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On a class of singular elliptic systems

Marius Ghergu

Abstract

We study the semilinear elliptic system

\[ \begin{cases} 
-\Delta u = u^{-p} + v^{-q}, & u > 0 \text{ in } \Omega, \\
-\Delta v = u^{-r} + v^{-s}, & v > 0 \text{ in } \Omega, \\
u = v = 0 & \text{ on } \partial \Omega,
\end{cases} \]

where \( \Omega \subset \mathbb{R}^N \) \((N \geq 1)\) is a smooth and bounded domain, \( p, q, r, s > 0 \). Under suitable ranges of exponents we obtain various results regarding the well posedness of our system.

Key words: Singular elliptic system, negative exponents, boundary behavior

1991 MSC: 35J55, 35B40, 35B50

1 Introduction and the main results

We are concerned in this paper with qualitative properties of solutions to the system

\[ \begin{cases} 
-\Delta u = u^{-p} + v^{-q}, & u > 0 \text{ in } \Omega, \\
-\Delta v = u^{-r} + v^{-s}, & v > 0 \text{ in } \Omega, \\
u = v = 0 & \text{ on } \partial \Omega,
\end{cases} \]  

where \( \Omega \subset \mathbb{R}^N \) \((N \geq 1)\) is a smooth and bounded domain, \( p, q, r, s > 0 \).

Solutions \((u, v)\) to (1) are understood in the classical sense, that is, \( u, v \in C^2(\Omega) \cap C(\overline{\Omega}) \).

The system (1) appears as a natural extension of the single singular problem

\[ \begin{cases} 
-\Delta u = u^{-p}, & u > 0 \text{ in } \Omega, \\
u = 0 & \text{ on } \partial \Omega,
\end{cases} \]  

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which was considered, among other works, in [3,9,15]. A particular feature of (2) in the case $p > 0$, and in contrast to the case $p < -1$ is that it has a unique solution. This fact will be used in dealing with (1) in order to study the existence of solutions.

Another singular elliptic system recently investigated in the literature is

\[
\begin{cases}
-\Delta u = u^{-p}v^{-q}, & u > 0 \text{ in } \Omega, \\
-\Delta v = u^{-r}v^{-s}, & v > 0 \text{ in } \Omega, \\
u = v = 0 & \text{on } \partial\Omega.
\end{cases}
\] (3)

The case $p, q, r, s > 0$ represents the singular counterpart of the standard Lane-Emden system and was discussed in [5,6,10,14,17].

The case $q, s > 0$ and $p, r < 0$ corresponds to the Gierer-Meinhardt system [11,12] with homogeneous Dirichlet boundary conditions (see [1,2,4,7–10]). Such a system describes the pattern formation of spatial tissue structures in morphogenesis, a biological phenomenon discovered by Trembley [16] in 1744.

Coming back to our system (1), we first state a nonexistence result.

**Theorem 1.1** (Nonexistence) Assume that one of the following two conditions hold.

(i) $2r < 1 + \max\{1,s\}$ and

\[
\min\left\{\frac{1}{1 + \max\{1,s\}}, 1 - \max\left\{\frac{1}{2}, 1 + \max\{1,p\}\right\}\right\} > \frac{1}{q}.
\] (4)

or

(ii) $2q < 1 + \max\{1,p\}$ and

\[
\min\left\{\frac{1}{1 + \max\{1,p\}}, 1 - \max\left\{\frac{1}{2}, 1 + \max\{1,s\}\right\}\right\} > \frac{1}{r}.
\] (5)

Then, the systems (1) has no solutions.

**Corollary 1.2** Assume that one of the following conditions hold:

$q > 1 + \max\{1,s\}$ and $2r < 1 + \max\{1,p\}$,

or

$r > 1 + \max\{1,p\}$ and $2q < 1 + \max\{1,s\}$.

Then, the systems (1) has no solutions.

More clearly but perhaps less precise, Corollary 1.2 states that if one of the exponents $q$ and $r$ is too small and the other is too big, then the systems (1) has no solutions.

In particular, from Corollary 1.2 we deduce that the system (1) has no solutions if

$q > 2 + s$ and $2r < 1 + p$,  

2
or
\[ r > 2 + p \quad \text{and} \quad 2q < 1 + s. \]

We shall next be concerned with the existence of a solution to (1). Our main result in this case is the following.

