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Auctioning Horizontally Differentiated Items

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Auctioning Horizontally Differentiated Items*

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Summary: This paper analyses strategic market allocation by two auctioneers holding substitutes. It characterizes both the cooperative and competitive outcomes. Under cooperation or competition with close substitutes, bidders are allocated according to the expected total surplus each generates. This market division is efficient if and only if the distribution of bidders' tastes is not skewed. If skewed, reserve prices distort participation towards the least preferred item. For greater degrees of product differentiation competition leads to multiple equilibria. Finally, competition with close substitutes sellers leave participation rents to their weakest bidder. They do not in other cases, whether they compete or cooperate.

1 Introduction

Auctions are an increasingly popular trading mechanism. While once reserved to sell rare or exclusive items, they are now employed for the sale of objects that have close substitutes. In Ireland second hand houses are sold through English auctions and this year's auction sales revenue is in excess of 1.6 billion euros[1]. Car rental companies and governmental agencies in the UK and in the United States auction their used cars. Finally auctions are also used to allocate shares of companies going public. In each of these cases, the number of bidders attending a particular auction and the revenue it raises critically depends on what other items are available and on how these are sold. This interdependence of auctions' performances complicates substantially the auction design problem sellers face. So far, the literature considering competing auctioneers focused at the sale of homogeneous items. This paper fills a gap by considering the case of horizontally differentiated items which is, I feel, more representative of some of the markets mentioned above.

Only a small number of papers examine markets with competing auctioneers. McAfee [5], Peters and Severinov [9] and Peters [8] address competing auctioneers holding homogeneous items. These papers examine markets with large numbers of sellers and buyers. The advantage of this assumption is that it gently restricts the strategic interactions between the market participants which permits to solve for the equilibrium. The drawback of this approach is that it is not clear whether the results are applicable to markets with a limited number of sellers.
A different strand of literature considers optimal nonlinear pricing with privately informed agents and competing principals. Within this strand, Stole [10] also considers horizontally differentiated products. Nonetheless, what separates Stole [10] from an auction setting is the fact that in the environment he considers a buyer's expected payoff does not depend on other buyers' decisions. In an auction setting, a bidder's probability of winning a specific item and the price he pays for it depend on the actions of other bidders. Thus, contrary to Stole [10], when an auctioneer alters one of his strategic variables, it affects both: the payoff bidders receive at his auction as well as the reservation payoff bidders can gather from attending any other auction.

[1] Figure obtained from selling agents "Douglas Newman Good" 2005 economic report.

This greater interdependence between sales' performances complicates substantially the problem.
In this paper we consider a market with 2 auctioneers. Thus, in contrast to the literature cited above, strategic interactions between sellers will play a more prominent role. The cost of this is that to make the problem tractable we have to restrict attention to competition in reserve prices in English auctions. In this sense, the paper is closest to Burguet and Sákovics [3] who analyze similar issues in a market for homogeneous items by restricting seller's strategies to setting reserve prices in second price auctions. It departs from it as I introduce product differentiation.

The model considered is based on Hotelling's [4] linear model. There are n buyers, located between the 2 sellers. Each buyer privately observes his location which gives a measure for his willingness to pay for each item. Sellers announce simultaneously their reserve prices which constitutes the lowest acceptable bid for their English auction. Given the reserve prices and their valuation, the buyers decide on which auction to attend. We consider that auctions are simultaneous so that one buyer can attend at most one auction. We search for sub-game perfect equilibria in which sellers perfectly anticipate the buyers' behavior which is described in section 3.

Interestingly, both Stole [10] and this paper show that the monopolistic outcome can still form an equilibrium provided the optimal monopolistic market shares do not overlap. This potentially occurs for highly differentiated items. For closer substitutes, the monopolistic outcome no longer forms an equilibrium. We analyze the sellers' profit maximizing decisions in sections 4 and 5. In section 4, we assume sellers set their reserve prices cooperatively, maximizing joint profits. We show that reserve prices are set so as to allocate bidders on the basis of the expected marginal revenues each is associated with. A necessary and sufficient condition for efficiency is that the bidder who values both objects equally must be equally likely to win either item. When the distribution of bidders is skewed, reserve prices distort participation towards the least preferred item. Sellers set a higher reserve price for the preferred item extracting more informational rents on its market while the lower reserve price encourages participation for the least preferred item. Finally, under cooperation sellers leave no rents to the weakest, marginal, bidder.
In section 5 we consider competing sellers. We show that for close substi-

tutes sellers leave some participation rents to the weakest, marginal bidder as they set their reserve prices below his valuations. The indifferent bidder is the one generating the same expected *total* surplus no matter which seller he attends. This surplus is the sum of marginal revenue and participation rents. Finally the same necessary and sufficient condition holds for efficiency. Once again, reserve prices are such that participation is biased towards the least preferred item if any.
For greater degrees of product differentiation, multiple equilibria arise. For each of these, the weakest, indifferent bidder gets no rents. These equilibria reflect the fact that for greater degrees of product differentiation sellers refrain from infringing their opponent's market share. Indeed, as products become more differentiated the cost of attracting bidders located further away increases while the benefit decreases. Indeed, while intrinsic marginal valuations decrease with product differentiation, the relative value of the outside option increases. In such cases, sellers set high reserve prices so as to save on participation rents.
Finally, this paper highlights the following two features. First, it shows entry is more severely restricted when two auctions are held simultaneously instead of sequentially. This is so because the presence of a substitute gives bidders an outside option which increases the cost of participation. As bidders are offered two items instead of one the marginal benefit from a lower reserve price is weakened while the marginal cost is the same. Thus, reserve prices are higher under simultaneous auctioning. Second, the paper formally compares oligopoly pricing with the optimal reserve prices. Bulow and Roberts [1] show that in a symmetric environment the optimal reserve price is equivalent to the monopoly price when considering the demand associated with a representative bidder. We prove and explain why this analogy does not extend to the oligopoly setting. Nonetheless their approach proved very useful in shedding light on the results.

Section 6 presents a conclusion.

2 The model

Consider 2 sellers (seller 1 and seller 2) each possessing a single item. We assume that sellers are risk neutral and that they have no interest in keeping or obtaining either item. The sellers face a market composed of n risk neutral consumers with $n > 2$. These consumers consider the items to be horizontally differentiated. Each consumer is characterized by his taste, $\theta \in [0,1]$ (refer to as a type), which gives a measure for his willingness to pay for each item. A consumer with taste θ is willing to pay $v_i(\theta)$ for seller $i's$ item ($i = 1,2$). Let

$$v_1(\theta) = 1 - t\theta,$$

$$v_2(\theta) = 1 - t(1-\theta),$$

where $t \in (0,1]$. Graphically, this situation can be represented as Hotelling's [4] model of horizontal product differentiation with sellers located at the extremities of a line of length 1.

Each buyer privately observes his own taste. However, it is common knowledge that types are identically and independently distributed according to a distribution function $F(\theta)$ defined over $[0,1]$. We assume that $F(.)$ is continuously differentiable and let $f(\theta)$ denote the density function, with $f(\theta) > 0$ almost everywhere.

We make the following, particularly usual, regularity assumption regarding the distribution of types:

Regularity assumption: The functions $\left(x + \dfrac{F(x)}{f(x)}\right)$ *and* $\left(x - \dfrac{1-F(x)}{f(x)}\right)$ *are increasing*[2] *and continuous.*

Given this formalization, there are three effects from a reduction in t. The first is to increase the value of each item to all buyers, the second is to decrease the amount of private information and the last is to reduce product differentiation. Implications from each of these effects, and the way they interact are particularly interesting for the case of competing sellers.

We consider that the sellers simultaneously auction their items using an English auction. The rule of this auction is such that sellers raise the bids until a unique bidder remains active. Each seller's strategic variable is his reservation price which corresponds to the lowest acceptable bid. Throughout the paper we will use the following notation:

Notation: Let γ_1 and γ_2 denote the reservation prices set by seller 1 and seller 2 respectively. Let $(r_1, r_2) \in \,]0, +\infty) \times (-\infty, 1[$ be defined such that $v_i(r_i) \equiv \gamma_i$ $(i = 1,2)$. Finally let $R = (r_1, r_2)$.

[2] Allowing for nondecreasing functions instead may lead to multiplicity of equilibria.

Notice that for any (γ_1, γ_2) there exists a unique (r_1, r_2) such that $v_i(r_i) \equiv \gamma_i$ $(i = 1, 2)$. Thus, considering γ_i $(i = 1, 2)$ as seller $i's$ strategic variable is equivalent to considering r_i $(i = 1, 2)$ as seller $i's$ strategic variable.

The timing of the game is the following. First, Nature draws the buyers' tastes. Then, both sellers simultaneously announce their reservation prices. Given this information buyers decide on which auction to attend. Not attending any auction leads to a reservation utility which we consider equal to zero.

