Developing problem-solving approaches to teaching: Theory and Practice
Four corner-stones in the cultivation of mathematical problem-solving
Aoibhinn Ní Shuilleabháin

Polya’s Plan
George Polya’s book, *How to solve it* (1945), is likely to have been one of the first books to focus on building students’ skills as problem solvers. Polya, a Hungarian professor of mathematics, realised that it was not sufficient that his students knew their mathematical facts — they also needed to have a relational understanding of the subject in order to use mathematics as a tool. While Polya’s book has provided much food for thought for mathematics educators at all levels throughout the decades, the legacy of his writing is in defining a heuristic or framework for students to solve problems:

1. Understanding the problem:
   What is the unknown? What are the data? Can you write them down?
2. Devising a plan:
   Have you seen it before or do you know a related problem? Do you know a more general problem? Can you derive anything useful from the question?
3. Carrying out the plan:
   Check each step and ensure you can prove it is correct.
4. Looking back:
   Check the result. Can you derive the result differently?

This heuristic has been used in varying forms in classrooms all over the world for students encountering questions or activities that are unfamiliar to them. However, providing students with this guiding framework is not necessarily enough in teaching students to become ‘problem solvers’. Educational research is pointing out more and more the need for a classroom environment that nurtures students’ thinking.

The Results of Research: The Four Corner-stones
Amalgamating this literature, it seems there are four corner-stones to cultivating mathematical problem solvers. From Goos’ (2004) study of developing a classroom community, Carpenter et al.’s (1989) research on teachers focusing on students’ mathematical thinking and Schoenfeld’s extensive writings on the teaching of mathematics these are summarised as the following:

- Building a classroom community which incorporates communication of students’ mathematical thinking (Boaler, 1998; Schoenfeld, 1992)
- Focusing on the processes of students’ thinking (van Es & Sherin, 2008)(Carpenter, Fennema, Peterson, Chiang, & Loef, 1989)
- Encouraging students to reflect on their work (Mason, 2002; Schoenfeld, 1994)
- Incorporating problems that are relevant to students’ learning (Stanic & Kilpatrick, 1989)
Encouraging more communication of mathematical thinking is an important element of developing classroom community. Teachers elicit whole class discussions and provide opportunity for students to work together on problems but also give students time to read, think about and attempt problems on their own (Goos, 2004). Encouraging students to collaborate on attempting problems also develops their mathematical understanding through necessitating students to articulate and validate their thinking (Boaler, 1999).

By incorporating more of a focus on how students are thinking within the mathematics classroom, teachers build on their own pedagogical content knowledge and can better equip themselves in facilitating classroom discussions and in guiding student learning. Noticing and interpreting the strategies and language that students use in attempting to solve a problem helps teachers in further building students’ mathematical thinking (Carpenter, Fennema, Peterson, Chiang, & Loef, 1989; van Es & Sherin, 2008).

Reflecting on conjectures and answers is, according to Schoenfeld and Mason, one of the most important features of becoming a problem solver and builds a student’s ability to perceive structures, see connections, capture patterns symbolically, conjecture and prove, abstract and generalise.

Finally, providing students with questions that are culturally relevant and provide meaning for their mathematical learning is important in order to engage students in solving unfamiliar problems (Galbraith, 2013; Stanic & Kilpatrick, 1989).

These points form the basis of my teaching to pre-service teachers who are studying Mathematics Pedagogy as part of their PME with the School of Mathematical Sciences in UCD.

Three such activities are suggested here which can be introduced to students as unfamiliar questions, but where students have enough prior knowledge to attempt the activity.

The first example may be presented as an initial introduction or further enquiry into quadratic patterns (modified from Driscoll, Mathematics Teacher magazine, 1999). The second example incorporates arithmetic series and the geometry of the circle and can be extended within calculus (modified from Paul Stephenson’s activity in the Journal of the Association of Teachers of Mathematics, 2014 (242)). A third example is suggested by Schoenfeld (1994) (originally from the work of Seymour Papert) as a basis for building classroom discussions.

**The Problems**

These activities are not proposed as a ‘finished product’ but as a starting point from which teachers can modify or extend for their own purposes and for their own students.
Window Business Task
Liberty Hall building in Dublin is 16 stories high and is covered entirely with windows on all four sides. Each floor has 38 windows and once a year all the windows are washed. The cost for washing the windows is €2.00 for each first floor window, €2.50 for each second floor window and this pricing pattern continues for each floor.
How much will it cost to wash all of the windows of this building?
What if the building were 30 stories high?
What if it were \( n \) stories high?

How long is the string?

The Concrete Wheel Problem
You are sitting in a room at ground level facing a floor-to-ceiling window which is 6 metres square. A huge solid concrete wheel, 160 kilometres in diameter, is rolling down the street and is about to pass right in front of the window from left to right. The centre of the wheel is moving to the right at 160 km per hour.
What does the view look like from inside the room as the wheel passes by?

Conclusion
For scientists, mathematicians, programmers, and people in all manner of other jobs, solving problems is an integral part of their work. Over the past two decades, mathematics curricula all over the world have reformed to incorporate more of a focus on problem-solving in order to provide students with opportunities to build their knowledge of content through solving problems but also to improve their skills in attempting unfamiliar questions. Building a classroom community where ideas are shared, conjectures made and suggestions trialled is an important part of preparing our students for an environment which rewards problem solvers who learn from failing as much as from succeeding in their mathematical endeavours.
The IMTA Archives

The IMTA have succeeded in establishing an archive of documents in the NUI Maynooth Library. Donated material becomes the property of the University. The material will be documented by the Library. The IMTA have full access and may borrow material. Other users and researchers may view the material by appointment.

A sub-committee of the Council viewed the current set of donated material before approving its lodgement in the archive. This chiefly consists of Minute-books of Council meetings and the Newsletter Editorial Committee from 1964 to 1970s as well as correspondence with Branches regarding the organisation of courses during the same time-period. A full set of Newsletters from 1964 to 2013 has also been donated to the archive and will be lodged in the near future.

Donations of material for the archive are, at present, coordinated by Neil Hallinan.
Email: hallinann@gmail.com