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QUADRATIC FORMS AND FOUR PARTITION FUNCTIONS MODULO 3

JEREMY LOVEJOY AND ROBERT OSBURN

Abstract. Recently, Andrews, Hirschhorn and Sellers have proven congruences modulo 3 for four types of partitions using elementary series manipulations. In this paper, we generalize their congruences using arithmetic properties of certain quadratic forms.

1. Introduction

A partition of a non-negative integer $n$ is a non-increasing sequence whose sum is $n$. An overpartition of $n$ is a partition of $n$ where we may overline the first occurrence of a part. Let $p(n)$ denote the number of overpartitions of $n$, $\overline{p}(n)$ the number of overpartitions of $n$ into odd parts, $\text{ped}(n)$ the number of partitions of $n$ without repeated even parts and $\text{pod}(n)$ the number of partitions of $n$ without repeated odd parts. The generating functions for these partitions are

\[ \sum_{n \geq 0} p(n)q^n = \frac{(-q;q)_\infty}{(q;q)_\infty}, \tag{1.1} \]
\[ \sum_{n \geq 0} \overline{p}(n)q^n = \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}, \tag{1.2} \]
\[ \sum_{n \geq 0} \text{ped}(n)q^n = \frac{(-q^2;q^2)_\infty}{(q^2;q^2)_\infty}, \tag{1.3} \]
\[ \sum_{n \geq 0} \text{pod}(n)q^n = \frac{(-q^2;q^2)_\infty}{(q^2;q^2)_\infty}. \tag{1.4} \]

where as usual

\[ (a;q)_n := (1 - a)(1 - aq) \cdots (1 - aq^{n-1}). \]

The infinite products in (1.1)–(1.4) are essentially the four different ways one can specialize the product $(-aq;q)_\infty/(bq;q)_\infty$ to obtain a modular form whose level is relatively prime to 3.

A series of four recent papers examined congruence properties for these partition functions modulo 3 \cite{1, 5, 6, 7}. Among the main theorems in these papers are the following congruences (see Theorem 1.3 in \cite{6}, Corollary 3.3 and Theorem 3.5 in \cite{1}, Theorem 1.1 in \cite{5} and Theorem 3.2 in \cite{7}, respectively). For all $n \geq 0$ and $\alpha \geq 0$ we have

\[ \overline{p}(3^{2\alpha}(An + B)) \equiv 0 \pmod{3}, \tag{1.5} \]

where $An + B = 9n + 6$ or $27n + 9$, 

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\[ \text{ped}(3^{2\alpha+3}n + \frac{17 \cdot 3^{2\alpha+2} - 1}{8}) \equiv \text{ped}(3^{2\alpha+2}n + \frac{19 \cdot 3^{2\alpha+1} - 1}{8}) \equiv 0 \pmod{3}, \quad (1.6) \]

\[ \bar{p}(3^{2\alpha}(27n + 18)) \equiv 0 \pmod{3} \quad (1.7) \]

and

\[ \text{pod}(3^{2\alpha+3} + \frac{23 \cdot 3^{2\alpha+2} + 1}{8}) \equiv 0 \pmod{3}. \quad (1.8) \]

We note that congruences modulo 3 for \( \bar{p}(n), \text{po}(n) \) and \( \text{ped}(n) \) are typically valid modulo 6 or 12. The powers of 2 enter trivially (or nearly so), however, so we do not mention them here.

The congruences in (1.5)–(1.8) are proven in [1, 5, 6, 7] using elementary series manipulations.

If we allow ourselves some elementary number theory, we find that much more is true.

With our first result we exhibit formulas for \( \text{po}(3n) \) and \( \text{ped}(3n + 1) \) modulo 3 for all \( n \geq 0 \).

These formulas depend on the factorization of \( n \), which we write as

\[ n = 2^a3^b \prod_{i=1}^{r} p_i^{v_i} \prod_{j=1}^{s} q_j^{w_j}, \quad (1.9) \]

where \( p_i \equiv 1, 5, 7 \) or 11 (mod 24) and \( q_j \equiv 13, 17, 19 \) or 23 (mod 24). Further, let \( t \) denote the number of prime factors of \( n \) (counting multiplicity) that are congruent to 5 or 11 (mod 24).