3 The buyers' sub-game

Bidding:

Consider a buyer who attends one of the two auctions. If he is the only one attending the auction, he wins and pays the reservation price. If there is excess demand at the reservation price, a dominant strategy equilibrium consists in dropping out whenever the price reaches his true valuation. Expected rents from participating necessarily decrease with t. Indeed, the price, which is equal to the second highest valuation, converges towards the winner's valuation as t decreases. This illustrates the fact that a bidder's private information decreases as t converges to zero.

Participation:

We consider that bidders can attend at most one auction and have a 0 utility is they attend none. When reservation prices are such that $r_1 \leq r_2$ buyers with a valuation greater than the reserve price for auction 1 have a valuation lower than the reserve price for auction 2 and vice-versa. In that case, all bidders with type $\theta \leq r_1$ attend seller 1 and all bidders such that $\theta \geq r_2$ attend seller 2. Buyers such that $\theta \in]r_1, r_2[$ do not participate. When $r_1 > r_2$, all $\theta \in [r_2, r_1]$ are such that their valuation for each item is greater than its reserve price. In that case, we have the following result:

Lemma 1: *If reservation prices are such that $r_1 > r_2$ there always exists a threshold value $\theta_R \in [0, 1]$ such that the following strategy forms a symmetric Nash equilibrium: all buyers with a valuation $\theta \leq \theta_R$ attend seller 1, while all buyers with a valuation $\theta \geq \theta_R$ attend seller 2.*

The variable θ_R characterizes the indifferent type. It is uniquely defined by:

$$(1 - F(\theta_R))^{n-1} (v_1(\theta_R) - \gamma_1) = F^{n-1}(\theta_R) (v_2(\theta_R) - \gamma_2). \qquad (1)$$

We have $\theta_R \in [0, 1]$ for any $(r_1, r_2) \in]0, +\infty) \times (-\infty, 1[$ such that $r_1 > r_2$.

Proof: see Appendix 1.

The variable θ_R determines each seller's market share. Equation (1) states that the indifferent type is the one for whom the expected surplus upon getting item 1 equals the expected surplus upon getting item 2.

4 The sellers' game

In this section, we analyze the strategic allocation of buyers for all possible degrees of product differentiation. Particular attention is paid to efficient allocation as defined below.

Definition: An allocation of buyers is *efficient* if all $\theta < \frac{1}{2}$ attend seller 1, while all $\theta > \frac{1}{2}$ attend seller 2.

In other words, bidders are efficiently allocated as they bid for the object they value the most.

Let the realization of types be ranked such that $\theta_1 < \theta_2 < ... < \theta_{n-1} < \theta_n$. The expected revenue to seller 1, when setting reserve price r_1 while seller 2 sets r_2 is given by

$$\begin{aligned}\pi_1 (r_1, r_2) &= v_1(r_1) n F(x) (1 - F(x))^{n-1} \\ &+ \int_0^x \int_{\theta_1}^x v_1(\theta_2) f_{1,2}(\theta_1, \theta_2) d\theta_2 d\theta_1\end{aligned}$$

where $x = r_1$ if $r_1 \leq r_2$ and $x = \theta_R$ if $r_1 > r_2$.and where distribution of (θ_1, θ_2) is given by

$$f_{1,2}(\theta_1, \theta_2) = n(n-1) (1 - F(\theta_2))^{n-1} f(\theta_2) f(\theta_1).$$

Seller 1 gets the reserve price if he faces a single bidder and the second highest bid if he faces at least 2 bidders. Similarly, seller 2 gets

$$\begin{aligned}\pi_2 (r_2, r_1) &= v_2(r_2) n (1 - F(x)) (F(x))^{n-1} \\ &+ \int_x^1 \int_x^{\theta_n} v_2(\theta_{n-1}) f_{n-1,n}(\theta_{n-1}, \theta_n) d\theta_{n-1} d\theta_n\end{aligned}$$

where $x = r_2$ when $r_2 \geq r_1$ and $x = \theta_R$ when $r_2 < r_1$.and where the distribution of (θ_{n-1}, θ_n) is

$$f_{n-1,n}(\theta_{n-1}, \theta_n) = n(n-1)(F(\theta_{n-1}))^{n-1} f(\theta_{n-1}) f(\theta_n).$$

Integrating by parts the expected profit functions can be re-written as:

$$\begin{aligned}\pi_1(r_1, r_2) &= \int_0^x MR_1(\theta_1)\, nf(\theta_1)(1 - F(\theta_1))^{n-1} d\theta_1 &(2)\\ &\quad -nF(x)(1 - F(x))^{n-1}[v_1(x) - v_1(r_1)], &(3)\end{aligned}$$

where $x = r_1$ if $r_1 \leq r_2$ and $x = \theta_R$ if $r_1 > r_2$ and where $MR(.)$ stands for marginal revenue and is given by

$$MR_1(\theta, t) = 1 - t\left(\theta + \frac{F(\theta)}{f(\theta)}\right). \tag{4}$$

Similarly for seller 2:

$$\begin{aligned}\pi_2(r_2, r_1) &= \int_x^1 MR_2(\theta_n)\, nf(\theta_n) F^{n-1}(\theta_n) d\theta_n &(5)\\ &\quad -nF^{n-1}(x)(1 - F(x))[v_2(x) - v_2(r_2)], &(6)\end{aligned}$$

where $x = r_2$ when $r_2 \geq r_1$ and $x = \theta_R$ when $r_2 < r_1$ and

$$MR_2(\theta, t) = 1 - t\left[1 - \left(\theta - \frac{1 - F(\theta)}{f(\theta)}\right)\right]. \tag{7}$$

As established in Myerson [6], a seller gets, on expectation, the valuation of the winning bidder shaded by the informational rents minus the potential surplus left to the marginal bidder.

The concept of marginal revenue is introduced in Bulow and Roberts [1]. This paper links mechanism design to monopoly pricing and provides an approach that is particularly helpful to understand the sellers' strategic decisions. An alternative approach to find these expressions consists in assuming that sellers for items 1 and 2 are independent monopolies. At a price $p_i = v_i(\theta)$ $(i = 1, 2)$ the quantity sold to a representative buyer is $q_1 = F(\theta)$ units of item 1 and $q_2 = (1 - F(\theta))$ units of item 2. Thus, we can express each seller's total revenue as a function of quantity as:

$$TR_1(q_1) = q_1\left[1 - tF^{-1}(q_1)\right],$$

and

$$TR_2(q_2) = q_2 \left[1 - t\left(1 - F^{-1}(1 - q_2)\right)\right].$$

Differentiating the above expressions with respect to q_i ($i = 1, 2$) leads to (4) and (7).

Under the regularity assumption, the marginal revenues are decreasing with taste. Let $\widehat{\theta}$ be defined as

$$MR_1\left(\widehat{\theta}, t\right) = MR_2\left(\widehat{\theta}, t\right).$$

Under the regularity assumption $\widehat{\theta}$ exists, it is unique and independent of t and it solves:

$$1 - 2\widehat{\theta} + \frac{1 - 2F(\widehat{\theta})}{f(\widehat{\theta})} \equiv 0. \tag{8}$$

Whether $MR_i(\widehat{\theta}, t) \geq 0$ ($i = 1, 2$), depends on t. Let $\bar{t} > 0$ be such that $MR_i(\widehat{\theta}, \bar{t}) = 0$ ($i = 1, 2$). Whether $\bar{t} < 1$ depends on the distribution function. Under the uniform distribution $\bar{t} = 1$. For any distribution function such that the median (denoted θ_M) is $1/2$ we have $\bar{t} < 1$ only when the population is polarized. Figure 1 below represents the marginal revenues in both cases: $t \leq \bar{t}$ and $t > \bar{t}$.

Insert figure 1 here.

As established in Bulow and Roberts [1] and given (2) and (5) it is optimal for a monopolistic seller to set the reserve price so that only bidders associated with a non-negative marginal revenue participate. Let θ_1 and $(1 - \theta_2)$ denote these optimal monopolistic market shares. They are the unique solution to $MR_i(\theta_i, t) = 0$ for $i = 1, 2$.

Proposition 1: If the distribution function is such that $\bar{t} < 1$, then for any $t > \bar{t}$, the optimal reserve prices are $\gamma_1 = v_1(\theta_1)$ and $\gamma_2 = v_2(\theta_2)$ whether sellers compete or cooperate. The market shares do not overlap.

Proof: see Appendix 2.

If the distribution function is such that $\bar{t} < 1$ then for sufficiently differentiated items sellers set the optimal monopolistic reserve prices and market shares do not overlap. For any $t < \bar{t}$ the optimal monopolistic reserve prices no longer form a solution as the monopolistic market shares would overlap. Interestingly Stole [10] exhibits the same threshold condition to separate the local monopoly outcome from the competition outcome for the particular case of horizontal differentiation.

4.1 Cooperating sellers

Consider that the 2 sellers decide on their reserve prices cooperatively, maximizing joint profit. Let γ_i^c refer to seller i's cooperative reserve price.