**Theorem 1.3** (Existence) Assume \( p, q, r, s > 0 \) satisfy
\[
q < 1 + \max\{1, s\} \quad \text{and} \quad r < 1 + \max\{1, p\}. \tag{6}
\]
Then, the system (1) has at least one classical solution.

**Corollary 1.4** Assume \( 0 < q, r < 2 \). Then, the system (1) has at least one classical solution.

In other words, and in contrast to Corollary 1.2, if \( q \) and \( r \) are both small, then a classical solution to system (1) always exists, regardless to the size of \( p \) and \( s \).

We should point out that there are regions for exponents \( p, q, r, s > 0 \) where we do not know whether the system (1) admits solutions. For instance, if
\[
q > \max\{1, s\} \quad \text{and} \quad r > 1 + \max\{1, p\}
\]
then, none of the conditions (4), (5) or (6) hold. In particular, for \( q \) and \( r \) large enough, we are not able to decide the (non)existence of a solution to (1).

**Theorem 1.5** (\( C^1 \)-regularity of solutions up to the boundary)
Let \((u, v)\) be a classical solution of (1).

(i) If \( p < 1 \) and \( 2q < 1 + \max\{1, s\} \) then \( u \in C^2(\Omega) \cap C^1(\overline{\Omega}) \);
(ii) If \( s < 1 \) and \( 2r < 1 + \max\{1, p\} \) then \( v \in C^2(\Omega) \cap C^1(\overline{\Omega}) \).

We are next concerned with the uniqueness of a solution to system (1). For the singular systems considered in [4–6,8] the uniqueness of the solution was deduced for some ranges of exponents \( p, q, r \) and \( s \) which imply either \( u \in C^1(\Omega) \) or \( v \in C^1(\Omega) \).

**Theorem 1.6** (Uniqueness) Assume \( p, q, r, s > 0 \) satisfy (6) and that one of the following holds.

(i) \( \frac{q}{1 + \max\{1, s\}} < \frac{\max\{1, p\}}{1 + \max\{1, p\}} \) and \( qr < 1 \);
(ii) \( \frac{r}{1 + \max\{1, p\}} < \frac{\max\{1, s\}}{1 + \max\{1, s\}} \) and \( qr < 1 \).

Then, the system (1) has a unique classical solution.
Corollary 1.7 Assume $p,q,r,s > 0$ satisfy
\[ 2q < 1 + \max\{1, s\}, \quad 2r < 1 + \max\{1, p\} \quad \text{and} \quad qr < 1. \] 
Then, the system (1) has a unique classical solution.

Corollary 1.8 Assume $p,s > 0$ and $0 < q,r < 1$. Then, the system (1) has a unique classical solution.

2 Some preliminary results

In this section we collect some basic results which will be useful in proving our main results. In the sequel $\Omega$ will be assumed to be a smooth and bounded domain of $\mathbb{R}^N$. We also denote by $\delta(x)$ the distance from $x \in \Omega$ to the boundary $\partial \Omega$. Given two positive functions $f,g$ defined in $\Omega$, we shall use $f \sim g$ to signify that $c^{-1}f \leq g \leq cf$ in $\Omega$ for some constant $c > 1$.

We start this section with the following comparison principle whose proof relies on the maximum principle.

Lemma 2.1 Let $K \in C(\Omega)$ and assume $u,v \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfy
\[ (i) \quad \Delta u + u^{-p} + K(x) \leq 0 \leq \Delta v + v^{-p} + K(x) \quad \text{in} \ \Omega, \]
\[ (ii) \quad u,v > 0 \quad \text{in} \ \Omega \quad \text{and} \quad v = 0 \quad \text{on} \ \partial \Omega. \]

Then $u \geq v$ in $\Omega$.

The following result stems from Crandall, Rabinowitz and Tartar [3].

