n+1}, a_n]$; i.e., this result is true when $\sigma$ is the transposition $(n \ n+1)$. From this it follows that

$$[a_1, \ldots, a_{n-1}, a_n, x] = \rho_x((a_1, \ldots, a_n))$$

$$= \rho_{a_n}((a_1, \ldots, a_{n-1}, x))$$

$$= \rho_{a_n}((a_1, \ldots, x^{-1}, a_{n-1}))$$

$$= [a_1, \ldots, x^{-1}, a_{n-1}, a_n] = [a_1, \ldots, x^{-1}, a_n, a_{n-1}]$$

$$= \rho_x((a_1, \ldots, a_n, a_{n-1}))$$

$$= [a_1, \ldots, a_n, a_{n-1}, x]$$

This fact, together with relation (ii), now implies that

$$[a_1, \ldots, a_i-1, a_i, \ldots, a_n, x] = \rho_x((a_1, \ldots, a_i-1, a_i, \ldots, a_n))$$

$$= \rho_x((a_1, \ldots, a_i, a_{i-1}, \ldots, a_n)) = [a_1, \ldots, a_{i-1}, a_i, \ldots, a_n, x]$$

proving the lemma.

Lemma 2.11. For all $x, y \in F^\times$, $\rho_x \rho_y = \rho_y \rho_x$.

Proof. Let $a_1, \ldots, a_n \in F^\times$. Then

$$\rho_y((a_1, \ldots, a_n)) = [a_1, \ldots, a_n, y] = [y, a_1, \ldots, a_n]$$
and hence
\[
\rho_x(\rho_y(\langle a_1, \ldots, a_n \rangle)) = \rho_x(\langle y, a_1, \ldots, a_n \rangle)
\]
\[
= \rho_x(\langle y, a_1, \ldots, a_{n-1}, a_n \rangle) - \rho_x(\langle y, a_1, \ldots, a_{n-2}, a_n \rangle) = [y, a_1, \ldots, a_{n-1}, x] - [y, a_1, \ldots, a_{n-1}, x - [y, a_1, \ldots, a_n, x]
\]
\[
= [x, a_1, \ldots, a_{n-1}, y] - [x, a_1, \ldots, a_{n-1}, y] - [x, a_1, \ldots, a_n, y]
\]
\[
= \rho_y(\rho_x(\langle a_1, \ldots, a_n \rangle)).
\]

\[\square\]

More generally, we define elements \([a_1, \ldots, a_r] \in K_n^{MM}(F) (r > n)\) recursively by the formula

\[[a_1, \ldots, a_r] := \rho_{a_{r+1}}([a_1, \ldots, a_r]).\]

We will also use the notation \([a_1, \ldots, a_n] := \langle a_1, \ldots, a_n \rangle\) (i.e., when \(r = n\)).

**Corollary 2.12.** Fix \(n \geq 2\). For all \(r > n\) and for all permutations, \(\sigma\), of \(\{1, \ldots, r\}\)

\[[a_1, \ldots, a_r] = [a_{\sigma(1)}, \ldots, a_{\sigma(r)}].\]

**Proof.** We use induction on \(r\). The case \(r = n + 1\) has already been proved.

For permutations of \(\{1, \ldots, r+1\}\) which fix \(r+1\), the result holds by induction since

\[[a_1, \ldots, a_r, a_{r+1}] = \rho_{a_{r+1}}([a_1, \ldots, a_r]).\]

On the other hand, when \(r > n\), the result holds for the permutation \(\sigma = (r+1)\) since

\[[a_1, \ldots, a_r, a_{r+1}] = \rho_{a_{r+1}}(\rho_{a_{r}}([a_1, \ldots, a_{r-1}])) = \rho_{a_r}(\rho_{a_{r+1}}([a_1, \ldots, a_{r-1}])) = [a_1, \ldots, a_{r+1}, a_r].\]

\[\square\]

**Remark 2.13.** Observe that it follows that the relations (i)-(v) extend to the symbols \([a_1, \ldots, a_r] (r \geq n)\) since we can always permute the key entries to before the \(n\)-th position and then use the fact that \([a_1, \ldots, a_r] = \phi(\langle a_1, \ldots, a_n \rangle)\) for an appropriate endomorphism \(\phi\). Furthermore, property (ii) and symmetry (Corollary 2.12) imply that

\([a_1, \ldots, a_i, \ldots, a_r] = [a_1, \ldots, a_i^{-1}, \ldots, a_r]\) for any \(i\).

**Corollary 2.14.** Let \(n \geq 2\). If \(r > n\) and if \(a_1, \ldots, a_r \in F^\times\) then \([a_1, \ldots, a_r] = 0\) if \(a_i\) is a square in \(F^\times\) for some \(i \leq r\).

**Proof.** By symmetry we can suppose that \(i > 1\). Suppose that \(a_i = b_i^2\). We thus have

\[
[a_1, \ldots, a_{i-1}, b_i^2, \ldots] = [a_1, \ldots, a_{i-1}, b_i, b_i, \ldots] + [a_1, \ldots, a_{i-1}, b_i, \ldots] - [a_1, \ldots, b_i, b_i, \ldots]
\]
\[
= [a_1, \ldots, a_{i-1}, b_i, b_i^{-1}, \ldots] + [a_1, \ldots, a_{i-1}, b_i, \ldots] - [a_1, \ldots, b_i, b_i^{-1}, \ldots]
\]
\[
= [a_1, \ldots, a_{i-1}, b_i^{-1}, \ldots] = [a_1, \ldots, a_{i-1}, 1, \ldots] = 0.
\]

\[\square\]

**Lemma 2.15.** Let \(n \geq 2\). There is a unique epimorphism \(\lambda : K_n^{MW}(F) \to K_n^{MM}(F)\) satisfying

\[\lambda(\eta^m[a_1] \cdots [a_r]) = [a_1, \ldots, a_r] \quad (r = n + m)\]
Proof. We must show that \( \lambda \) preserves relations (1)-(3) of Lemma 2.2.