Proposition 2: For any $t \leq \bar{t}$ the optimal reserve prices for each object are such that the market is entirely covered. The indifferent type extracts no rents and bidders are allocated with priority based on the expected surplus each generates.
More precisely, we have $\gamma_1^c = v_1(r^c)$ and $\gamma_2^c = v_2(r^c)$ where $r^c \in]0,1[$ is unique and characterized by

$$(1 - F(r^c))^{n-1} MR_1(r^c, t) = (F(r^c))^{n-1} MR_2(r^c, t). \qquad (9)$$

Proof: See appendix 3.

Setting high reserve prices (such that $r_1 < r_2$) for any $t \leq \bar{t}$ prevents participation of bidders associated with a positive marginal revenue and is therefore sub-optimal. Indeed, when $t \leq \bar{t}$ and $r_1 < r_2$, we either have $MR_1(r_1, t) > 0$ or $MR_2(r_2, t) > 0$ or both[3]. Thus to maximize revenue sellers cover the entire market.

It is then obvious that any reservation prices such that $r_1 > r_2$ cannot be optimal. Sellers are better-off raising both reserve prices to $v_i(\theta_R)$ $(i = 1, 2)$ as they extract more surplus without affecting participation. Thus, we must have $r_1 = r_2 = r$ to maximize joint profits, and

$$\begin{aligned} \pi_1 + \pi_2 &= \int_0^r MR_1(\theta_1)\, nf(\theta_1)\,(1 - F(\theta_1))^{n-1}\, d\theta_1 \\ &\quad + \int_r^1 MR_2(\theta_n)\, nf(\theta_n)\, F^{n-1}(\theta_n)\, d\theta_n \end{aligned}$$

Differentiating the above expression with respect to r leads to (9). Notice that at the solution, $MR_i(r^c, t) > 0$. Thus the cooperative reserve prices are greater than the monopolistic ones. Bidders would be better-off under sequential, monopolistic auctioning.

Lemma 2: A necessary and sufficient condition for bidders to be allocated efficiently is that type $\theta = \frac{1}{2}$, who values the items equally, must be

[3] The remaining possibility: $MR_i(r_i, t) < 0$ for $i = 1, 2$ requires $r_1 > r_2$.

equally likely to win either object. Technically, it requires $\theta_M = 1/2$, where θ_M refers to the median type ($F(\theta_M) = 1/2$). If the distribution of tastes is skewed, participation is distorted towards the least preferred item.

Proof: See Appendix 4.

Assume that the distribution is skewed such that $\theta_M < \frac{1}{2}$ meaning that it is more likely to have a greater number of bidders preferring item 1. In that case, $\widehat{\theta} < \frac{1}{2}$ and cooperating sellers set the marginal bidder, type r^c, such that $\theta_M < r^c < \widehat{\theta}$ so that $MR_1(r^c, t) > MR_2(r^c, t)$. Basically, sellers set a higher reserve price for the preferred item which reduces the informational rents left on its market. The lower reserve price, set for the least preferred object, has the advantage of promoting participation.

5 Competing sellers.

I proceed as follows. First I describe the analytical solution, then explain the underlying intuition and finally describe its properties.

Assume there exists an equilibrium such that $r_1 \geq r_2$. The first order conditions from maximizing the expected profits lead to[4]:

$$n\left(1 - F(\theta_R)\right)^{n-1}\left[v_1(\theta_R) - \gamma_1\right] = g_1\left(\theta_R, t\right),$$

and

$$n\left(F(\theta_R)\right)^{n-1}\left[v_2(\theta_R) - \gamma_2\right] = g_2(\theta_R, t).$$

where $g_1(\theta, t)$ and $g_2(\theta, t)$ are given by:

$$g_1\left(\theta, t\right) = MR_1\left(\theta, t\right)\left[1 - F\left(\theta\right)\right]^{n-1} - t\frac{\left[F(\theta)\right]^n}{f(\theta)}, \tag{10}$$

and

$$g_2\left(\theta, t\right) = MR_2\left(\theta, t\right)\left[F\left(\theta\right)\right]^{n-1} - t\frac{\left[1 - F\left(\theta\right)\right]^n}{f(\theta)}. \tag{11}$$

[4]See Appendix for details.

Since (1) must hold in equilibrium, the optimal reserve prices must guarantee that

$$g_1(\theta_R, t) = g_2(\theta_R, t). \tag{12}$$

Moreover, in any equilibrium where reserve prices are set such that $r_1 > r_2$, the indifferent type be granted a non-negative expected utility to participate. Thus, in equilibrium we must also have:

$$g_i(\theta_R, t) \geq 0 \text{ for } i = 1, 2. \tag{13}$$

Depending on whether both (12) and (13) hold, two different types of equilibria arise. For each the market is entirely covered. Let γ_i^* denote seller i's reserve price in equilibrium ($i = 1, 2$).

Proposition 4: Equilibrium reserve prices for close substitutes.

There exists a unique $\underline{t} \in \left]0, \bar{t}\right[$ *such that for any given degree of product differentiation* $t < \underline{t}$ *there exists a unique equilibrium. It is such that the market is entirely covered (i.e.* $r_1^* > r_2^*$*) and each seller leaves some participation rents to the indifferent (weakest) type (*$\gamma_i^* < v_i(\theta_{R^*})$ *for* $i = 1, 2$*). Bidders are allocated with priority based on the expected total surplus each type generates. This allocation of bidders is efficient if and only if* $\theta_M = \frac{1}{2}$.

More precisely, the optimal reserve prices are such that

$$\begin{cases} \gamma_1^* = v_1(\theta_{R^*}) - \dfrac{g_1(\theta_{R^*}, t)}{n\left[1 - F(\theta_{R^*})\right]^{n-1}} \\ \gamma_2^* = v_2(\theta_{R^*}) - \dfrac{g_2(\theta_{R^*}, t)}{n\left[F(\theta_{R^*})\right]^{n-1}}. \end{cases}$$

The indifferent consumer θ_{R^*} *is defined such that:*

$$\begin{aligned} &[1 - F(\theta_{R^*})]^{n-1} \left\{ MR_1(\theta_{R^*}, t) + \left[v_1(\theta_{R^*} - \frac{1 - F(\theta_{R^*})}{f(\theta_{R^*})}) - \gamma_1^* \right] \right\} \\ &= [F(\theta_{R^*})]^{n-1} \left\{ MR_2(\theta_{R^*}, t) + \left[v_2\left(\theta_{R^*} + \frac{F(\theta_{R^*})}{f(\theta_{R^*})} \right) - \gamma_2^* \right] \right\} \end{aligned} \tag{14}$$

Proof: See Appendix 5.

Finally, for $t \in \left[\underline{t}, \overline{t}\right]$ different type of equilibria arise.

Proposition 5: For any $t \in \left]\underline{t}, \overline{t}\right[$ *there exists multiple equilibria. In any of these equilibria the indifferent bidder gets no rents.*
For any given $t \in \left]\underline{t}, \overline{t}\right[$ *there exists a non-empty range of types* Θ_t *defined as follows:*

$$\Theta_t = \{\theta \in [0,1] : MR_i(\theta, t) \geq 0 \text{ and } g_i(\theta, t) \leq 0 \text{ for } i = 1, 2\}.$$

Any $\gamma_1^* = v_1(r^*)$ *and* $\gamma_2^* = v_2(r^*)$ *with* $r^* \in \Theta_t$ *forms a Nash equilibrium. (Proof: See Appendix 6.)*

Lemma 4 completes the characterization of the optimal reserve prices for the cases $t = \overline{t}$ and $t = \underline{t}$.

Lemma 4: At $t = \overline{t}$ *the unique solution is such that* $\gamma_1^* = v_1(r^*)$ *and* $\gamma_2^* = v_2(r^*)$ *with* $r^* = \widehat{\theta}$. *At* $t = \underline{t}$, *the unique solution is such that* $\gamma_1^* = v_1(\theta_{\underline{t}})$ *and* $\gamma_2^* = v_2(\theta_{\underline{t}})$ *with* $\theta_{\underline{t}}$ *defined as:*

$$g_1(\theta_{\underline{t}}, \underline{t}) = g_2(\theta_{\underline{t}}, \underline{t}).$$

(The proof follows from proofs of proposition 4 and 5.)