Lemma 2.2 For any $p > 0$ there exists a unique solution $w \in C^2(\Omega) \cap C(\overline{\Omega})$ such that
\[ \begin{cases} 
-\Delta w = w^{-p} & \text{in} \ \Omega, \\
w > 0 & \text{in} \ \Omega, \\
w = 0 & \text{on} \ \partial \Omega, 
\end{cases} \] 

Furthermore, $w$ has the following asymptotic behavior
\[ w(x) \sim \begin{cases} 
\delta(x) & \text{if} \ 0 < p < 1, \\
\delta(x) \log^{1/2} \frac{1}{\delta(x)} & \text{if} \ p = 1, \\
\delta(x)^{\frac{1}{1+p}} & \text{if} \ p > 1.
\end{cases} \] 

In particular, there exists $C > 0$ such that
\[ w \geq C\delta(x)^{\frac{2}{1+\max\{1,p\}}} \quad \text{in} \ \Omega, \]
and for all $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$w \leq C_\varepsilon \delta(x)^{2-\varepsilon} \quad \text{in } \Omega.$$  \hfill (11)

**Lemma 2.3** Let $w \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfy

$$0 \leq -\Delta w \leq C\delta(x)^{-a} \quad \text{in } \Omega, \quad w = 0 \text{ in } \partial \Omega,$$

where $a \in (0, 2)$ and $C > 0$. Then, $w \in C^{0,\gamma}(\overline{\Omega})$ for some $\gamma \in (0, 1)$ and

(i) If $a \in (0, 1)$ then $w \in C^2(\Omega) \cap C^{1-a}(\Omega)$ and $w(x) \leq c\delta(x) \text{ in } \Omega$, for some $c > 0$;

(ii) If $a = 1$ then for all $\varepsilon > 0$ there exists $c_\varepsilon > 0$ such that $w(x) \leq c_\varepsilon \delta(x)^{1-\varepsilon} \text{ in } \Omega$;

(iii) If $a \in (1, 2)$ then $w(x) \leq c\delta(x)^{2-a} \text{ in } \Omega$, for some $c > 0$.

**Proof.** Let $G$ be the Green’s function for the negative Laplace operator. This yields

$$u(x) = -\int_{\Omega} G(x, y) \Delta u(y) dy,$$

for all $x \in \Omega$. Hence

$$|u(x_1) - u(x_2)| \leq -\int_{\Omega} |G(x_1, y) - G(x_2, y)| \Delta u(y) dy \leq c \int_{\Omega} |G(x_1, y) - G(x_2, y)| \delta(y)^{-a} dy,$$

for all $x_1, x_2 \in \Omega$. Next, using the method in [13, Theorem 1.1] we have

$$|u(x_1) - u(x_2)| \leq C|x_1 - x_2|^\gamma \quad \text{for some } 0 < \gamma < 1.$$

Hence $u \in C^{0,\gamma}(\overline{\Omega})$.

(i) Suppose $0 < a < 1$. Then,

$$\nabla u(x) = -\int_{\Omega} G_x(x, y) \Delta u(y) dy \quad \text{for all } x \in \Omega,$$

and

$$|
abla u(x_1) - \nabla u(x_2)| \leq -\int_{\Omega} |G_x(x_1, y) - G_x(x_2, y)| \Delta u(y) dy \leq c \int_{\Omega} |G_x(x_1, y) - G_x(x_2, y)| \delta(y)^{-a} dy.$$

The same technique as in [13, Theorem 1.1] yields

$$|
abla u(x_1) - \nabla u(x_2)| \leq C|x_1 - x_2|^{1-a} \quad \text{for all } x_1, x_2 \in \Omega.$$

Therefore $u \in C^{1,1-a}(\overline{\Omega})$. This also implies $w(x) \leq c\delta(x) \text{ in } \Omega$, for some $c > 0$.

(iii) Denote by $\lambda_1$ (resp. $\varphi_1$) the first eigenvalue (resp. eigenfunction) of $-\Delta$ in $\Omega$. By normalization, we can assume $\varphi_1 > 0$ in $\Omega$. Let now $\overline{w} := M\varphi_1^{2-a}$. A straightforward calculation yields

$$-\Delta \overline{w} = M(2-a)\varphi_1^{-a}[(a-1)|\nabla \varphi_1|^2 + \lambda_1 \varphi_1^2] \quad \text{in } \Omega.$$
By Hopf boundary point lemma and the maximum principle, we have 
\((a - 1)\left|\nabla \varphi_1\right|^2 + \lambda_1 \varphi_1^2 > c > 0\) in \(\Omega\), for some positive constant \(c > 0\). Thus, we can choose \(M > 1\) suitably large such that

\[-\Delta \overline{w} \geq C\delta^{-a}(x) \geq -\Delta w\quad \text{in } \Omega.\]

This yields \(w(x) \leq \overline{w}(x) \leq c\delta(x)^{2-a}\) in \(\Omega\).

