A NOTE ON MILNOR-WITT $K$-THEORY AND A THEOREM OF SUSLIN

KEVIN HUTCHINSON, LIQUN TAO

Abstract. We give a simple presentation of the additive Milnor-Witt $K$-theory groups $K_{MW}^n(F)$ of the field $F$, for $n \geq 2$, in terms of the natural small set of generators. When $n = 2$, this specialises to a theorem of Suslin which essentially says that $K_{MW}^2(F) \cong H_2(\text{Sp}(F), \mathbb{Z})$.

1. Introduction

In [7], Suslin proved that for an infinite field $F$, $H_2(\text{Sl}(2, F), \mathbb{Z})$ is isomorphic to the fibre product $K_{MW}^2(F) \times_{I^2/F} I^2(F)$, where $K_{MW}^n(F)$ is the $n$-th Milnor $K$-group of $F$ and $I = I(F)$ is the ideal of even-dimensional forms in the Witt ring $W(F)$. The proof uses the Matsumoto-Moore presentation of the group $H_2(\text{Sp}(F), \mathbb{Z}) = H_2(\text{Sl}(2, F), \mathbb{Z})$ as well as the characterisation of the 2-torsion of $K_{MW}^2(F)$ as the set of all elements of the form $\{-1, a\}$. (More recently, Mazzoleni, [3], has given an alternative proof of this theorem which by-passes the theorem of Matsumoto-Moore.)

More recently, F. Morel has introduced the Milnor-Witt $K$-theory, $K_{MW}^*(F)$ ([4], [5]). This is a graded algebra given by a simple presentation, due to Morel and M. Hopkins, from which the following properties are easily deduced: $K_{MW}^n(F) \cong W(F)$ for all $n < 0$; $K_{MW}^0(F) \cong GW(F)$, the Grothendieck-Witt ring of isometry classes of quadratic forms over $F$; there is an element $\eta$, of degree $-1$, such that $K_{MW}^*(F)/\langle \eta \rangle \cong K_{M}^*(F)$. The main result about Milnor-Witt $K$-theory is that it gives an exact description of certain operations in stable motivic homotopy theory; namely there is a natural isomorphism of graded rings

$$K_{MW}^*(F) \cong [S^0, (\mathbb{G}_m)^*]$$

where $S^0$ is the ‘motivic’ sphere spectrum, and $[\ ,\ ]$ denotes the group of morphisms in the stable $\mathbb{A}^1$-homotopy category ([4], section 6).

Morel has shown (see [5], for example) that, for all $n \geq 0$,

$$K_{MW}^n(F) \cong K_{M}^n(F) \times_{I^n/I^{n+1}} I^n(F).$$

In fact this result is essentially a reformulation of some of the main results of Arason and Elman, [1], on the powers of $I(F)$. Their work, in turn, relies heavily on the work of Voevodsky, Orlov and Vishik on the Milnor conjecture. In view of Morel’s result, Suslin’s theorem can be re-formulated as the statement that $H_2(\text{Sl}(2, F), \mathbb{Z}) \cong K_{MW}^2(F)$, at least when $F$ is infinite. Elsewhere ([6]), Morel has sketched a direct proof of this fact, using the machinery of $\mathbb{A}^1$-homotopy theory.

In this note, which is more elementary in nature than any of the references above, we prove that the Matsumoto-Moore relations give a simple presentation of the additive

\begin{itemize}
  \item \textbf{Date:} July 3, 2007.
  \item 1991 \textit{Mathematics Subject Classification.} 19G99, 20G10.
  \item \textit{Key words and phrases.} $K$-theory, Witt Rings, Group Homology.
\end{itemize}
group $K_n^{MW}(F)$, for all $n \geq 2$, in terms of the natural set of generators. When $n = 2$, this statement specializes to Suslin’s theorem, as re-formulated above.

As another application of our main theorem, we give an abstract additive presentation of the group $I^n(F)$ with $n$-fold Pfister forms as generators. (Corollary 2.16).