Intuitively, we can explain the above results as follows. Assume seller 2 sets his reserve price r_2 equal to r_2^* which belongs to the set of equilibria, and let us focus at seller 1. Since $MR_1(r_2^*, t) \geq 0$ for all $t \in \left(0, \overline{t}\right]$, seller 1 wants to allow entry of all $\theta \in [0, r_2^*]$. Whether he wants to overlap on his opponent's market and set $r_1^* > r_2^*$ depends on t.
In the presence of a competitor, the marginal benefit from lowering the reserve price is weakened as a seller's market share extends only up to θ_R. Yet the marginal cost is the same. Thus competing sellers are overall less inclined to lower their reserve prices.
For relatively low values of t, the intrinsic valuations and the marginal revenue associated with each bidder are both high. Moreover, the relative valuation of bidders in absolute value, that is $|v_1 - v_2|$, increases with t. Extrapolating a little: it is "cheaper" for seller 1 to deter a bidder from attending the auction for item 2 when the two items are almost identical (t low). It is

therefore beneficial for seller 1 to set low reserve prices ($r_1^* > r_2^*$) for close substitutes. By opposition, as products become more differentiated, the intrinsic value of an additional bidder lowers while the relative value of their outside option increases. In such cases, seller 1 does not benefit from infringing on his opponent's market and sets $r_1^* = r_2^*$.

For $t \in \left[\underline{t}, \overline{t}\right]$ a seller's revenue is not differentiable at the solution as it forms a peak. Figure 2 describes the equilibrium:

Insert figure 2 here.

In equilibrium the following properties hold.

Entry restriction: In equilibrium we have $MR_i\left(\theta_{R^*}, t\right) > 0$ $(i = 1, 2)$ for any $t \in \left(0, \overline{t}\right]$, where $\theta_{R^*} = r^*$ for $t \in \left[\underline{t}, \overline{t}\right]$. Thus the equilibrium reserve prices under competition leads to more restricted entry than the monopolistic ones. Clearly, participation is more costly when there is an outside option. This constrains each seller's strategic choices.

Participation rents: Competition for close substitutes lead sellers to abandon rents to their weakest bidder since $\gamma_i^* < v_i\left(\theta_{R^*}\right)$ for $i = 1, 2$. To evaluate these rents, notice that seller 1 evaluates

$$v_1(\theta_{R^*} - \frac{1 - F(\theta_{R^*})}{f(\theta_{R^*})}) - \gamma_1^*$$

while seller 2 evaluates

$$v_2\left(\theta_{R^*} + \frac{F(\theta_{R^*})}{f(\theta_{R^*})}\right) - \gamma_2^*.$$

Indeed to lure away a bidder from his opponent's auction each seller must take into account the value of the marginal bidder not to him but to to his opponent. More precisely, to dissuade the marginal bidder from attending his opponent, seller 1 takes into account the fact that type θ_{R^*} is worth $\left(\theta_{R^*} - \dfrac{1 - F(\theta_{R^*})}{f(\theta_{R^*})}\right)$ to seller 2. Similarly, seller 2's treats type θ_{R^*} as $\left(\theta_{R^*} + \dfrac{F(\theta_{R^*})}{f(\theta_{R^*})}\right)$.
Leaving participation rents is clearly a Pareto dominated strategy as

both sellers would be better-off setting their reserve prices equal to the valuations of the indifferent type.

Efficiency: Efficiency for $t < \underline{t}$ relies once again on the same necessary and sufficient condition. In the proof of proposition 4 we show that if it is more likely that consumers prefer item 1 on average ($\theta_M < \frac{1}{2}$) then the indifferent consumer will be closer to item 1. Despite participation rents, this suggests that once again, the seller with the "preferred" item is able to sustain higher reservation prices than his opponent. For higher values of t, we have the following result:

Lemma 2: When $\theta_M = \frac{1}{2}$, the cooperative, efficient reserve prices $r^c = \frac{1}{2}$ forms an equilibrium for any $t \in \left[\underline{t}, \overline{t}\right]$.

Proof: When $\theta_M = \frac{1}{2}$, we have $r^c = \widehat{\theta} = \frac{1}{2}$ and $g_1\left(\frac{1}{2}, t\right) = g_2\left(\frac{1}{2}, t\right)$ for any t. Finally, figure 4 in the appendix shows that for any $t \in \left[\underline{t}, \overline{t}\right]$, we have $\frac{1}{2} \in \Theta_t$ as $g_i\left(\frac{1}{2}, t\right) < 0$ for $i = 1, 2$.

The link between Auction and Oligopoly Pricing

Bulow and Roberts [1] teach us that computing the optimal reserve price for an auction with symmetric bidders is equivalent to computing the monopoly price when considering the demand associated with a representative bidder (as shown above). As we have seen, for substantially differentiated items ($t \geq \overline{t}$) this remains true. A natural question is then whether this analogy holds for closer substitutes.

Applying the Bulow and Roberts' [1] approach consists in constructing each seller's residual demand considering that a bidder's opportunity cost of not attending one seller's auction is not only his valuation but this minus what he expects at the other auction. For instance a representative buyer will purchase item 1 when prices are given by $p_i = v_i\left(r_i\right)$ $(i = 1, 2)$ if and only if the forllowing two conditions hold:

$$v_1\left(\theta\right) - p_1 \geq 0 \text{ and } v_1\left(\theta\right) - p_1 \geq v_2\left(\theta\right) - p_2.$$

From there we can derive a demand function for each seller:

$$q_1 = \begin{cases} F(r_1) \text{ if } r_1 \leq r_2, \\ F\left(\dfrac{r_1 + r_2}{2}\right) \text{ if } r_1 > r_2, \end{cases}$$

while

$$q_2 = \begin{cases} 1 - F(r_2) \text{ if } r_1 \leq r_2 \\ 1 - F\left(\dfrac{r_1 + r_2}{2}\right) \text{ if } r_1 > r_2. \end{cases}$$

Condering a uniform distribution of types, we can evaluate the equilibirum prices and reserve prices. We get the following result:

The optimal (symmetric) reserve prices are given by

$$\gamma^* = 1 - \frac{t}{2} - \frac{2-3t}{2n} \text{ for } t \in \left]0, \frac{2}{3}\right]$$

and a range of different reserve prices among which $\gamma^* = 1 - \dfrac{t}{2}$ for $t \in \left[\frac{2}{3}, 1\right]$.

The optimal (symmetric) prices are given by

$$p^* = \begin{cases} 1 - t \text{ for } t \in \left]0, \frac{1}{2}\right] \\ \frac{1-t}{t} \text{ for } t \in \left[\frac{1}{2}, \frac{2}{3}\right] \\ 1 - \frac{t}{2} \text{ for t} \in \left[\frac{2}{3}, 1\right] \end{cases}$$

Thus, the two solutions do not coincide. From Burguet and Sákovics [3] we learn that the analogy established in Bulow and Roberts [1] is present exclusively when a marginal change in the reserve price affects the revenue only when a single bidder attended the auction. Consider the case of a monopolistic seller. If, prior to the change, the seller attracted more than a single bidder, then it is quite obvious that a marginal change in his reserve price will not affect his revenue. When competing sellers are considered, a marginal change in the reserve price may affect a seller's revenue even conditional on him attracting more than a single seller initially. In Burguet and Sákovics [3] this is the case as reserve prices not only determine the price paid by a single bidder but also the composition of demand. In the case analyzed here, a marginal change in the reserve price affects a seller's revenue conditional on gathering one but also two, and exactly two, bidders. Indeed, the first order condition taken at all $r_1 \geq r_2$ from maximizing seller 1's revenue can be written as:

$$\begin{aligned} \frac{\partial \pi_1}{\partial r_1} &= n\left(1 - F\left(\theta_R\right)\right)^{n-1} F\left(\theta_R\right)\left[v_1(r_1)\frac{f\left(\theta_R\right)}{F\left(\theta_R\right)}\frac{\partial \theta_R}{\partial r_1} - t\right] \\ &+ \frac{n(n-1)}{2}\left(1 - F\left(\theta_R\right)\right)^{n-2}\left(F\left(\theta_R\right)\right)^2\left[2\left(v_1(\theta_R) - \gamma_1\right)\frac{f(\theta_R)}{F(\theta_R)}\frac{\partial \theta_R}{\partial r_1}\right]. \end{aligned}$$

The first term expresses the marginal change in revenue when a single bidder was present at the auction. The probability of attracting one and only one bidder is given by $n\left(1-F\left(\theta_R\right)\right)^{n-1}F\left(\theta_R\right)$.

The term $\dfrac{n(n-1)}{2}\left(1-F\left(\theta_R\right)\right)^{n-2}\left(F\left(\theta_R\right)\right)^2$ is the probability that exactly 2 bidders attended the auction. In any equilibrium where $r_1 > r_2$ seller 1's market share is given by $[0, \theta_R]$ where $\theta_R < r_1$. All of the bidders he gathers have valuations *strictly* greater than the reserve price. When 2 bidders attend the same auction the minimum bid (and thus price) is at least equal to $v_1\left(\theta_R\right) > \gamma_1$. Thus, conditional on attracting 2 bidders, the seller saves $2\left(v_1(\theta_R) - \gamma_1\right)$ which is subject to marginal changes in the reserve price. Conditional on attracting 3 bidders, marginal changes in the reserve price no longer affects the seller's revenue. The lowest possible bid (potentially equal to $v_1(\theta_R)$) is never the price.