(ii) This follows directly from part (iii) by noting that for \(a = 1\) and \(\epsilon > 0\) we have 
\(-\Delta w \leq C\epsilon \delta(x)^{-1-\epsilon}\) in \(\Omega\).

Our last result in this section concerns the problem

\[
\begin{cases}
-\Delta w = w^{-p} + K(x) & \text{in } \Omega, \\
w > 0 & \text{in } \Omega, \\
w = 0 & \text{on } \partial \Omega,
\end{cases}
\tag{12}
\]

where \(p > 0\) is a constant and \(K \in C(\Omega)\) is a positive function in \(\Omega\).

**Lemma 2.4** (i) If \(K(x)\) satisfies

\[
\int_\Omega \delta(x)K(x)dx = \infty,
\]

then (12) has no solutions.

(ii) If \(K(x) \geq c\delta(x)^{-a}\) in \(\Omega\), for some \(a \geq 2\) then (12) has no solutions.

(iii) If \(K(x) \leq C\delta(x)^{-a}\) in \(\Omega\), for some \(a \in (0, 2)\), then (12) has a unique solution 
\(w \in C^2(\Omega) \cap C(\overline{\Omega})\). Furthermore, for any \(\epsilon > 0\) there exists \(C_\epsilon > 0\) such that

\[
w(x) \leq C_\epsilon \delta(x)^{\min \left\{ -\frac{2-\epsilon}{1+p}, 2-\epsilon-\max\{1, a\} \right\}} \quad \text{in } \Omega.
\tag{13}
\]

**Proof.** Assume there exists a classical solution \(w \in C^2(\Omega) \cap C(\overline{\Omega})\) of (12). For \(\epsilon > 0\) denote

\[
\Omega_\epsilon := \{ x \in \Omega : \delta(x) > \epsilon \}.
\]

Thus, \(\Omega_\epsilon\) is a smooth domain provided \(\epsilon > 0\) is small enough. Denote by \(\lambda_{1,\epsilon}\) (resp. \(\varphi_{1,\epsilon}\)) the first eigenvalue (resp. eigenfunction) of \(-\Delta\) in \(\Omega_\epsilon\). We can normalize \(\varphi_1\) and \(\varphi_{1,\epsilon}\) and assume \(\|\varphi_1\|_\infty = \|\varphi_{1,\epsilon}\|_\infty = 1\). Consider the problem

\[
\begin{cases}
-\Delta w = (w + \epsilon)^{-p} + K(x) & \text{in } \Omega_\epsilon, \\
w > 0 & \text{in } \Omega_\epsilon, \\
w = 0 & \text{on } \partial \Omega_\epsilon.
\end{cases}
\tag{14}
\]

Clearly, the solution \(w\) of (12) is a supersolution to (14) while \(\underline{w} = m\varphi_{1,\epsilon}\) is a subso- lution of (14) provided \(m > 0\) is small enough. By Lemma 2.1 we have \(w \geq \underline{w}\) in \(\Omega_\epsilon\).
Thus, (14) has a solution \( w_\varepsilon \) which, by elliptic regularity, satisfies \( w_\varepsilon \in C^2(\Omega_\varepsilon) \). Let us next multiply by \( \varphi_{1,\varepsilon} \) in (14) and integrate over \( \Omega_\varepsilon \). We obtain

\[
\lambda_{1,\varepsilon} \int_{\Omega_\varepsilon} w \, dx \geq \lambda_{1,\varepsilon} \int_{\Omega_\varepsilon} \varphi_{1,\varepsilon} w \, dx \geq \lambda_{1,\varepsilon} \int_{\Omega_\varepsilon} \varphi_{1,\varepsilon} w_\varepsilon \, dx \geq \int_{\Omega_\varepsilon} \varphi_{1,\varepsilon} K(x) \, dx.
\]