Relation (1) follows from (i) and (v) (see Remark 2.13).

Relation (2): We must prove that, for \( r \geq n \) and \( i \leq r \),
\[
[a_1, \ldots, a_i a'_i, \ldots, a_r] = [a_1, \ldots, a_i, \ldots, a_r] + [a_1, \ldots, a'_i, \ldots, a_r] + [a_1, \ldots, a_i, a'_i, \ldots, a_r].
\]
By symmetry we can assume \( i \leq n \) and thus we reduce to the key case \( r = n \); i.e. we must prove
\[
\langle a_1, \ldots, a_i a'_i, \ldots, a_n \rangle = \langle a_1, \ldots, a_i, \ldots, a_n \rangle + \langle a_1, \ldots, a'_i, \ldots, a_n \rangle + \langle a_1, \ldots, a_i, a'_i, \ldots, a_n \rangle.
\]
By property (ii), and by symmetry, we can assume \( i = n \). The identity is now just the definition of \( [a_1, \ldots, a_n, a'_n] \).

Relation (3): We must prove that for \( r \geq n \)
\[
[a_1, \ldots, a_{i-1}, -1, a_{i+1}, \ldots, a_{r+2}] = -2[a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_{r+2}].
\]
By symmetry, we can suppose that \( r = n \) and \( i = n + 1 \). So we must show that
\[
[a_1, \ldots, a_n, -1, a_{n+2}] = -2[a_1, \ldots, a_n, a_{n+2}].
\]
Now
\[
[a_1, \ldots, a_n, -1] = [a_1, \ldots, a_n, a_n] \quad \text{(by (iv))}
\]
and thus
\[
[a_1, \ldots, a_n, -1, a_{n+2}] = [a_1, \ldots, a_n^2,a_{n+2}] - 2[a_1, \ldots, a_n, a_{n+2}] = -2[a_1, \ldots, a_n, a_{n+2}]
\]
by Corollary 2.14. \( \square \)

As an application, we derive a simple additive presentation of the ideals \( I^n(F) \), \( n \geq 2 \), in the Witt Ring of a field \( F \):

**Corollary 2.16.** For any field \( F \), let \( I(F) \) be the ideal of even-dimensional forms in the Witt Ring, \( W(F) \), of the field \( F \). As an additive group, \( I^n(F) = I(F)^n \) has the following abstract presentation:

It is generated by the classes of Pfister forms \( \langle< a_1, \ldots, a_n >\rangle \), \( a_i \in F^\times \) subject to the following relations:

(i) \( \langle< a_1, \ldots, a_n >\rangle = 0 \) if \( a_i \) is a square for some \( i \)

(ii) \( \langle< a_1, \ldots, a_{i-1}, a_i, \ldots, a_n >\rangle = \langle< a_1, \ldots, a_i, a_i, a_{i-1}, \ldots, a_n >\rangle \)

(iii) \( \langle< a_1, \ldots, a_{n-1}, a_n a'_n >\rangle + \langle< a_1, \ldots, a_n, a'_n >\rangle = \langle< a_1, \ldots, a_{n-1} a_n, a'_n >\rangle + \langle< a_1, \ldots, a_{n-1}, a_n >\rangle \)

(iv) \( \langle< a_1, \ldots, a_{n-1}, a_n >\rangle = \langle< a_1, \ldots, a_{n-1}, (1 - a_{n-1}) a_n >\rangle \)

Proof. Morel’s theorem ([5], Théorème 5.3), shows that there is an exact sequence
\[
0 \to K^M_n(F)^2 \to K^MW_n(F) \to I^n(F) \to 0
\]
where the first (nontrivial) homomorphism maps \( \{a_1, \ldots, a_n\}^2 = \{a_1, \ldots, a_i^2, \ldots, a_n\} \) to \( [a_1] \cdots [a_i^2] \cdots [a_n] \) (for any \( i \)) and the next homomorphism sends \( [a_1] \cdots [a_n] \) to the class of the Pfister form \( \langle< a_1, \ldots, a_n >\rangle \). Combining this with Theorem 2.5 give the result, since the identity \( -a = (1 - a)/(1 - a^{-1}) \) shows that (i) and (iv) imply the identity \( \langle< a_1, \ldots, a_{n-1}, a_n >\rangle = \langle< a_1, \ldots, a_{n-1}, -a_{n-1} a_n >\rangle \).

\( \square \)

**Remark 2.17.** Compare this with the presentation of \( I^n(F) \) given by Arason and Elman ([1], Theorem 3.1). Of course, Corollary 2.16 – like the result of Arason and Elman – requires the proof of the Milnor conjecture (since it is needed for Morel’s theorem), and conversely easily implies the Milnor conjecture.
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References


School of Mathematical Sciences, University College Dublin

E-mail address: kevin.hutchinson@ucd.ie, lqtao@ucd.ie