6 Conclusion

This paper deals with strategic market allocation achieved by two sellers using auctions instead of prices. More precisely sellers strategically set their reserve prices which in turn determine the bidders' attendance. We give a necessary and sufficient condition to reach an efficient allocation of bidders, where by efficient we mean that buyers bid for the item they most value. The results can be summarized as follows.

Efficiency is reached if and only if the distribution of bidders is not skewed, whether sellers compete or cooperate. If an object is preferred on average, sellers are able to take advantage of this by setting a higher reserve price for this item. This allows to reduce informational rents for this item while it promotes participation for the least preferred item's auction. In general, market shares are determined with priority based on the expected surplus a bidder generates.

The cooperative solution is such that reserve prices are equal to the indifferent bidder's valuations so that he gets no rents. Market splits so that the expected value of the marginal bidder is the same to either seller.

The competitive solution departs from the cooperative one in different ways according to the degree of product differentiation. As sellers hold close substitutes the weakest, indifferent bidder gathers participation rents. As product differentiation increases, sellers save on these rents as they refrain from infringing their opponent's market share. In such cases there exists several equilibria for which sellers are constrained by the choice of their opponent's reserve price. We show that the cooperative equilibrium is one of the potential outcomes.

Several interesting extensions could be considered. First, a natural question is what mechanism would sellers use in equilibrium if this was part of their strategic choices. Such an issue is complex not only because of the level of interdependence auction performances exhibit mentioned in the introduction. The consideration of horizontally differentiated substitutes potentially triggers countervailing incentives. Indeed, as a buyer lies about his type it affects the expected utility from attending a specific auction as well as his reservation utility. If sellers cooperate, it can be in their interest to resort to inefficient allocations (see, for instance, Parlane [7]) to play with these countervailing incentives so as to save on informational rents. If sellers compete it is not clear whether the best reply to an efficient mechanism, such as the English auction, is also an efficient mechanism.
A second interesting extension would be to analyze sequential versus simultaneous auctioning. The analysis performed in this paper shows that entry is more severely restricted under simultaneous auctions. The obvious is that competition for each item is then weaker. Yet, higher reserve prices also mean less informational rents.

7 Appendix

- Appendix 1: Proof of proposition 1.

If reserve prices are such that $r_1 \leq r_2$, the proof is trivial since the reservation utility is zero.

Let the reserve prices be such that $r_1 > r_2$.

Claim 1:For any $R \in]0, +\infty) \times (-\infty, 1[$ with $r_1 > r_2$, θ_R, defined by (1), is unique and always within the interval $[0, 1]$.

Proof: Consider the function

$$H(\theta) = (1 - F(\theta))^{n-1} (r_1 - \theta) - F^{n-1}(\theta) (\theta - r_2),$$

defined over $[0, 1]$. Given (1),we have:

$$H(\theta_R) \equiv 0.$$

Given any R such that $r_1 > r_2$, we have[5] $H(0) > 0$ and $H(1) < 0$. Since $H(.)$ is continuous, there exists at least one value $\theta_R \in [0, 1]$ such that $H(\theta_R) = 0$. For any $\theta \geq \min\{r_1, 1\}$, $H(\theta) < 0$. For any $\theta \leq \max\{0, r_2\}$, $H(\theta) > 0$. Thus, all solutions to $H(\theta) = 0$ lie within the range $[r_2, r_1]$. Given this, we have

$$\left.\frac{dH}{d\theta}\right|_{\theta=\theta_R} < 0.$$

Thus, there exists at most one $\theta_R \in [0, 1]$ such that $H(\theta_R) \equiv 0$.

Claim 2: The buyers' strategy depicted in proposition 1 forms a Nash equilibrium.

Proof: Assume that $(n-1)$ buyers adopt the strategy depicted in proposition 1. Consider a buyer of type θ. Let $U_i(\theta)$ with $i = 1, 2$ denote this bidder's expected payoff when attending seller i $(i = 1, 2)$. (We focus at the case where $0 < r_2 < r_1 < 1$. The extension to $r_1 > 1$ and/or $r_2 < 0$ is trivial.)

-If $\theta \in [r_1, 1]$ then $U_2(\theta) > 0 > U_1(\theta)$: attending seller 2 is a best reply.

-If $\theta \in [0, r_2]$ then $U_1(\theta) > 0 > U_2(\theta)$: attending seller 1 is a best reply.

-If $\theta \in [r_1, r_2]$. We have

$$U_1(\theta) = \begin{cases} (1 - F(\theta_R))^{n-1} t(r_1 - \theta) \\ \quad + \int_{\theta}^{\theta_R} t(x - \theta)(n-1)(1 - F(x))^{n-2} f(x)dx \text{ if } \theta < \theta_R \\ (1 - F(\theta_R))^{n-1} t(r_1 - \theta) \text{ if } \theta \geq \theta_R, \end{cases}$$

[5] Note that $r_1 > 0$ and $r_2 < 1$.

and

$$U_2(\theta) = \begin{cases} (F(\theta_R))^{n-1} t(\theta - r_2) \text{ if } \theta \leq \theta_R \\ (F(\theta_R))^{n-1} t(\theta - r_2) \\ \qquad + \int_\theta^{\theta_R} t(\theta - x)(n-1)(F(x))^{n-2} f(x)dx \text{ if } \theta > \theta_R. \end{cases}$$

For any $\theta \in [r_1, r_2]$, we have $\dfrac{dU_1}{d\theta} < 0$ and $\dfrac{dU_2}{d\theta} > 0$. Moreover we have $U_1(\theta_R) = U_2(\theta_R)$ by definition of θ_R, and $U_i(\theta_R) > 0$ for $i = 1, 2$. Thus for any $\theta < \theta_R$ (respectively $\theta > \theta_R$) attending seller 1 (respectively seller 2) forms a best reply. Therefore the strategy depicted in proposition 1 forms a Nash equilibrium.

- Appendix 2: Proof of proposition 2.

To clarify the presentation I will write the sellers' profits as a function of (r_1, r_2) instead of (γ_1, γ_2) and consider r as a strategic variable instead of γ. There is no loss in generalities in doing so.

Let $\pi_i(r_i, r_j)$ denote seller i's expected profit function ($i = 1, 2$ and $j \neq i$). Let $MR_i(., t)$ ($i = 1, 2$) and $g_i(., t)$ ($i = 1, 2$) be the functions defined by (4), (7), (10) and (11) in the text.

The first order condition (FOC hereafter) for each seller are given by:

Seller 1:

$$\frac{\partial \pi_1}{\partial r_1} = \begin{cases} n(1 - F(r_1))^{n-1} f(r_1) MR_1(r_1, t) \text{ for any } r_1 \leq r_2, \\ nf(\theta_R)\dfrac{\partial \theta_R}{\partial r_1}\left[g_1(\theta_R, t) - nt(1 - F(\theta_R))^{n-1}(r_1 - \theta_R)\right] \text{ for } r_1 > r_2. \end{cases} \tag{15}$$

Seller 2:

$$\frac{\partial \pi_2}{\partial r_2} = \begin{cases} -n(F(r_2))^{n-1} f(r_2) MR_2(r_2, t) \text{ for any } r_2 \geq r_1, \\ nf(\theta_R)\dfrac{\partial \theta_R}{\partial r_2}\left[nt(F(\theta_R))^{n-1}(\theta_R - r_2) - g_2(\theta_R, t)\right] \text{ for } r_2 < r_1. \end{cases} \tag{16}$$

Assume $t > \bar{t}$. As figure 1 shows, there always exist θ_1 and θ_2 such that $MR_i(\theta_i, t) = 0$ for $i = 1, 2$ and such that $\theta_1 < \theta_2$.

Assume seller 2 sets $\gamma_2 = v_2(\theta_2)$. Over the interval $[0, \theta_2]$ $\pi_1(r_1, \theta_2)$ reaches a maximum at θ_1 since $\dfrac{\partial \pi_1}{\partial r_1} = 0$ at θ_1 and concavity is ensured under the regularity assumption. The expected profit π_1 is continuous at $r_1 = \theta_2$. We

have $\dfrac{\partial \pi_1}{\partial r_1} < 0$ for all $r_1 \geq \theta_2$ since $\left(r_1 - \theta_{(r_1,\theta_2)}\right) > 0$ and $g_1(\theta_{(r_1,\theta_2)}, t) < MR_1(\theta_{(r_1,\theta_2)}, t) < 0$.
Assume seller 1 sets $\gamma_1 = v_1(\theta_1)$. Over the interval $[\theta_1, 1]$ $\pi_2(r_2, \theta_1)$ reaches a maximum at θ_2 since $\dfrac{\partial \pi_2}{\partial r_2} = 0$ at θ_2 and concavity is ensured under the regularity condition. The expected profit π_2 is continuous at $r_2 = \theta_1$. We have $\dfrac{\partial \pi_2}{\partial r_2} > 0$ for all $r_2 \leq \theta_1$ since $\left(\theta_{(\theta_1,r_2)} - r_2\right) > 0$ and $g_2(\theta_{(\theta_1,r_2)}, t) < MR_2(\theta_{(\theta_1,r_2)}, t) < 0$. Thus, independently on whether they compete or maximize joints profits, seller i's optimal reserve price when $t > \bar{t}$ is given by $\gamma_i = v_i(\theta_i)$ for $i = 1, 2$.