2. Milnor-Witt $K$-theory

**Definition 2.1** (Hopkins-Morel, [4]). The Milnor-Witt $K$-theory of the field $F$ is the graded associative ring $K_n^{MW}(F)$ generated by the symbols $[u], u \in F^\times$, of degree +1 and one symbol $\eta$ of degree $-1$ subject to the following relations:

1. For each $a \in F^\times \setminus \{1\}$, $[a] \cdot [1-a] = 0$.
2. For each $a, b \in F^\times$, $[ab] = [a] + [b] + \eta[a][b]$.
3. For each $u \in F^\times$, $[u]\eta = \eta[u]$.
4. $\eta^2[-1] + 2\eta = 0$.

The following result is easily deduced ([6], Lemma 2.4):

**Lemma 2.2.** For all $n \in \mathbb{Z}$, $K_n^{MW}(F)$ has the following presentation as an additive group: It is generated by the elements $\eta^m[a_1] \cdots [a_r], m \geq 0, r = n + m \geq 0$ subject to the following relations:

1. $\eta^m[a_1] \cdots [a_r] = 0$ if $r \geq 2$ and $a_{i-1} + a_i = 1$ for some $i \geq 2$.
2. $\eta^m[a_1] \cdots [a_{i-1}]a_i[a'_i] \cdots [a_r] = \eta^m[a_1] \cdots [a_{i-1}][a_i'] \cdots [a_r] + \eta^m[a_1] \cdots [a_{i-1}]a_i[a_i'] \cdots [a_r] + \eta^{m+1}[a_1] \cdots [a_{i-1}]a_i[a'_i] \cdots [a_r] + \eta^{m+1}[a_1] \cdots [a_{i-1}]a_i[a_i'] \cdots [a_r]$.
3. $\eta^{m+2}[a_1] \cdots [a_{i-1}][-1][a_{i+1}] \cdots [a_{r+2}] + 2\eta^{m+1}[a_1] \cdots [a_{i-1}]a_{i+1} \cdots [a_{r+2}] = 0$.

However, in view of the relation $\eta[a_1][a_2] = [a_1a_2] - [a_1] - [a_2]$, it is clear that $K_n^{MW}(F)$ can be generated by the elements $[a_1] \cdots [a_n]$ whenever $n \geq 1$. Our main theorem is a presentation of $K_n^{MW}(F)$ in terms of these generators when $n \geq 2$.

The theorem of Matsumoto and Moore ([2]), for the case of the symplectic group $\text{Sp}(F)$, gives a presentation of the group $H_2(\text{Sp}(F), \mathbb{Z})$. It has the following form: The generators are symbols $\langle a_1, a_2 \rangle$, $a_i \in F^\times$, subject to the relations:

1. $\langle a_1, a_2 \rangle = 0$ if $a_i = 1$ for some $i$
2. $\langle a_1, a_2 \rangle = \langle a_2^{-1}, a_1 \rangle$
3. $\langle a_1, a_2a_2' \rangle + \langle a_2, a_2' \rangle = \langle a_1a_2, a_2' \rangle + \langle a_1, a_2 \rangle$
4. $\langle a_1, a_2 \rangle = \langle a_1, -a_1a_2 \rangle$
5. $\langle a_1, a_2 \rangle = \langle a_1, (1-a_1)a_2 \rangle$

This motivates the following (provisional) definition:

**Definition 2.3.** Let $n \geq 2$. For a field $F$, $K_n^{MM}(F)$ (MM is for ‘Matsumoto-Moore’) will denote the additive group which has the following presentation: the generators are $\langle a_1, \ldots, a_n \rangle$, $a_i \in F^\times$, subject to the following relations:

1. $\langle a_1, \ldots, a_n \rangle = 0$ if $a_i = 1$ for some $i$
2. $\langle a_1, \ldots, a_i-1, a_i, \ldots, a_n \rangle = \langle a_1, \ldots, a_i^{-1}, a_i-1, \ldots, a_n \rangle$
3. $\langle a_1, \ldots, a_i-1, a_i a_i' \rangle + \langle a_1, \ldots, a_i, a_i' \rangle = \langle a_1, \ldots, a_{i-1}a_n, a_i' \rangle + \langle a_1, \ldots, a_{i-1}, a_i \rangle$
4. $\langle a_1, \ldots, a_{n-1}, a_n \rangle = \langle a_1, \ldots, a_{n-1}, -a_{n-1}a_n \rangle$
5. $\langle a_1, \ldots, a_{n-1}, a_n \rangle = \langle a_1, \ldots, a_{n-1}, (1-a_{n-1})a_n \rangle$