Let \( R(n, Q) \) denote the number of representations of \( n \) by the quadratic form \( Q \).

**Theorem 1.1.** For all \( n \geq 0 \) we have

\[ \text{po}(3n) \equiv f(n)R(n, x^2 + 6y^2) \pmod{3} \]

and

\[ \text{ped}(3n + 1) \equiv (-1)^{n+1}R(8n + 3, 2x^2 + 3y^2) \pmod{3}, \]

where \( f(n) \) is defined by

\[ f(n) = \begin{cases} -1, & n \equiv 1, 6, 9, 10 \pmod{12}, \\ 1, & \text{otherwise}. \end{cases} \]

Moreover, we have

\[ \text{po}(3n) \equiv f(n)(1 + (-1)^a\cdot b+t) \prod_{i=1}^{r} (1 + v_i) \prod_{j=1}^{s} \left( \frac{1 + (-1)^{w_j}}{2} \right) \pmod{3} \quad (1.10) \]

and

\[ (-1)^n \text{ped}(3n + 1) \equiv \text{po}(48n + 18) \pmod{3}. \quad (1.11) \]

There are many ways to deduce congruences from Theorem 1.1. For example, calculating the possible residues of \( x^2 + 6y^2 \) modulo 9 we see that

\[ R(3n + 2, x^2 + 6y^2) = R(9n + 3, x^2 + 6y^2) = 0, \]

and then (1.10) implies that \( \text{po}(27n) \equiv \text{po}(3n) \pmod{3} \). This gives (1.5). The congruences in (1.6) follow from those in (1.5) after replacing \( 48n + 18 \) by \( 3^{2\alpha}(48(3n + 2) + 18) \) and \( 3^{2\alpha}(48(9n + 6) + 18) \) in (1.11). We record two more corollaries, which also follow readily from Theorem 1.1.
Corollary 1.2. For all \( n \geq 0 \) and \( \alpha \geq 0 \) we have
\[
\bar{p}_o(2^{2\alpha}(An + B)) \equiv 0 \pmod{3},
\]
where \( An + B = 24n + 9 \) or \( 24n + 15 \).

Corollary 1.3. If \( \ell \equiv 1, 5, 7 \) or \( 11 \pmod{24} \) is prime, then for all \( n \) with \( \ell \nmid n \) we have
\[
\bar{p}_o(3\ell^2n) \equiv 0 \pmod{3}. \quad (1.12)
\]

For the functions \( \bar{p}(3n) \) and \( pod(3n + 2) \) we have relations not to binary quadratic forms but to \( r_5(n) \), the number of representations of \( n \) as the sum of five squares. Our second result is the following.

Theorem 1.4. For all \( n \geq 0 \) we have
\[
\bar{p}(3n) \equiv (-1)^n r_5(n) \pmod{3}
\]
and
\[
\text{pod}(3n + 2) \equiv (-1)^n r_5(8n + 5) \pmod{3}.
\]
Moreover, for all odd primes \( \ell \) and \( n \geq 0 \), we have
\[
\bar{p}(3\ell^2n) \equiv \left( \ell - \frac{n}{\ell} \right) + 1 \pmod{3}
\]
\[
\bar{p}(3\ell^2n) \equiv \ell \bar{p}(3n) - \ell \bar{p}(3n/\ell^2) \pmod{3} \quad (1.13)
\]
and
\[
(-1)^{n+1} \text{pod}(3n + 2) \equiv \bar{p}(24n + 15) \pmod{3}, \quad (1.14)
\]
where \( \left( \frac{n}{\ell} \right) \) denotes the Legendre symbol.