- Appendix 3: Proof of proposition 3.

Let $t \leq \bar{t}$ and assume sellers maximize joint profits.

In the text we prove formally that, joint profit maximization requires $r_1 = r_2 = r$. Using this result, we can write the derivative of joint profits as:

$$\frac{d(\pi_1 + \pi_2)}{dr} = f(r)\left[(1 - F(r))^{n-1} MR_1(r,t) - (F(r))^{n-1} MR_2(r,t)\right].$$

Consider the function

$$h(r) = (1 - F(r))^{n-1} MR_1(r,t) - (F(r))^{n-1} MR_2(r,t). \tag{17}$$

Let θ_i be such that $MR_i(\theta_i, t) \equiv 0$ for $i = 1, 2$. For any $t \leq \bar{t}$ we have $\theta_1 \geq \theta_2$. The function $h(r)$ is strictly decreasing over $[\theta_2, \theta_1]$. We have $h(\theta_2) > 0$ and $h(\theta_1) < 0$. Thus, over the range $[\theta_2, \theta_1]$ there is one and only one value r^c such that $h(r^c) = 0$. For any $r < \theta_2$, we have $h(r) > 0$, and for any $r > \theta_1$ we have $h(r) < 0$. Thus, r^c is a unique solution to $\dfrac{d(\pi_1 + \pi_2)}{dr} = 0$ such that

$$\frac{d\pi}{dr} \underset{<}{\overset{>}{-}} 0 \text{ as } r \underset{>}{\overset{<}{-}} r^c.$$

Thus the sellers' joint profits reach a maximum at $r = r^c$.

- Appendix 4: Proof of lemma 1: $r^c = \dfrac{1}{2} \Leftrightarrow \theta_M = \dfrac{1}{2}$.

($\Leftarrow$) Assume $\theta_M = \frac{1}{2}$, then we have $\widehat{\theta} = \frac{1}{2}$ and thus $h(r^c) = 0$ at $r^c = \frac{1}{2}$.

($\Rightarrow$) To prove this point we will show that if the median is not situated at $\frac{1}{2}$ then the allocation of bidders cannot be efficient. In other words we will show that $\theta_M \neq \frac{1}{2} \Rightarrow r^c \neq \frac{1}{2}$.

Assume $F(\frac{1}{2}) > \frac{1}{2}$, then it is true that $\left(1 - F(\frac{1}{2})\right)^{n-1} < \left(F(\frac{1}{2})\right)^{n-1}$. Moreover, considering (8), we would have $MR_1\left(\frac{1}{2}, t\right) < MR_2\left(\frac{1}{2}, t\right)$. Thus, we necessarily have $h(\frac{1}{2}) < 0$, which proves that $r^c = \frac{1}{2}$ is not a solution.

Similar steps show that if $F\left(\frac{1}{2}\right) < \frac{1}{2}$, then $h\left(\frac{1}{2}\right) > 0$.

- Appendix 5: Proofs of proposition 4.

The first step consists in proving that there exists a range of degree of product differentiation below which both, (12) and (13) can hold.

Lemma 3: There exists a unique $\underline{t} \in \left]0, \bar{t}\right[$ such that for any $t \in \left]0, \underline{t}\right[$ there exists a unique $\theta_t \in [\theta_2, \theta_1]$ for which $g_1(\theta_t, t) = g_2(\theta_t, t)$. The variable $\underline{t}$ solves

$$g_1\left(\theta_{\underline{t}}, \underline{t}\right) = g_2\left(\theta_{\underline{t}}, \underline{t}\right) = 0.$$

Proof:

(1) Existence and uniqueness of θ_t.

Under the regularity assumption, the function $(g_1(\theta, t) - g_2(\theta, t))$ is continuous over $[0, 1]$ and decreasing in θ over $[\theta_2, \theta_1]$. Let t_l be defined as the highest value for t such that for any $t \in (0, t_l]$, $g_1(\theta_2, t) - g_2(\theta_2, t) > 0$ and $g_1(\theta_1, t) - g_2(\theta_1, t) < 0$ (it is obvious that $t_l > 0$). For any $t \in (0, t_l]$, we necessarily have $g_1(\theta, t) - g_2(\theta, t) > 0$ for any $\theta \in [0, \theta_2]$ while $g_1(\theta, t) - g_2(\theta, t) < 0$ for any $\theta \in [\theta_1, 1]$. Thus, for any such t, there exists *at most one* θ_t such that:

$$g_1(\theta_t, t) - g_2(\theta_t, t) = 0. \tag{18}$$

(2) Existence and uniqueness of $\underline{t}$.

Let $\underline{t}$ denote the degree of product differentiation such that:

$$g_1(\theta_{\underline{t}}, \underline{t}) = g_2(\theta_{\underline{t}}, \underline{t}) = 0.$$

Given that $g_i(\theta_{\underline{t}}, \underline{t}) = 0$ (for $i = 1, 2$) we necessarily have $MR_i(\theta_{\underline{t}}, \underline{t}) > 0$, for $i = 1, 2$, which means that $\theta_{\underline{t}} \in [\theta_2, \theta_1]$. This implies that $\underline{t} \in (0, t_l]$ for which existence and uniqueness of $\theta_{\underline{t}}$ has been established. The variable $\underline{t}$ is then solution to

$$\underline{t} = MR_1(\theta_{\underline{t}}, \underline{t}) \left[\frac{1 - F(\theta_{\underline{t}})}{F(\theta_{\underline{t}})}\right]^{n-1} f(\theta_{\underline{t}}), \text{ where } \theta_{\underline{t}} \in [\theta_2, \theta_1].$$

It is obvious that $\underline{t} > 0$.

We necessarily have $\underline{t} < \bar{t}$ since at $t = \bar{t}$, we have $\theta_1 = \theta_2 = \widehat{\theta}$, and $g_i(\widehat{\theta}, \bar{t}) < 0$. The function $g_1(\theta, \bar{t})$ is decreasing over $\left[0, \widehat{\theta}\right]$, and $g_2(\theta, \bar{t})$ increases over $\left[\widehat{\theta}, 1\right]$. Thus, at $t = \bar{t}$, there exist $\widetilde{\theta}_1$ and $\widetilde{\theta}_2$, such that $g_i\left(\widetilde{\theta}_i, t\right) = 0$ for $i = 1, 2$ with $\widetilde{\theta}_1 < \widetilde{\theta}_2$. Thus, wherever, they meet, the functions $g_1(.)$ and $g_2(.)$ are necessarily strictly negative.

Uniqueness of $\underline{t}$. Consider any $t \in (0, t_l]$. For such t let $G(t) \equiv g_i(\theta_t, t)$ where $i = 1$ or 2. The function $G(t)$ is continuous. As $t \to 0$, $G(t) > 0$ since $g_i(\theta, 0) > 0$ for any $\theta \in]0, 1[$. To prove uniqueness of $\underline{t}$ we prove that $G(t)$ is decreasing in t. We have

$$\frac{dG}{dt} = \frac{\partial g_i}{\partial \theta}\frac{d\theta_t}{dt} + \left.\frac{\partial g_i}{\partial t}\right|_{\theta = \theta_t}.$$

Using the implicit function theorem we have

$$\frac{d\theta_t}{dt} = -\left.\frac{\dfrac{\partial g_1}{\partial t} - \dfrac{\partial g_2}{\partial t}}{\dfrac{\partial g_1}{\partial \theta} - \dfrac{\partial g_2}{\partial \theta}}\right|_{\theta = \theta_t}.$$

Since $\left(\dfrac{\partial g_1}{\partial \theta} - \dfrac{\partial g_2}{\partial \theta}\right) < 0$ at θ_t, it is true that

$$\text{sign of } \frac{dG}{dt} = \text{sign of } \left[\frac{\partial g_i}{\partial \theta}\left(\frac{\partial g_1}{\partial t} - \frac{\partial g_2}{\partial t}\right) + \frac{\partial g_i}{\partial t}\left(\frac{\partial g_2}{\partial \theta} - \frac{\partial g_1}{\partial \theta}\right)\right],$$

for $i = 1$ this leads to

$$\text{sign of } \frac{dG}{dt} = \text{ sign of } \left[\frac{\partial g_1}{\partial \theta}\left(-\frac{\partial g_2}{\partial t}\right) + \frac{\partial g_1}{\partial t}\left(\frac{\partial g_2}{\partial \theta}\right)\right],$$

for $i = 2$ it leads to

$$\text{sign of } \frac{dG}{dt} = \text{ sign of } \left[\frac{\partial g_2}{\partial \theta}\left(\frac{\partial g_1}{\partial t}\right) + \frac{\partial g_2}{\partial t}\left(-\frac{\partial g_1}{\partial \theta}\right)\right].$$

Over the range $[\theta_2, \theta_1]$ we have $\frac{\partial g_1}{\partial \theta} < 0$ and $\frac{\partial g_2}{\partial \theta} > 0$. We always have $\frac{\partial g_i}{\partial t} < 0$ for $i = 1, 2$. Thus, whether one sets $i = 1$ or $i = 2$ in the expression above, we always have that the sign of $\frac{dG}{dt}$ is negative.