**Remark 2.4.** In particular, $K_2^{MM}(F) \cong H_2(\text{Sp}(F), \mathbb{Z}) = H_2(\text{Sl}(2, F), \mathbb{Z})$ if $F$ is infinite or at least sufficiently large) by the theorem of Matsumoto-Moore.
Observe that, using relation (ii) together with (iii), (iv) and (v), we easily deduce the following relations in $K_n^\text{MM}(F)$:

(iii)' $\langle a_1, \ldots, a_{i-1}, a_i a'_i, \ldots, a_n \rangle + \langle a_1, \ldots, a_i, a'_i, \ldots, a_n \rangle = \langle a_1, \ldots, a_{i-1} a_i, a'_i, \ldots, a_n \rangle + \langle a_1, \ldots, a_{i-1}, a_i, \ldots, a_n \rangle$

(iv)' $\langle a_1, \ldots, a_{i-1}, a_i, \ldots, a_n \rangle = \langle a_1, \ldots, a_{i-1}, -a_{i-1} a_i, \ldots, a_n \rangle$

(v)' $\langle a_1, \ldots, a_{i-1}, a_i, \ldots, a_n \rangle = \langle a_1, \ldots, a_{i-1}, (1 - a_{i-1}) a_i, \ldots, a_n \rangle$

Theorem 2.5. $K_n^\text{MW}(F) \cong K_n^\text{MM}(F)$ for all $n \geq 2$ via an isomorphism sending $[a_1] \cdots [a_n]$ to $\langle a_1, \ldots, a_n \rangle$.

Proof. The theorem follows from Lemmas 2.8 and 2.15 below. \hfill \square

Corollary 2.6. For all infinite fields $F$, $K_2^\text{MW}(F) \cong H_2(\text{Sl}(2, F), \mathbb{Z})$.

Lemma 2.7. The relations $[a][-a] = 0$ and $[a][b] = [b^{-1}]a$ hold in $K_\ast^\text{MW}(F)$ for all $a, b \in F^\times$.

Proof. Using the identity $-a(1 - a^{-1}) = 1 - a$ together with 2. gives

$$[-a] = [1 - a] - [1 - a^{-1}] - \eta[-a][1 - a^{-1}]$$

Hence

$$[a][-a] = -[a](1 + \eta[-a])[1 - a^{-1}] \quad \text{(using 1.)}$$

$$= -(1 + \eta[-a])[a][1 - a^{-1}] \quad \text{(since } \eta[u][v] = \eta[v][u] \text{ by 2.)}$$

$$= (1 + \eta[-a])(1 + \eta[a])[a^{-1}][1 - a^{-1}] \quad \text{(since } [a] = (1 + \eta[a])[a^{-1}] \text{ by 2.)}$$

$$= 0 \quad \text{(by 1.)}$$

We will need the identity

$$\eta([a] + [-a])[-1] = -2[-1]$$

obtained by letting $x = a$ and $x = -a$ in the identity $[x] = [-1] + [-x] + \eta[-x][-1]$ and adding.

Observe also that $[a]^2 = [a][-a]$ for all $a \in F^\times$ (let $x = a$ above and use $[a][-a] = 0$).

Now, for any $a, b \in F^\times$ we have

$$0 = [ab][-ab]$$

$$= ([a] + [b] + \eta[a][b])([-a] + [b] + \eta[-a][b])$$

$$= [a][b] + [b][-a] + \eta([a] + [-a])[b^2] + [b]^2$$

$$= [a][b] + [b][-a] + \eta([a] + [-a])[-1][b] + [b][-1]$$

$$= [a][b] + [b][-a] - [b][-1]$$

$$= [a][b] + [b]([-a] - [-1])$$

$$= [a][b] + [b][a](1 + \eta[-1])$$

and hence

$$[a][b] = -[b][a](1 + \eta[-1]) = -(1 + \eta[-1])[b][a]$$.
Thus
\[
[a][b] - [b^{-1}][a] = -(1 + \eta[-1])[b] + [b^{-1}][a]
\]
\[
= -(b) + [b^{-1}] + \eta[-1][b][a]
\]
\[
= -(b) + [b^{-1}] + \eta[b^{-1}][b][a] = -[1][a] = 0
\]
(where we have used \([-1][b] = [b^{-1}][b]\) which follows from \([-1] = [-b] + [b^{-1}] + \eta[-b][b^{-1}]
and \([b][b] = 0\)).

\[\square\]

**Lemma 2.8.** Let \(n \geq 2\). The map \(\phi\) which sends the element \(\langle a_1, \ldots, a_n \rangle\) of \(K_n^{\text{MM}}(F)\) to \([a_1] \cdots [a_n]\) in \(K_n^{\text{MW}}(F)\) extends uniquely to a well-defined epimorphism of groups.