A passage to the limit in the above estimate together with Fatou lemma yield

\[
M := \lambda_1 \int_\Omega w \, dx \geq \liminf_{\varepsilon \to 0} \int_{\Omega_\varepsilon} \varphi_{1,\varepsilon} K(x) \, dx \geq \int_\Omega \liminf_{\varepsilon \to 0} \varphi_{1,\varepsilon} \chi_{\Omega_\varepsilon} K(x) \, dx = \int_\Omega \varphi_1 K(x) \, dx \geq C \int_\Omega \delta(x) K(x) \, dx = \infty,
\]

contradiction. Hence, (12) has no classical solution.

(ii) This follows from part (i) since

\[
\int_\Omega \delta(x)^{-\gamma} \, dx = \infty, \quad \text{for all } \gamma \geq -1.
\]

This can be seen by using local co-ordinates near the boundary of \( \Omega \) as explained in [15].

(iii) Let \( w_1 \) be the solution of (8) and denote by \( \tilde{w} \) be solution of

\[
\begin{cases}
-\Delta \tilde{w} = K(x) & \text{in } \Omega, \\
\tilde{w} > 0 & \text{in } \Omega, \\
\tilde{w} = 0 & \text{on } \partial\Omega.
\end{cases}
\]

(15)

Note that such a solution \( \tilde{w} \) always exist due to the fact that \( K(x) \leq C\delta(x)^{-a} \) in \( \Omega \), with \( a \in (0, 2) \). It is easy to check that \( M \varphi_1^{2-a} \) is a supersolution of (15) while the zero function is a subsolution. This simple observation together with the maximum principle yield the existence and uniqueness of a solution \( \tilde{w} \in C^2(\Omega) \cap C(\overline{\Omega}) \) of (15). By Lemma 2.3 we have

\[
\tilde{w}(x) \leq \begin{cases}
C\delta(x) & \text{if } 0 < a < 1, \\
C_\varepsilon \delta(x)^{1-a} & \text{if } a = 1, \\
C\delta(x)^{2-a} & \text{if } 1 < a < 2.
\end{cases}
\]

We can summarize the above estimates by noting that for all \( \varepsilon > 0 \) there exists \( C_\varepsilon > 0 \) such that

\[
\tilde{w}(x) \leq C_\varepsilon \delta(x)^{2-\varepsilon-\max\{1,a\}} \quad \text{in } \Omega.
\]

(16)

Let \( w_1 \) be the unique solution of (8). Thus, \( w_1 \) is a subsolution while \( w_1 + \tilde{w} \) is a supersolution of (12). Thus, (12) admits a solution \( w \) such that

\[
w_1 \leq w \leq w_1 + \tilde{w} \quad \text{in } \Omega.
\]

(17)
The uniqueness of the solution to (12) follows from Lemma 2.1. The estimate (13) follows from (17), (11) and (16).

3 Proof of Theorem 1.1.

Assume that condition (i) in Theorem 1.1 holds and there exists a solution \((u,v)\) of system (1). From (4) we can find \(\varepsilon > 0\) small such that

\[
\sigma := \min \left\{ \frac{2 - \varepsilon}{1 + \max \{1, s\}}, 2 - \varepsilon - \max \left\{ 1, \frac{2r}{1 + \max \{1, p\}} \right\} \right\} > \frac{2}{q}.
\]

Let \(w_1\) be the unique solution of (8). Then, by the comparison principle and (10), there exists a constant \(C > 0\) such that

\[
u \geq w_1 \geq C\delta(x)^{\frac{2}{1 + \max \{1, p\}}} \text{ in } \Omega.
\]

Using this fact in the second equation of (1) we find

\[
-\Delta v \leq v^{-s} + c\delta(x)^{-\frac{2r}{1 + \max \{1, p\}}} \text{ in } \Omega,
\]

for some constant \(c > 0\). Since \(r < 1 + \max \{1, p\}\), we can use Lemma 2.4(iii) to deduce

\[
v \leq C\delta(x)^{\sigma} \text{ in } \Omega,
\]

where \(C > 0\) is a constant. Using now this last estimate in the first equation of (1) we find

\[
-\Delta u \geq u^{-p} + c\delta(x)^{-\sigma q} \text{ in } \Omega,
\]

for some constant \(c > 0\). But this is impossible according to Lemma 2.4(ii), since \(\sigma q > 2\).