We may now prove that the reservation prices described in proposition 4 form an equilibrium. Let $t \leq \underline{t}$. Consider (γ_1^*, γ_2^*) presented in proposition 4. They satisfy the first order conditions. Moreover, since these reserve prices are set such that $\theta_{R^*} = \theta_t$ for any $t < \underline{t}$, the additional restrictions (12) and (13) hold. Thus, all we must prove is that they also satisfy the second order conditions. In order to do so, we need more information on the functions $g_i(\theta, t)$ $(i = 1, 2)$.

(1) There is a unique $\widetilde{\theta}_1$ such that $g_1\left(\widetilde{\theta}_1, t\right) = 0$ for any t.

Proof: We have $g_1(0, t) = 1$, and $g_1(1, t) = -\frac{t}{f(1)} < 0$. Since g_1 is a continuous function, there exists at least one $\widetilde{\theta}_1$ such that $g_1\left(\widetilde{\theta}_1, t\right) = 0$. Moreover, because $g_1(\theta, t)$ is decreasing in θ for all θ such that $MR_1(\theta, t) \geq 0$ and since $MR_1\left(\widetilde{\theta}_1, t\right) > 0$, we always have $\frac{d}{d\theta} g_1(\theta, t) < 0$ at $\widetilde{\theta}_1$. Thus, $\widetilde{\theta}_1$ is unique.

(2) There is a unique $\widetilde{\theta}_2$ such that $g_2\left(\widetilde{\theta}_2, t\right) = 0$ for any t.

Proof: We have $g_2(1, t) = 1$, and $g_2(0, t) = -\frac{t}{f(0)} < 0$. Since g_2 is a continuous function, there exists at least one $\widetilde{\theta}_2$ such that $g_2\left(\widetilde{\theta}_2, t\right) = 0$.

Moreover, because the function $g_2(\theta, t)$ is increasing in θ for all θ such that $MR_2(\theta, t) \geq 0$ and since $MR_2\left(\widetilde{\theta}_2, t\right) > 0$, we always have $\frac{d}{d\theta} g_2(\theta, t) > 0$ at $\widetilde{\theta}_2$. Thus, $\widetilde{\theta}_2$ is unique.

Given the previous points we know that the curves of marginal revenues and $g_i(.)$ interact as follows for $t < \underline{t}$:

Insert figure 3 here.

We can now prove that the proposed solution maximizes each seller's expected profit. For any of these reserve prices we have $r_2^* < \theta_t < r_1^*$.

Consider seller 1. He takes as given r_2^*, such that $r_2^* < \theta_t$. Clearly, we have $\frac{\partial \pi_1}{\partial r_1} > 0$ for all $r_1 < r_2^*$ since $MR_1(r, t) > 0$ for all $r \leq \theta_t$. For any $r_1 \in [r_2^*, +\infty)$, we have (given (15)):

$$\frac{\partial \pi_1}{\partial r_1} = nf(\theta_R) \frac{\partial \theta_R}{\partial r_1} (1 - F(\theta_R))^{n-1} H_1(\theta_R, r_1, t) = 0,$$

at $r_1 = r_1^*$ and $R = (r_1, r_2^*)$, with

$$H_1(\theta_R, r_1, t) = \left[MR_1(\theta_R, t) - t\frac{F(\theta_R)}{f(\theta_R)} \left[\frac{F(\theta_R)}{1 - F(\theta_R)} \right]^{n-1} - nt(r_1^* - \theta_R) \right].$$

Since $0 < \frac{\partial \theta_R}{\partial r_i} < 1$ (trivial), and under the regularity assumption we have $\frac{\partial H_1}{\partial r_1} < 0$. Thus:

$$\frac{\partial \pi_1}{\partial r_1} \overset{>}{\underset{<}{-}} 0 \text{ as } r_1 \overset{<}{\underset{>}{-}} r_1^*$$

which proves that r_1^* is best reply to r_2^*.

Consider now seller 2. He takes as given r_1^*, such that $r_1^* > \theta_t$. Clearly, we have $\frac{\partial \pi_2}{\partial r_2} < 0$ for all $r_2 > r_1^*$ since $MR_2(r, t) > 0$ for all $r \geq \theta_t$. For any $r_2 \in (-\infty, r_1^*]$, we have (given(16)) :

$$\frac{\partial \pi_2}{\partial r_2} = nf(\theta_R) \frac{\partial \theta_R}{\partial r_1} (F(\theta_R))^{n-1} H_2(\theta_R, r_2, t) = 0 \text{ at } r_2 = r_2^* \text{ and } R = (r_1^*, r_2),$$

with

$$H_2(\theta_R, r_2, t) = \left[nt(\theta_R - r_2) - MR_2\left(\theta_R, t\right) + t\frac{1 - F(\theta_R)}{f(\theta_R)} \left[\frac{1 - F\left(\theta_R\right)}{F\left(\theta_R\right)}\right]^{n-1} \right].$$

Since $0 < \dfrac{\partial \theta_R}{\partial r_i} < 1$, and under the regularity assumption we have $\dfrac{\partial H}{\partial r_2} < 0$. Thus:

$$\frac{\partial \pi_2}{\partial r_2} \overset{>}{\underset{<}{-}} 0 \text{ as } r_2 \overset{<}{\underset{>}{-}} r_2^*$$

which proves that r_2^* is best reply to r_1^*. (Expression (14) can be found easily using (12).)

Efficiency: $\theta_M = \dfrac{1}{2} \Leftrightarrow \theta_{R^*} = \dfrac{1}{2}$.

($\Rightarrow$) It is straightforward to show that when $\theta_M = \dfrac{1}{2}$, the equilibrium is symmetric. We have $\widehat{\theta} = \dfrac{1}{2}$ and setting $\theta_{R^*} = \dfrac{1}{2}$, satisfies the first order conditions as well as (14).

($\Leftarrow$) We use a similar approach as the one presented in Appendix 3, for the proof of lemma 1. We prove that $\theta_M \neq \dfrac{1}{2} \Rightarrow \theta_{R^*} \neq \dfrac{1}{2}$, but this time we use contradiction in the proof. Assume that $\theta_M \neq \dfrac{1}{2}$, but that $\theta_{R^*} = \dfrac{1}{2}$ forms a solution. We show that a contradiction arises as $\theta_{R^*} = \dfrac{1}{2}$ cannot possibly solve (14). Consider in details the case where $\theta_M < \dfrac{1}{2}$, that is $F\left(\dfrac{1}{2}\right) > \dfrac{1}{2}$.

In the proof of lemma 1, we established that if $F(\dfrac{1}{2}) > \dfrac{1}{2}$, then it is true that $\left(1 - F(\dfrac{1}{2})\right)^{n-1} < \left(F(\dfrac{1}{2})\right)^{n-1}$ and that $MR_1\left(\dfrac{1}{2}, t\right) < MR_2\left(\dfrac{1}{2}, t\right)$. Assume now that $\theta_{R^*} = \dfrac{1}{2}$ forms a solution. Under this assumption and by using (1), showing that

$$[1 - F(\theta_{R^*})]^{n-1} \left[v_1(\theta_{R^*} - \frac{1 - F(\theta_{R^*})}{f(\theta_{R^*})}) - \gamma_1^* \right]$$

$$< [F(\theta_{R^*})]^{n-1} \left[v_2 \left(\theta_{R^*} + \frac{F(\theta_{R^*})}{f(\theta_{R^*})} \right) - \gamma_2^* \right]$$

at $\theta_{R^*} = \dfrac{1}{2}$ is equivalent to showing that

$$[1-F(\theta_{R^*})]^{n-1}\left[v_1(\theta_{R^*}-\frac{1-F(\theta_{R^*})}{f(\theta_{R^*})})-v_1\left(\theta_{R^*}\right)\right]$$
$$<[F(\theta_{R^*})]^{n-1}\left[v_2\left(\theta_{R^*}+\frac{F(\theta_{R^*})}{f(\theta_{R^*})}\right)-v_2\left(\theta_{R^*}\right)\right]$$

at $\theta_{R^*} = \dfrac{1}{2}$. Indeed, all that was done was to add and subtract $v_1\left(\theta_{R^*}\right)$ in the bracket on the left hand side, and add and subtract $v_2\left(\theta_{R^*}\right)$ in the brackets on the right hand side. This inequality holds since at $\theta_{R^*} = \dfrac{1}{2}$, we have $v_1\left(\theta_{R^*}\right) = v_2\left(\theta_{R^*}\right)$ and $v_1(\theta_{R^*} - \dfrac{1-F(\theta_{R^*})}{f(\theta_{R^*})}) < v_2\left(\theta_{R^*} + \dfrac{F(\theta_{R^*})}{f(\theta_{R^*})}\right)$ when $F(\dfrac{1}{2}) > \dfrac{1}{2}$. Thus, $\theta_{R^*} = \dfrac{1}{2}$ cannot form a solution as all terms on the right hand side of (14) are less than those of the right hand side.