**Proof.** Well-definedness is the issue; we must prove that relations (i)-(v) are preserved by \(\phi\).

Relation (i): This follows from the identity \([1] = 0\) in \(K_n^{\text{MW}}(F)\) (since \([1] = 2[-1] + \eta[-1]^2\) by 2. and hence \(\eta[1] = 0\) by 4. and thus \([1] = 2[1]\) by 2. again).

Relation (ii): This follows immediately from the relation \([a_{i+1}][a_i] = [a_i^{-1}][a_{i+1}]\) (Lemma 2.7).

Relation (iii): We have
\[
[a_{n-1}][a_n]^2 + [a_n][a_n'] = [a_{n-1}][a_n] + [a_n] + \eta[a_n][a_n'] + [a_n][a_n']
\]
\[
= ([a_{n-1}] + [a_n] + \eta[a_{n-1}][a_n])[a_n'] + [a_n][a_n]
\]
\[
= [a_{n-1}][a_n'][a_n'] + [a_n][a_n]
\]

Relation (iv): We have \([a_{n-1}][-a_{n-1}a_n] = [a_{n-1}][-a_{n-1}a_n + [a_n] + \eta[-a_{n-1}][a_n]] = [a_{n-1}][a_n]\) by Lemma 2.7.

Relation (v): Similarly, \([a_{n-1}][(1 - a_{n-1})a_n] = [a_{n-1}][a_n]\) using 1. and 2.

\[\square\]

**Lemma 2.9.** Let \(n \geq 2\). For \(a_1, \ldots, a_n, x \in F^\times\) let
\[
\rho_x(\langle a_1, \ldots, a_n \rangle) := \langle a_1, \ldots, a_n x \rangle - \langle a_1, \ldots, a_n \rangle - \langle a_1, \ldots, x \rangle.
\]

Then \(\rho_x\) extends uniquely to an endomorphism of \(K_n^{\text{MM}}(F)\).

**Proof.** We must prove that \(\rho_x\) preserves defining relations (i)-(v).

Relation (i) is clear.

Relation (ii): When \(i < n\) in (ii), the result is clear. For the case \(i = n\), we find:
\[
\rho_x(\langle a_1, \ldots, a_{n-1}, a_n \rangle) = \langle a_1, \ldots, a_{n-1}, xa_n \rangle - \langle a_1, \ldots, a_{n-1}, a_n \rangle - \langle a_1, \ldots, a_{n-1}, x \rangle
\]
\[
= (\langle a_1, \ldots, a_{n-1}, x, a_n \rangle + \langle a_1, \ldots, a_{n-1}, x \rangle - \langle a_1, \ldots, x, a_n \rangle)
\]
\[
- \langle a_1, \ldots, a_{n-1}, a_n \rangle - \langle a_1, \ldots, a_{n-1}, x \rangle \quad \text{(using (iii))}
\]
\[
= \langle a_1, \ldots, a_{n-1}, x, a_n \rangle - \langle a_1, \ldots, x, a_n \rangle - \langle a_1, \ldots, a_{n-1}, a_n \rangle
\]
\[
= \langle a_1, \ldots, a_{n-1}^{-1}, a_{n-1}^{-1}x \rangle - \langle a_1, \ldots, a_{n-1}^{-1}, a_{n-1} \rangle - \langle a_1, \ldots, a_{n-1}^{-1}, x \rangle
\]
\[
= \rho_x(\langle a_1, \ldots, a_{n-1}^{-1}, a_{n-1}^{-1} \rangle).
\]
Relation (iii):