4 Proof of Theorems 1.3 and Theorem 1.5

Proof of Theorem 1.3. Let \(w_1, w_2\) be solutions of (8) and

\[
\begin{aligned}
-\Delta w_2 &= w_2^{-s} \quad \text{in } \Omega, \\
w_2 &= 0 \quad \text{on } \partial \Omega,
\end{aligned}
\]

respectively. By Lemma 2.2 we have

\[
w_1(x) \geq C\delta(x)^{\frac{2}{1 + \max \{1, p\}}} \quad \text{and} \quad w_2(x) \geq C\delta(x)^{\frac{2}{1 + \max \{1, p\}}} \text{ in } \Omega.
\]
Hence, by (6) and Lemma 2.4, we may find \(w_3, w_4 \in C^2(\Omega) \cap C(\overline{\Omega})\) such that

\[
\begin{align*}
-\Delta w_3 &= w_3^{-p} + w_2^{-q} \quad \text{in } \Omega, \\
w_3 &= 0 \quad \text{on } \partial \Omega,
\end{align*}
\]

and

\[
\begin{align*}
-\Delta w_4 &= w_4^{-s} + w_1^{-r} \quad \text{in } \Omega, \\
w_4 &= 0 \quad \text{on } \partial \Omega.
\end{align*}
\]

(20)

Set

\[
A = \left\{ (u, v) \in C(\overline{\Omega}) \times C(\overline{\Omega}) : w_1 \leq u \leq w_3 \quad \text{in } \Omega \right\},
\]

For any \((u, v) \in A\), we consider \((Tu, Tv)\) the unique solution of the decoupled system

\[
\begin{align*}
-\Delta (Tu) &= (Tu)^{-p} + v^{-q}, \quad Tu > 0 \quad \text{in } \Omega, \\
-\Delta (Tv) &= (Tv)^{-s} + u^{-r}, \quad Tv > 0 \quad \text{in } \Omega, \\
Tu &= Tv = 0 \quad \text{on } \partial \Omega.
\end{align*}
\]

(21)

Since \(u, v \in C(\overline{\Omega})\) and \(u \geq w_1, v \geq w_2\) in \(\Omega\), by (19) we have

\[
w_1^{-r}(x) \leq c\delta(x)^{-\frac{2r}{1+p(1-r)}}, \quad w_2^{-q}(x) \leq c\delta(x)^{-\frac{2r}{1+q(1-r)}} \quad \text{in } \Omega.
\]

Thus, from (6) and Lemma 2.4, we deduce that the solution \((Tu, Tv)\) of (21) is well posed. Define next

\[
\mathcal{F} : A \to C(\overline{\Omega}) \times C(\overline{\Omega}) \quad \text{by} \quad \mathcal{F}(u, v) = (Tu, Tv) \quad \text{for any } (u, v) \in A.
\]

(22)

Thus, the existence of a solution to system (1) follows once we prove that \(\mathcal{F}\) has a fixed point in \(A\). To this aim, we shall prove that \(\mathcal{F}\) satisfies the conditions:

\[
\mathcal{F}(A) \subseteq A, \quad \mathcal{F}\text{ is compact and continuous.}
\]

Then, by Schauder’s fixed point theorem we deduce that \(\mathcal{F}\) has a fixed point in \(A\), which, by standard elliptic estimates, is a classical solution to (1).

Let us show first that \(\mathcal{F}(A) \subseteq A\). Indeed, comparing (8), (21) and using Lemma 2.1 we easily deduce \(Tu \geq w_1\) in \(\Omega\). Further, since \(v \geq w_2\) we have

\[
\Delta w_3 + w_3^{-p} + w_2^{-q} \leq 0 \leq \Delta (Tu) + (Tu)^{-p} + w_2^{-q} \quad \text{in } \Omega
\]

and \(Tu, w_3 > 0\) in \(\Omega\), \(Tu = w_3 = 0\) on \(\partial \Omega\). By Lemma 2.1 it follows that \(Tu \leq w_3\) in \(\Omega\), and thus \(w_1 \leq Tu \leq w_3\) in \(\Omega\). Similarly \(w_2 \leq Tv \leq w_4\) in \(\Omega\) which shows that \(\mathcal{F}(A) \subseteq A\).