Here we have taken \( \bar{p}(3n/\ell^2) \) to be 0 unless \( \ell^2 \mid 3n \). Again there are many ways to deduce congruences. For example, (1.7) follows readily upon combining (1.13) in the case \( \ell = 3 \) with the fact that
\[
r_5(9n + 6) \equiv 0 \pmod{3},
\]
which is a consequence of the fact that \( R(9n + 6, x^2 + y^2 + 3z^2) = 0 \). One can check that (1.8) follows similarly. For another example, we may apply (1.13) with \( n \) replaced by \( n\ell \) for \( \ell \equiv 2 \pmod{3} \) to obtain

Corollary 1.5. If \( \ell \equiv 2 \pmod{3} \) is prime and \( \ell \nmid n \), then
\[
\bar{p}(3\ell^3n) \equiv 0 \pmod{3}.
\]

2. Proofs of Theorems 1.1 and 1.4

Proof of Theorem 1.1. On page 364 of [6] we find the identity
\[
\sum_{n \geq 0} \bar{p}_o(3n)q^n = \frac{D(q^3)D(q^6)}{D(q)^2},
\]
where
\[
D(q) := \sum_{n \in \mathbb{Z}} (-1)^n q^n^2.
\]
Reducing modulo 3, this implies that
\[ \sum_{n \geq 0} \bar{p}(3n)q^n \equiv \sum_{x,y \in \mathbb{Z}} (-1)^{x+y} q^{x^2+6y^2} \pmod{3} \]
\[ \equiv \sum_{n \geq 0} f(n)R(n,x^2+6y^2)q^n \pmod{3}. \]

Now it is known (see Corollary 4.2 of [3], for example) that if \( n \) has the factorization in (1.9), then
\[ R(n,x^2+6y^2) = (1 + (-1)^{a+b+t}) \prod_{i=1}^{r} (1 + v_i) \prod_{j=1}^{s} \left( \frac{1 + (-1)^{w_j}}{2} \right). \] (2.1)

This gives (1.10). Next, from [1] we find the identity
\[ \sum_{n \geq 0} p_{ed}(3n+1)q^n = \frac{D(q^3)\psi(-q^3)}{D(q)^2}, \]
where
\[ \psi(q) := \sum_{n=0}^{\infty} q^{n(n+1)/2}. \]

Reducing modulo 3, replacing \( q \) by \(-q^8\) and multiplying by \( q^3 \) gives
\[ \sum_{n \geq 0} (-1)^{n+1} p_{ed}(3n+1)q^{8n+3} \equiv \sum_{n \geq 0} R(8n+3, 2x^2 + 3y^2)q^{8n+3} \pmod{3}. \]

It is known (see Corollary 4.3 of [3], for example) that if \( n \) has the factorization given in (1.9), then
\[ R(n,2x^2 + 3y^2) = (1 - (-1)^{a+b+t}) \prod_{i=1}^{r} (1 + v_i) \prod_{j=1}^{s} \left( \frac{1 + (-1)^{w_j}}{2} \right). \]

Comparing with (2.1) finishes the proof of (1.11). \( \Box \)

Proof of Theorem 1.4. On page 3 of [5] we find the identity
\[ \sum_{n \geq 0} \bar{p}(3n)q^n \equiv \frac{D(q^3)^2}{D(q)} \pmod{3}. \]

Reducing modulo 3 and replacing \( q \) by \(-q\) yields
\[ \sum_{n \geq 0} (-1)^n \bar{p}(3n)q^n \equiv \sum_{n \geq 0} r_5(n)q^n \pmod{3}. \]

It is known (see Lemma 1 in [4], for example) that for any odd prime \( \ell \) we have
\[ r_5(\ell^2n) = \left( \ell^3 - \ell \left( \frac{n}{\ell} \right) + 1 \right) r_5(n) - \ell^3 r_5(n/\ell^2). \]
Here $r_5(n/\ell^2) = 0$ unless $\ell^2 \mid n$. Replacing $r_5(n)$ by $(-1)^n\overline{p}(3n)$ throughout gives (1.13). Now equation (1) of [7] reads

$$\sum_{n \geq 0} (-1)^n pod(3n + 2)q^n = \frac{\psi(q^3)^3}{\psi(q)^4}.$$ 

Reducing modulo 3 we have

$$\sum_{n \geq 0} (-1)^n pod(3n + 2)q^n \equiv \psi(q)^5 \pmod{3}$$

$$\equiv \sum_{n \geq 0} r_5(8n + 5)q^n \pmod{3}$$

$$\equiv -\sum_{n \geq 0} p(24n + 15)q^n \pmod{3},$$

where the second congruence follows from Theorem 1.1 in [2]. This implies (1.14) and thus the proof of Theorem 1.4 is complete.

\[\square\]

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