\[
\rho_x((a_1, \ldots, a_{n-1}, a_n, a'_n)) + \rho_x((a_1, \ldots, a_n, a'_n)) = \langle a_1, \ldots, a_{n-1}, a_n(a'_n \rangle - \langle a_1, \ldots, a_{n-1}, x \rangle - \langle a_1, \ldots, a_{n-1}, a_n a'_n \rangle \\
+ (a_1, \ldots, a_n, a'_n x) - \langle a_1, \ldots, a_n, x \rangle - \langle a_1, \ldots, a_n, a'_n \rangle \\
= (\langle a_1, \ldots, a_{n-1}, a_n(a'_n x) + \langle a_1, \ldots, a_n, a'_n x \rangle - (\langle a_1, \ldots, a_{n-1}, a_n a'_n \rangle + \langle a_1, \ldots, a_n, a'_n \rangle) \\
- \langle a_1, \ldots, a_{n-1}, x \rangle - \langle a_1, \ldots, a_n, x \rangle \\
= (\langle a_1, \ldots, a_{n-1} a_n, a'_n x \rangle + \langle a_1, \ldots, a_{n-1}, a_n \rangle) - (\langle a_1, \ldots, a_{n-1} a_n, a'_n \rangle + \langle a_1, \ldots, a_{n-1}, a_n \rangle) \\
- \langle a_1, \ldots, a_{n-1}, x \rangle - \langle a_1, \ldots, a_n, x \rangle \\
= \rho_x((a_1, \ldots, a_{n-1} a_n, a'_n x)) + \rho_x((a_1, \ldots, a_{n-1}, a_n))
\]

Relations (iv) and (v) are immediate.

For \( n \geq 2 \), we will denote the element \( \rho_x((a_1, \ldots, a_n)) \) in \( K_n^{\text{MW}}(F) \) by \( [a_1, \ldots, a_n, x] \).

**Lemma 2.10.** Let \( n \geq 2 \). For any permutation, \( \sigma \), of \( \{1, \ldots, n+1\} \), \( [a_1, \ldots, a_{n+1}] = [a_{\sigma(1)}, \ldots, a_{\sigma(n+1)}] \).

**Proof.** It is immediate from the definition that \( [a_1, \ldots, a_n, a_{n+1}] = [a_1, \ldots, a_{n+1}, a_n] \); i.e. the result is true when \( \sigma \) is the transposition \( (n \, n+1) \). From this it follows that

\[
[a_1, \ldots, a_{n-1}, a_n, x] = \rho_x((a_1, \ldots, a_n)) \\
= \rho_{a_n}(\langle a_1, \ldots, a_{n-1}, x \rangle) \\
= \rho_{a_n}(\langle a_1, \ldots, x^{-1}, a_{n-1} \rangle) \\
= [a_1, \ldots, x^{-1}, a_{n-1}, a_n] = [a_1, \ldots, x^{-1}, a_n, a_{n-1}] \\
= \rho_x(\langle a_1, \ldots, a_n, a_{n-1} \rangle) \\
= [a_1, \ldots, a_n, a_{n-1}, x]
\]

This fact, together with relation (ii), now implies that

\[
[a_1, \ldots, a_{i-1}, a_i, \ldots, a_n, x] = \rho_x(\langle a_1, \ldots, a_{i-1}, a_i, \ldots, a_n \rangle) \\
= \rho_x(\langle a_1, \ldots, a_i, a_{i-1}, \ldots, a_n \rangle) = [a_1, \ldots, a_{i-1}, a_i, \ldots, a_n, x]
\]

proving the lemma. \( \square \)

**Lemma 2.11.** For all \( x, y \in F^\times \), \( \rho_x \rho_y = \rho_y \rho_x \).

**Proof.** Let \( a_1, \ldots, a_n \in F^\times \). Then

\[
\rho_y(\langle a_1, \ldots, a_n \rangle) = [a_1, \ldots, a_n, y] = [y, a_1, \ldots, a_n]
\]
and hence
\[
\rho_x(\rho_y(\langle a_1, \ldots, a_n \rangle)) = \rho_x([y, a_1, \ldots, a_n])
\]
\[
= \rho_x(\langle y, a_1, \ldots, a_{n-1}, a_n \rangle) - \rho_x(\langle y, a_1, \ldots, a_{n-1} \rangle) - \rho_x(\langle y, a_1, \ldots, a_{n-2}, a_n \rangle)
\]
\[
= [y, \ldots, a_{n-1}a_n, x] - [y, a_1, \ldots, a_{n-1}, x] - [y, a_1, \ldots, a_n, x]
\]
\[
= [x, \ldots, a_{n-1}a_n, y] - [x, a_1, \ldots, a_{n-1}, y] - [x, a_1, \ldots, a_n, y]
\]
\[
= \rho_y(\rho_x(\langle a_1, \ldots, a_n \rangle)).
\]