Our next aim is to show that \(\mathcal{F}\) is compact and continuous. Let \((u, v) \in A\). Since \(u\) and \(v\) are bounded, we deduce from (21) that \(Tu, Tv \in C^{0,\gamma}(\overline{\Omega})\) for some \(\gamma \in (0, 1)\).

Since the embedding \(C^{0,\gamma}(\overline{\Omega}) \hookrightarrow C(\overline{\Omega})\) is compact, it follows that \(\mathcal{F}\) is also compact.

It remains to prove that \(\mathcal{F}\) is continuous. To this aim, let \(\{(u_n, v_n)\} \subseteq A\) be such that \(u_n \to u\) and \(v_n \to v\) in \(C(\overline{\Omega})\) as \(n \to \infty\). Using the fact that \(\mathcal{F}\) is compact, there exists
\((U, V) \in \mathcal{A}\) such that up to a subsequence we have

\[
Tu_n \to U, \quad Tv_n \to V \quad \text{in } C(\bar{\Omega}) \quad \text{as } n \to \infty.
\]

On the other hand, by standard elliptic estimates, the sequences \(\{Tu_n\}\) and \(\{Tv_n\}\) are bounded in \(C^{2,\beta}(\bar{\omega})\) \((0 < \beta < 1)\) for any smooth open set \(\omega \subset \subset \Omega\). Therefore, up to a diagonally subsequence, we have

\[
Tu_n \to U, \quad Tv_n \to V \quad \text{in } C^2(\bar{\omega}) \quad \text{as } n \to \infty,
\]

for any smooth open set \(\omega \subset \subset \Omega\). Passing to the limit in the definition of \(Tu_n\) and \(Tv_n\) we find that \((U, V)\) satisfies

\[
\begin{cases}
-\Delta U = U^{-p} + v^{-q}, \quad U > 0 & \text{in } \Omega, \\
-\Delta V = V^{-s} + u^{-r}, \quad V > 0 & \text{in } \Omega, \\
U = V = 0 & \text{on } \partial\Omega.
\end{cases}
\]

By uniqueness of (21), we have that \(Tu = U\) and \(Tv = V\). Hence

\[
Tu_n \to Tu, \quad Tv_n \to Tv \quad \text{in } C(\bar{\Omega}) \quad \text{as } n \to \infty.
\]

This proves that \(\mathcal{F}\) is continuous.

We are now in a position to apply the Schauder’s fixed point theorem. Thus, there exists \((u, v) \in \mathcal{A}\) such that \(\mathcal{F}(u, v) = (u, v)\), that is, \(Tu = u\) and \(Tv = v\). By standard elliptic estimates, it follows that \((u, v)\) is a classical solution of system (1).

**Proof of Theorem 1.5.** (i) Assume \(p < 1, 2q < 1 + \max\{1, s\}\) and let \((u, v)\) be a solution to (1). By Lemma 2.1 we have \(u \geq w_1, v \geq w_2\) in \(\Omega\), where \(w_1, w_2\) are solutions of (8) and (18) respectively. Using the asymptotic behavior described in (10) it follows that

\[
u(x) \geq C\delta(x), \quad v(x) \geq C\delta(x)^{\frac{2}{1 + \max\{1, s\}}} \quad \text{in } \Omega,
\]

where \(C > 0\) is a constant. We next use these estimates for \(u\) and \(v\) in the first equation of our system (1). We find

\[
-\Delta u \leq C\left[\delta(x)^{-p} + \delta(x)^{\frac{-2}{1 + \max\{1, s\}}}\right] \quad \text{in } \Omega.
\]

By our assumption on \(p, q, r, s\) and Lemma 2.3(i) it follows that \(u \in C^2(\Omega) \cap C^1(\bar{\Omega})\).

The proof of (ii) is similar.
5 Proof of Theorem 1.6

We shall prove Theorem 1.6 under the assumption (i). The case where (ii) holds can be treated similarly. Our arguments are divided into two steps.