A symmetric argumentation can be used to prove that a contradiction arises when $\theta_M > \dfrac{1}{2}$, that is $F\left(\dfrac{1}{2}\right) < \dfrac{1}{2}$.

- Appendix 6: Proof of proposition 5.

Let $t \in \left]\underline{t}, \bar{t}\right[$. For such t, the marginal revenue and $g_i(.,.)$ curves interact as follows:

Insert figure 4 here.

(1) The set Θ_t is always non-empty. Let

$$\Omega_i = \{\theta : MR_i\left(\theta, t\right) \geq 0 \text{ and } g_i\left(\theta, t\right) \leq 0\}$$

for $i = 1, 2$. Since $g_1\left(\theta, t\right)$ is decreasing over $[0, \theta_1]$ with $g_1(\theta_1, t) < 0$, Ω_1 is non empty and $\Omega_1 = \left[\widetilde{\theta}_1, \theta_1\right]$, with $g_1\left(\widetilde{\theta}_1, t\right) = 0$. Since $g_2\left(\theta, t\right)$ is increasing over $[\theta_2, 1]$ with $g_2\left(\theta_2, t\right) < 0$, Ω_2 is also non-empty and $\Omega_2 = \left[\theta_2, \widetilde{\theta}_2\right]$ with $g_2\left(\widetilde{\theta}_2, t\right) = 0$.

By definition we have $\Theta_t = \Omega_1 \cap \Omega_2$. As $t < \bar{t}$ we have $\theta_2 < \theta_1$. Thus, Θ_t would be empty if and only if $\widetilde{\theta}_2 < \widetilde{\theta}_1$. However, for $t > \underline{t}$, we have $g_i\left(\theta_t, t\right) < 0$ for $i = 1, 2$ and therefore $\widetilde{\theta}_2 > \widetilde{\theta}_1$.

Consider now any $r_2^* \in \Theta_t$. For any $r_1 < r_2^*$ we have $\dfrac{\partial \pi_1}{\partial r_1} > 0$ since $MR_1(r,t) > 0$ for all $r < r_2^*$. For all $r_1 > r_2^*$ we have $\left(\theta_{(r_1, r_2^*)} - r_1\right) < 0$ and $g_1(\theta_{(r_1,r_2^*)}, t) < 0$ which implies that $\dfrac{\partial \pi_1}{\partial r_1} < 0$ for all $r_1 \geq r_2^*$. At r_2^*, π_1 is not differentiable and forms a peak since $\left.\dfrac{\partial \pi_1}{\partial r_1}\right|_{r_2^*-\varepsilon} > \left.\dfrac{\partial \pi_1}{\partial r_1}\right|_{r_2^*+\varepsilon}$ as $\varepsilon \to 0$. Yet, it also forms a maximum (as it appears on picture 2 in the text)..

Consider now any $r_1^* \in \Theta_t$. For any $r_2 > r_1^*$ we have $\dfrac{\partial \pi_2}{\partial r_2} < 0$ since $MR_2(r,t) > 0$ for all $r > r_1^*$. For all $r_2 < r_1^*$ we have $\left(\theta_{(r_1^*, r_2)} - r_2\right) > 0$ and $g_2(\theta_{(r_1^*,r_2)}, t) < 0$ which implies that $\dfrac{\partial \pi_2}{\partial r_2} > 0$ for all $r_2 < r_1^*$. Once again at $r_2 = r_1^*$ the profit function forms a peak.

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Figures

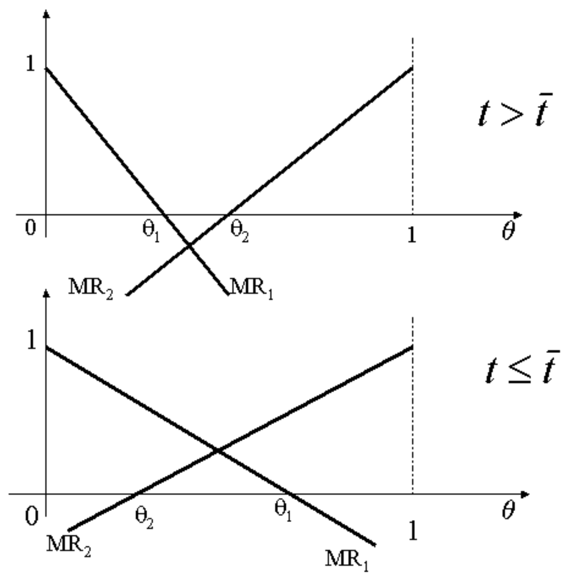

Figure 1: Represents the marginal revenue curves for different degrees of product differentiation.

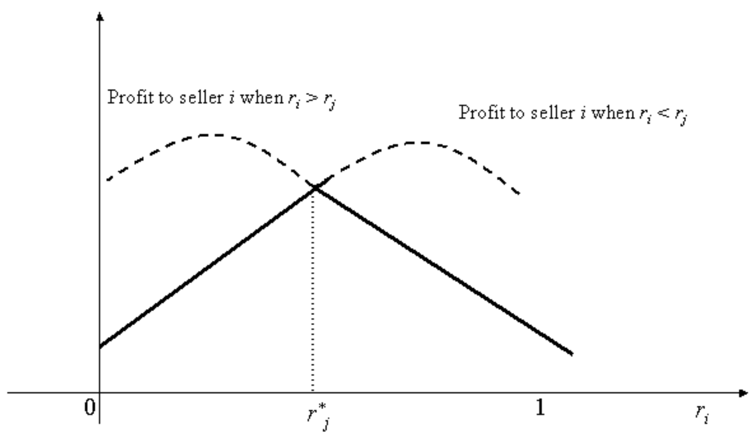

Figure 2: Illustration of proposition 5.

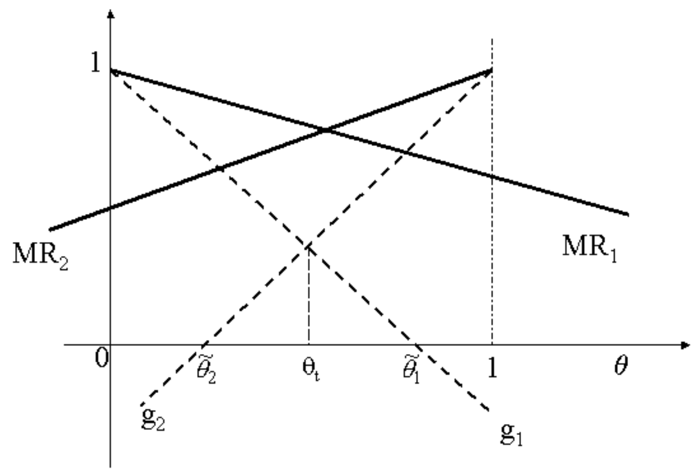

Figure 3: Representation of the functions $MR_i(\theta, t)$ and $g_i(\theta, t)$ for $t \in [0, \underline{t}]$.

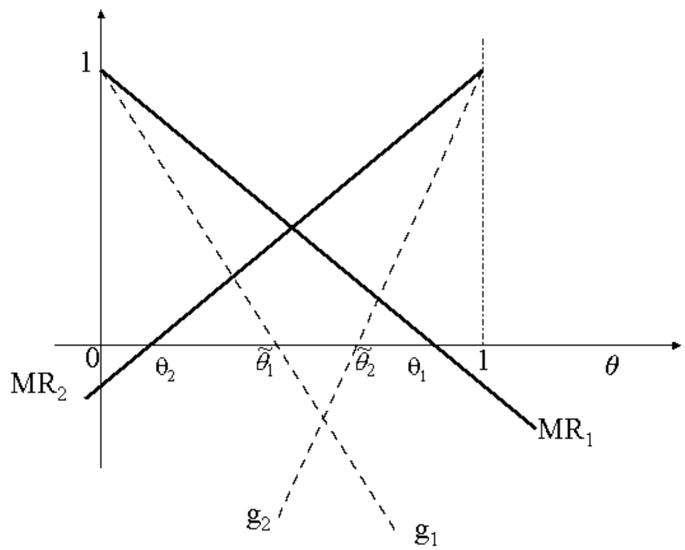

Figure 4: Representation of the functions $MR_i(\theta, t)$ and $g_i(\theta, t)$ for $t \in \left[\underline{t}, \overline{t}\right]$.