More generally, we define elements \([a_1, \ldots, a_r] \in K_n^{\text{MM}}(F) \ (r > n)\) recursively by the formula
\[
[a_1, \ldots, a_{r+1}] := \rho_{a_{r+1}}([a_1, \ldots, a_r]).
\]
We will also use the notation \([a_1, \ldots, a_n] := \langle a_1, \ldots, a_n \rangle \) (i.e., when \(r = n\)).

**Corollary 2.12.** Fix \(n \geq 2\). For all \(r > n\) and for all permutations, \(\sigma\), of \(\{1, \ldots, r\}\)
\[
[a_1, \ldots, a_r] = [a_{\sigma(1)}, \ldots, a_{\sigma(r)}].
\]

**Proof.** We use induction on \(r\). The case \(r = n + 1\) has already been proved.
For permutations of \(\{1, \ldots, r + 1\}\) which fix \(r + 1\), the result holds by induction since
\[
[a_1, \ldots, a_r, a_{r+1}] = \rho_{a_{r+1}}([a_1, \ldots, a_r]).
\]
On the other hand, when \(r > n\), the result holds for the permutation \(\sigma = (r + 1)\) since
\[
[a_1, \ldots, a_r, a_{r+1}] = \rho_{a_{r+1}}(\rho_{a_r}([a_1, \ldots, a_{r-1}])) = \rho_{a_r}(\rho_{a_{r+1}}([a_1, \ldots, a_{r-1}])) = [a_1, \ldots, a_{r+1}, a_r].
\]

**Remark 2.13.** Observe that it follows that the relations (i)-(v) extend to the symbols \([a_1, \ldots, a_r] \ (r \geq n)\) since we can always permute the key entries to before the \(n\)-th position and then use the fact that \([a_1, \ldots, a_r] = \phi(\langle a_1, \ldots, a_n \rangle)\) for an appropriate endomorphism \(\phi\). Furthermore, property (ii) and symmetry (Corollary 2.12) imply that
\[
[a_1, \ldots, a_i, \ldots, a_r] = [a_1, \ldots, a_i^{-1}, \ldots, a_r]
\]
for any \(i\).

**Corollary 2.14.** Let \(n \geq 2\). If \(r > n\) and if \(a_1, \ldots, a_r \in F^\times\) then \([a_1, \ldots, a_r] = 0\) if \(a_i\) is a square in \(F^\times\) for some \(i \leq r\).

**Proof.** By symmetry we can suppose that \(i > 1\). Suppose that \(a_i = b_i^2\). We thus have
\[
[a_1, \ldots, a_{i-1}, b_i^2, \ldots] = [a_1, \ldots, a_{i-1}b_i, b_i, \ldots] + [a_1, \ldots, a_{i-1}, b_i, \ldots] - [a_1, \ldots, a_{i-1}, b_i, b_i, \ldots]
\]
\[
= [a_1, \ldots, a_{i-1}b_i, b_i^{-1}, \ldots] + [a_1, \ldots, a_{i-1}, b_i, \ldots] - [a_1, \ldots, a_{i-1}b_i, b_i^{-1}, \ldots]
\]
\[
= [a_1, \ldots, a_{i-1}, b_i, b_i^{-1}, \ldots] = [a_1, \ldots, a_{i-1}, 1, \ldots] = 0.
\]

**Lemma 2.15.** Let \(n \geq 2\). There is a unique epimorphism \(\lambda : K_n^{\text{MW}}(F) \rightarrow K_n^{\text{MM}}(F)\) satisfying
\[
\lambda(\eta^m[a_1] \cdots [a_r]) = [a_1, \ldots, a_r] \quad (r = n + m)
\]
By property (ii), and by symmetry, we can assume $i \leq n$ and thus we reduce to the key case $r = n$; i.e. we must prove
\[ [a_1, \ldots, a_i a_i', \ldots, a_r] = [a_1, \ldots, a_i, \ldots, a_r] + [a_1, \ldots, a_i', \ldots, a_r] + [a_1, \ldots, a_i, a_i', \ldots, a_r]. \]
By symmetry we can suppose that $a_i$ and by symmetry, we can assume $i = n$. The identity is now just the definition of $[a_1, \ldots, a_n, a_n']$.