**Step 1:** For any solution \((u, v)\) of (1) we have \(u \sim w_1\), where \(w_1\) satisfies (8).

With similar arguments to those used in the proof of Theorem 1.3 we have

\[
w_1 \leq u \leq w_3 \quad \text{in } \Omega, \quad w_2 \leq v \leq w_4 \quad \text{in } \Omega,
\]

where \(w_2, w_3\) and \(w_4\) are solutions of (18) and (20). Using the above estimates, Lemma 2.2 and (19) we have

\[
-\Delta u \leq w_1^{-p} + \delta(x)^{-\frac{2q}{1+\max\{1, s\}}} \quad \text{in } \Omega. \tag{23}
\]

If \(0 < p \leq 1\) then condition (i) reads \(\frac{2q}{1+\max\{1, s\}} < 1\) so by (17) we have

\[
\begin{cases}
\delta(x) & \text{if } 0 < p < 1, \\
\delta(x) \log^{1/2} \frac{1}{\delta(x)} & \text{if } p = 1.
\end{cases}
\]

If \(p > 1\) then from (i) and (23) we deduce

\[
-\Delta u \leq c\delta(x)^{-\frac{2q}{1+p}} + \delta(x)^{-\frac{2q}{1+\max\{1, s\}}} \leq C\delta(x)^{-\frac{2p}{1+p}} \quad \text{in } \Omega,
\]

where \(c, C > 0\) are constants. Thus, by Lemma ... we find

\[
u(x) \leq C\delta(x)^{\frac{1}{1+p}} \leq C_0 w_1(x) \quad \text{in } \Omega.
\]

Hence, in both the above cases \(u \sim w_1\).

**Step 2:** System (1) has a unique classical solution.

Let \((u_1, v_1)\) and \((u_2, v_2)\) be two solutions of system (1). From Step 1 we have \(u_1 \sim w_1 \sim u_2\). This means that we can find a constant \(C > 1\) such that \(Cu_1 \geq u_2\) and \(Cu_2 \geq u_1\) in \(\Omega\).

We claim that \(u_1 \leq u_2\) in \(\Omega\). Supposing the contrary, let

\[
M := \inf\{A > 1 : u_1 \leq Au_2 \quad \text{in } \Omega\}.
\]

By our assumption, we have \(M > 1\). From \(u_1 \leq Mu_2 \quad \text{in } \Omega\), it follows that

\[
-\Delta v_1 = u_1^{-r} + v_1^{-s} \geq M^{-r}u_2^{-r} + v_1^{-s} \quad \text{in } \Omega.
\]

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Therefore $v_2$ is a solution and $M^rv_1$ is a supersolution of
\[
\begin{cases}
-\Delta w = w^{-s} + u_2^{-r}, & w > 0 \quad \text{in } \Omega, \\
w = 0 & \text{on } \partial \Omega.
\end{cases}
\]
By Lemma 2.1 we obtain $M^rv_1 \geq v_2$ in $\Omega$ which reads
\[v_1 \geq M^{-r}v_2 \quad \text{in } \Omega.\]
The above estimate yields
\[-\Delta u_1 = u_1^{-p} + v_1^{-q} \leq u_1^{-p} + M^{rq}v_2^{-q} \quad \text{in } \Omega.\]
It follows that $u_2$ is a solution and $M^{-rq}u_1$ is a subsolution of
\[
\begin{cases}
-\Delta w = w^{-p} + v_2^{-q}, & w > 0 \quad \text{in } \Omega, \\
w = 0 & \text{on } \partial \Omega.
\end{cases}
\]
By Lemma 2.1 we now deduce
\[u_1 \leq M^{rq}u_2 \quad \text{in } \Omega.\]
Since $M > 1$ and $qr < 1$, the above inequality contradicts the minimality of $M$. Hence, $u_1 \geq u_2$ in $\Omega$. Similarly we deduce $u_1 \leq u_2$ in $\Omega$, so $u_1 \equiv u_2$ which also yields $v_1 \equiv v_2$. Therefore, the system (1) has a unique solution. This completes the proof of Theorem 1.6.

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**References**