Relation (3): We must prove that for $r \geq n$
\[ [a_1, \ldots, a_{i-1}, -1, a_{i+1}, \ldots, a_{r+2}] = -2[a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_{r+2}]. \]
By symmetry, we can suppose that $r = n$ and $i = n + 1$. So we must show that
\[ [a_1, \ldots, a_n, -1, a_{n+2}] = -2[a_1, \ldots, a_n, a_{n+2}]. \]
Now
\[ [a_1, \ldots, a_n, -1] = [a_1, \ldots, a_n, a_n] \quad \text{(by (iv))} \]
\[ = \langle a_1, \ldots, a_n^2 \rangle - 2\langle a_1, \ldots, a_n \rangle \]
and thus
\[ [a_1, \ldots, a_n, -1, a_{n+2}] = [a_1, \ldots, a_n^2, a_{n+2}] - 2[a_1, \ldots, a_n, a_{n+2}] = -2[a_1, \ldots, a_n, a_{n+2}] \]
by Corollary 2.14.

As an application, we derive a simple additive presentation of the ideals $I^n(F)$, $n \geq 2$, in the Witt Ring of a field $F$:

**Corollary 2.16.** For any field $F$, let $I(F)$ be the ideal of even-dimensional forms in the Witt Ring, $W(F)$, of the field $F$. As an additive group, $I^n(F) = I(F)^n$ has the following abstract presentation:

It is generated by the classes of Pfister forms $<< a_1, \ldots, a_n >>$, $a_i \in F^\times$ subject to the following relations:

(i) $<< a_1, \ldots, a_n >> = 0$ if $a_i$ is a square for some $i$
(ii) $<< a_1, \ldots, a_{i-1}, a_i, a_{i+1}, \ldots, a_n >> = -<< a_1, \ldots, a_{i-1}, a_i, a_{i-1}, \ldots, a_n >>$
(iii) $<< a_1, \ldots, a_{n-1}, a_n a_n' >> + << a_1, \ldots, a_n, a_n' >> = << a_1, \ldots, a_{n-1} a_n, a_n' >>$
(iv) $<< a_1, \ldots, a_{n-1}, a_n >> = << a_1, \ldots, a_{n-1}, (1 - a_{n-1})a_n >>$

**Proof.** Morel’s theorem ([5], Théorème 5.3), shows that there is an exact sequence
\[ 0 \rightarrow K^M_n(F)^2 \rightarrow K^MW_n(F) \rightarrow I^n(F) \rightarrow 0 \]
where the first (nontrivial) homomorphism maps $\{a_1, \ldots, a_n\}^2 = \{a_1, a_1^2, \ldots, a_n\}$ to $[a_1] \cdots [a_i^2] \cdots [a_n]$ (for any $i$) and the next homomorphism sends $[a_1] \cdots [a_n]$ to the class of the Pfister form $<< a_1, \ldots, a_n >>$. Combining this with Theorem 2.5 give the result, since the identity $-a = (1 - a)/(1 - a^{-1})$ shows that (i) and (iv) imply the identity $<< a_1, \ldots, a_{n-1}, a_n >> = << a_1, \ldots, a_{n-1}, -a_{n-1} a_n >>$. □

**Remark 2.17.** Compare this with the presentation of $I^n(F)$ given by Arason and Elman ([1], Theorem 3.1). Of course, Corollary 2.16 – like the result of Arason and Elman – requires the proof of the Milnor conjecture (since it is needed for Morel’s theorem), and conversely easily implies the Milnor conjecture.
3. Acknowledgements

The work in this article was partially funded by the Science Foundation Ireland Research Frontiers Programme grant 05/RFP/MAT0022.

References


School of Mathematical Sciences, University College Dublin

E-mail address: kevin.hutchinson@ucd.ie, lqtao@ucd.ie